Real Analysis Exchange
Vol. (), , pp. 919–922

Krzysztof Ciesielski,* Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA (K_Cies@math.wvu.edu)
http://www.math.wvu.edu/~kcies

MEASURE ZERO SETS WHOSE ALGEBRAIC SUM IS NON-MEASURABLE

Abstract

In this note we will show that for every natural number $n>0$ there exists an $S\subset[0,1]$ such that its n-th algebraic sum $nS=S+\cdots+S$ is a nowhere dense measure zero set, but its $n+1$-st algebraic sum $nS+S$ is neither measurable nor it has the Baire property. In addition, the set S will be also a Hamel base, that is, a linear base of $\mathbb{R}$ over $\mathbb{Q}$.

We use the standard notation as in [2]. Thus symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathfrak{c}$ stand for the set of real numbers, the set of rational numbers, the set of integers, and the cardinality of $\mathbb{R}$, respectively. The set of natural numbers $\{0,1,2,\ldots\}$ will be denoted by either $\mathbb{N}$ or ω, and $|X|$ will stand for the cardinality of a set X. For $A,B\subseteq\mathbb{R}$ we put $A+B=\{a+b\colon a\in A \ \&\ b\in B\}$ and $\mathrm{LIN}_{\mathbb{Q}}(A)$ will stand for the linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by A. In addition for $0<n<\omega$ symbol $[X]^n$ will stand for the family of all n-element subsets of X and nA for the n-th algebraic sum of A, that is,

$$nA=\left\{\sum_{i=1}^{n}a_i\colon a_i\in A \ \text{ for all } i=1,2,\ldots,n\right\}.$$

Thus $2S=S+S$. For a Polish space X we say that $B\subset X$ is a *Bernstein set (in X)* provided B and $X\setminus B$ intersect every perfect set $P\subset X$. Clearly every Bernstein subset of an interval in $\mathbb{R}$ is neither measurable nor it has the Baire property.

For $0<p<q<\omega$ let Z_q^p stand for the set of all numbers $x\in[0,1]$ which in base q have representations formed with digits $0,\ldots,p$, that is,

$$Z_q^p=\left\{\sum_{i=1}^{\infty}\frac{a(i)}{q^i}\colon a(i)\in\{0,\ldots,p\} \text{ for every } i=1,2,\ldots\right\}.$$

Key Words: sets, algebraic sum, measurability.
Mathematical Reviews subject classification: 28A05
Received by the editors December 27, 2000

*Papers authored or co-authored by a Contributing Editor are managed by a Managing Editor or one of the other Contributing Editors.

Clearly if $p < q-1$ then the set Z_q^p is closed, nowhere dense, and has Lebesgue measure zero. Also, for every $k \in \mathbb{N}$

$$\text{if } 0 < kp < q \text{ then } kZ_q^p = Z_q^{kp}. \tag{1}$$

In what follows we will also need the following lemma.

Lemma 1. *Let $n > 0$ be a natural number, $q = 2n+2$, and $A \subset \mathbb{R}$ with $|A| < \mathfrak{c}$. Then for every $x \in [0,1] \setminus \mathrm{LIN}_{\mathbb{Q}}(A)$ there exists $\{x_0, \dots, x_n\} \in [Z_q^2]^{n+1} \setminus A$ such that $x = x_0 + \dots + x_n$ and $\{x_0, \dots, x_n\} \cup A$ is linearly independent over $\mathbb{Q}$.*

PROOF. In the proof any sequence $\langle z(i) \in \{0, \dots, q-1\} \colon i = 1, 2, \dots \rangle$ will be treated as a base q representation of a number $z \in [0,1]$, that is, $z = \sum_{i=1}^{\infty} \frac{z(i)}{q^i}$.

Let $x \in [0,1] \setminus \mathrm{LIN}_{\mathbb{Q}}(A)$. Then $x > 0$ and it can be represented as sequence $\langle x(i) \in \{0, \dots, q-1\} \colon i = 1, 2, \dots \rangle$. We can also assume that the set $T = \{i \colon x(i) > 0\}$ is infinite, since any number with almost all $x(i)$'s being zero has also another representation with almost all $x(i)$'s equal to $q-1$.

Let $\{T_0, \dots, T_{n-1}\}$ be a partition of T onto infinite sets. For each $i \notin T$ and $j \le n$ define $x_j^*(i) = 0$. For $i \in T_k$ we choose $x_j^*(i) \in \{0,1,2\}$ such that

$$x_k^*(i) \in \{0,1\},\ x_n^*(i) \in \{1,2\}, \text{ and } x_0^*(i) + \dots + x_n^*(i) = x(i). \tag{2}$$

Such a choice can be made since $0 < x(i) \le 2n+1$ for any such i. Next, by induction on $k < n$, we will choose the sequences $\langle s_k(i) \in \{0,1\} \colon i < \omega \rangle$ such that

$$s_k(i) = 0 \text{ for all } i \notin T_k. \tag{3}$$

We aim for $x_n = x_n^* - \sum_{j=0}^{n-1} s_j$ and $x_k = x_k^* + s_k$ for every $k < n$. Notice that, by (2) and (3), such a definition ensures that x_j's belong to Z_q^2 and that their sum is equal to x. The freedom of choice of s_i's will allow us to enforce required linear independence.

In our inductive construction we will use the following notation for $Z \subset \mathbb{R}$ and $T \subset \{1, 2, 3, \dots\}$:

$$Z \restriction T = \left\{ z \restriction T \colon\ z \text{ maps } \{1,2,3,\dots\} \text{ into } \{0, \dots, q-1\} \ \ \& \ \ \sum_{i=1}^{\infty} \frac{z(i)}{q^i} \in Z \right\}.$$

Now, since $A_0 = (\mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\})) \restriction T_0$ has cardinality less than $\mathfrak{c}$ we can easily choose s_0 as in (3) for which $\langle x_0^*(i) + s_0(i) \colon i \in T_0 \rangle \notin A_0$. This clearly implies that $x_0 = x_0^* + s_0 \notin \mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\})$. In general, if for some $0 < k < n$ the sequences s_i's (so also x_i's) are already defined for all $i < k$

we put $A_k = (\mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\} \cup \{x_i : i < k\})) \restriction T_k$ and choose s_k as in (3) for which $\langle x_k^*(i) + s_k(i) : i \in T_k \rangle \notin A_k$. This ensures that

$$x_k = x_k^* + s_k \notin \mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\} \cup \{x_i : i < k\}). \tag{4}$$

This finishes the inductive construction.

To finish the proof it is enough to notice that, by our assumption that $x \notin \mathrm{LIN}_{\mathbb{Q}}(A)$ and (4), $x_n = x - (x_0 + \cdots + x_{n-1}) \notin \mathrm{LIN}_{\mathbb{Q}}(A \cup \{x_i : i < n\})$. ■

Theorem 2. *For every natural number $n > 0$ there exists a Hamel basis $H \subset Z_{2n+2}^2$ such that for every natural number $m > n$ the set $[0,1] \cap mH$ is Bernstein in $[0,1]$.*

Before we prove the theorem, we like to list some of its corollaries. First note that if H is as in the theorem then, by (1), for every natural number $k \leq n$ we have $kH \subset Z_{2n+2}^{2k}$. So kH is nowhere dense and it has measure 0. In particular,

Corollary 3. *For every natural number $n > 0$ there exists an $S \subset [0,1]$ such that for every $k \in \mathbb{N}$, kS is Lebesgue measurable if and only if $k \leq n$.*

Corollary 3 used with $n = 1$ implies that there exists a measure 0 subset S of $[0,1]$ such that $S + S$ is non-measurable. This fact has been known for quite a while and was used by several authors. (See e.g. [5, 3, 1].) However, we have not been able to locate its proof in the literature. This prompted the author to write this note. At the same time we should note here that it is very easy to find a meager measure zero set $E \subset \mathbb{R}$ such that for some natural number n the n-th algebraic sum nE of E is neither measurable nor it has the Baire property. For this take a meager measure zero Hamel basis $H \subset \mathbb{R}$ and note that $E = \{qh : q \in \mathbb{Q} \ \&\ h \in H\}$ has these properties, since $\mathbb{R} = \bigcup_{n<\omega} nE$. Sets with the properties similar to these of our set E has been also investigated in [4].

Notice also, that if we put $S = \mathbb{Z} + H$ then we get the following

Corollary 4. *For every natural number $n > 0$ there exists an $S \subset \mathbb{R}$ such that for every $k \in \mathbb{N}$: if $k \leq n$ then kS is meager and it has Lebesgue measure 0; and, if $k > n$ then kS is a Bernstein set in $\mathbb{R}$.*

It is also not difficult to complicate a bit shifts in the definition of S from Corollary 4 to ensure that S is still a Hamel basis.

Corollary 5. *There exists a Bernstein set $B \subset \mathbb{R}$ such that kB is also Bernstein for every natural $k > 0$.*

PROOF. Just put $B = S + S$ where S is from Corollary 4 used with $n = 1$. ■

PROOF OF THEOREM 2. Fix an $n > 0$ and let $\{P_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of all perfect subsets of $[0, 1]$. By transfinite induction on $\xi < \mathfrak{c}$ construct a sequence $\langle H_\xi \in [Z^2_{2n+2}]^{n+1} \colon \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$, $\sum_{y \in H_\xi} y \in P_\xi$ and H_ξ is linearly independent over $\mathrm{LIN}_{\mathbb{Q}}\left(\bigcup_{\zeta<\xi} H_\zeta\right)$.

To make such a choice, just put $A = \bigcup_{\zeta<\xi} H_\zeta$, pick an $x \in P_\xi \setminus \lim(A)$, and apply Lemma 1 to find H_ξ.

The set $\bar{H} = \bigcup_{\xi<\mathfrak{c}} H_\xi \subset Z^2_{2n+2}$ is linearly independent over $\mathbb{Q}$. Since $(n + 1)Z^2_{2n+2} \supset [0, 1]$ we can find an $H \subset Z^2_{2n+2}$ containing $\bar{H}$ which is a Hamel basis. We will show that H is as desired.

For this notice that for every $m > n$ the set mH intersects every $P \subset [0, 1]$. This is clear for $m = n + 1$, since for $\xi < \mathfrak{c}$ such that $P = P_\xi$ the number $\sum_{y \in H_\xi} y \in P_\xi$ belongs to P and $(n + 1)H_\xi \subset (n + 1)H$.

So, take an $m > n + 1$ and let $k = m - (n + 1)$. Taking a subset of P, if necessary, we can assume that there exists an $\varepsilon > 0$ such that $P \subset (\varepsilon, 1]$. By the part with $m = n+1$ the set H contains arbitrary small elements. So, there exists a $z \in (0, \varepsilon) \cap kH$. Then $P - z \subset [0, 1]$ and, again by case $m = n + 1$, there exists an $x \in (P - z) \cap (n + 1)H$. But then $x + z \in P \cap mH$. So $(*)$ is proved.

To finish the theorem it is enough to note that for every $m > n$ and perfect set $P \subset [0, 1]$ by $(*)$ we have $P \cap (m + 1)H \neq \emptyset$, while $(m + 1)H \subset \mathbb{R} \setminus mH$ (since H is a Hamel basis). ■

References

[1] J. Marshall Ash, Stefan Catoiu, and Ricardo Ríos-Collantes-de-Terán, *On the n-th quantum derivative*, preprint.

[2] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Stud. Texts **39**, Cambridge Univ. Press 1997.

[3] H. Fejzić, C. E. Weil, *Repairing the proof of a classical differentiation result*, Real Anal. Exchange **19** (1993-94), 639–643.

[4] F. B. Jones, *Measure and other properties of a Hamel basis*, Bull. Amer. Math. Soc. **48** (1942), 472–481.

[5] B. Thomson, *Symmetric Properties of Real Functions*, Marcel Dekker, 1994.