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THE DETERMINATION OF A HARMONIC FUNCTION BY ITS SIGN

Abstract

We give an improvement of the result that if $hP \geq 0$ on $\mathbb{R}^n$, where h is a harmonic function and P a non-trivial harmonic polynomial, then h is proportional to P.

The classical Liouville theorem for harmonic functions may be stated as follows: if h is a non-negative harmonic function on $\mathbb{R}^n$, where $n \geq 2$, then h is constant. Many years ago, Kuran [6, Theorem 1] generalized this result by showing that if h is harmonic on $\mathbb{R}^n$ and P is a harmonic polynomial such that $P \not\equiv 0$ and $hP \geq 0$ on the complement of some compact set, then h is a constant multiple of P. In this note we show that the same conclusion holds under milder hypotheses.

Let $S(r)$ denote the boundary of the open ball $B(r)$ of radius r centered at the origin of $\mathbb{R}^n$.

Theorem. *Let (r_j) be an unbounded sequence of positive numbers. If h is harmonic on $\mathbb{R}^n$ and $P \not\equiv 0$ is a harmonic polynomial such that $hP \geq 0$ on $\cup_{j=1}^{\infty} S(r_j)$, then h is a constant multiple of P.*

We shall prove the Theorem by showing that if its hypotheses hold, then in fact $hP \geq 0$ in the complement of some compact set; the conclusion will then follow from the theorem of Kuran cited above. After giving the proof, we discuss possible relaxations of the hypotheses.

For a positive integer p, let Δ^p denote the p^{th} iterate of the Laplacian operator on $\mathbb{R}^n$. Recall that a real-analytic function u on $\mathbb{R}^n$ is said to be *polyharmonic of order p* if $\Delta^p \equiv 0$. We shall need the following result of Nakai and Tada [7, Theorem 1]. (For a short, elementary proof of a somewhat stronger result, see [1].)

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Lemma 1. *If u is polyharmonic of order p on $\mathbb{R}^n$ and there exists an increasing divergent sequence (r_j) of positive numbers such that*

$$\liminf_{j\to\infty} r_j^{-s} \min\{u(x) : x \in S(r_j)\} \geq 0,$$

where $s > 2p-2$, then u is a polynomial of degree less than s.

Now let h and P be as in the Theorem, and let m denote the degree of P. Denoting the coordinates of a point x by $(x_1, \ldots, x_n)$, we calculate that if v is polyharmonic of order q on $\mathbb{R}^n$, then

$$\Delta^{q+1}(x_k v) = \Delta^q(x_k \Delta v) = \Delta^{q-1}(x_k \Delta^2 v) = \cdots = \Delta(x_k \Delta^q v) \equiv 0$$

for $k \in \{1, 2, \ldots, n\}$, so the function $x \mapsto x_k v(x)$ is polyhormonic of order $q+1$. Hence, by induction, if M is a monomial of degree m, then Mv is polyharmonic of order $m+q$, and the same holds if M is a polynomial of degree m. In particular, hP is polyharmonic of order $m+1$.

By working with a subsequence, if necessary, we may suppose that the sequence (r_j) is increasing. Since $hP \geq 0$ on $S(r_j)$ for each j, we can apply Lemma 1 with $u = hP$, $p = m+1$ and $s = 2m+1$. Thus we find that hP is a polynomial of degree at most $2m$.

Let Ω denote the open set $\{x \in \mathbb{R}^n : h(x)P(x) < 0\}$. Since hP is a polynomial, it follows from a theorem of Whitney [8, Theorem 4] that Ω has at most finitely many connected components. Moreover, since $\Omega \cap \cup_{j=1}^{\infty} S(r_j) = \emptyset$, each such component is bounded. Hence Ω is bounded, which is to say that $hP \geq 0$ on the complement of some compact set. It now follows from Kuran's theorem [6, Theorem 1] that h is a constant multiple of P.

The question naturally arises whether the spheres $S(r_j)$ can be replaced by more general sets. If (ω_j) is an expanding sequence of bounded domains with $\cup_{j=1}^{\infty} \omega_j = \mathbb{R}^n$, does the Theorem hold with $\partial\omega_j$ in place of $S(r_j)$? Example 1 below shows that in general the answer to this question is negative, even if $\partial\omega_j$ is close to being spherical. The verification of the example requires the following result of Gauthier, Goldstein and Ow ([4] for the case $n = 2$ and [5] for $n \geq 3$) or see e.g. [3, Corollary 3.8].

Lemma 2. *Let E be a closed subset of $\mathbb{R}^n$ such that the complement of E in the one-point compactifiction of $\mathbb{R}^n$ is connected and locally connected. If g is a harmonic function on some open set containing E and $\epsilon > 0$, then there exists a harmonic function h on $\mathbb{R}^n$ such that $|h - g| < \epsilon$ on E.*

Example 1. *For positive numbers r, ϵ we define*

$$\omega(r, \epsilon) = B(r) \cup \{x \in B(r+\epsilon) : x_n < 0\}.$$

Let P be the harmonic polynomial given by $P(x) = x_n$, and let (r_j) and (ϵ_j) be sequences of positive numbers such that $r_j \to +\infty$ and $r_j + \epsilon_j < r_{j+1}$ for each j. We claim that there exists a harmonic function h on $\mathbb{R}^n$ such that $hP \geq 0$ on $\cup_{j=1}^{\infty} \partial\omega(r_j, \epsilon_j)$ but h is not a constant multiple of P.

To verify this, let $E_1,\ E_2$ be the closed sets given by

$$E_1 = \cup_{j=1}^{\infty} \{x \in S(r_j) : x_n \geq 0\} \text{ and } E_2 = \cup_{j=1}^{\infty} \{x \in S(r_j + \epsilon_j) : x_n \leq 0\}.$$

Let $E = E_1 \cup E_2$ and let $\Omega_1,\ \Omega_2$ be disjoint open sets containing $E_1,\ E_2$ respectively. We define a function g by $g(x) = \begin{cases} 1 & \text{if } x \in \Omega_1 \\ -1 & \text{if } x \in \Omega_2 \end{cases}$. Then g is harmonic on an open set containing E. The topological hypotheses of Lemma 2 are clearly satisfied, and we conclude that there is a harmonic function h on $\mathbb{R}^n$ such that $|h - g| < 1$ on E. Thus $h > 0$ on E_1 and $h < 0$ on E_2. It follows that $hP \geq 0$ on $E_1 \cup E_2 \cup \{x \in \mathbb{R}^n : x_n = 0\}$, which contains $\cup_{j=1}^{\infty} \partial\omega(r_j, \epsilon_j)$. Since $0 < h(0, \dots, 0, r_j) < 2$, we see that h is not a constant multiple of P.

It is unclear whether the Theorem remains true if $\cup_{j=1}^{\infty} S(r_j)$ is replaced by $\cup_{j=1}^{\infty} \partial\omega_j$ in the case where, for example, (ω_j) is an expanding sequence of (not necessarily concentric) balls or, more generally, ellipsoids, with $\cup_{j=1}^{\infty} \omega_j = \mathbb{R}^n$.

The question whether the Theorem holds if P is merely supposed to be harmonic, not necessarily a polynomial, is easily answered. An observation similar to that in the following example was made in [2, Example 4].

Example 2. *Let $a = (a_1, \dots, a_2)$ be a point in the unit sphere Σ in $\mathbb{R}^{n-1}$ and let h_a be defined on $\mathbb{R}^n$ by*

$$h_a(x) = \exp(a_1 x_1 + \cdots + a_{n-1} x_{n-1}) \sin x_n.$$

Then h_a is harmonic on $\mathbb{R}^n$. If $a,\ b \in \Sigma$, then $h_a h_b \geq 0$ on $\mathbb{R}^n$, and if $a \neq b$, then the functions $h_a,\ h_b$ are not constant multiples of one another.

This example shows that the Theorem fails if P is one of the function h_a. We note in passing that in the case $n \geq 3$ there are uncountably many distinct functions h_a, whereas in the case $n = 2$ there are only two such functions. Much insight into the differences between the plane and higher dimensions as regards the set where a harmonic function is positive is given in [2].

The functions h_a in Example 2 are of exponential growth. The question, first raised in [2, p. 215] in relation to Kuran's theorem [6, Theorem1], as to whether P can be taken to be a harmonic function of slow growth remains open, to the best of my knowledge: is it enough, for example, to suppose that $\max_{S(r)} \log(1 + |P|) = o(r)$ as $r \to +\infty$?

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