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A FUNDAMENTAL THEOREM OF CALCULUS FOR THE KURZWEIL-HENSTOCK INTEGRAL IN $\mathbb{R}^m$

Abstract

In this paper, we give a characterization of the Kurzweil-Henstock integral in n-dimensional space.

1 Introduction

The Kurzweil-Henstock integral is now well-known. See, for example, [1] or [5]. It includes the Rieman integral, Riemann-Stieltjes integral, Lebesgue integral, Denjoy-Perron integral among others. The Kurzweil-Henstock theory on the real line has been fully developed. However, its extension to functions defined on an interval in the m-dimensional space is not straightforward. One difficulty is that the primitive function is no longer differentiable almost everywhere as it is in the one-dimensional case. An example can be found in [2], p.95. Some years ago, Henstock [3], p.143, showed that the primitive function is in fact differentiable everywhere except for a set of inner variation zero. We shall make precise the definition of inner variation later in the paper. Lu Jitan [6] proved the same result independently and in essence gave the converse. Lu described his result in terms of Γ-measure zero, and regarded Γ-measure zero as an extension of Lebesgue measure zero. In other words, Lu provided a version of the fundamental theorem of calculus for the Kurzweil-Henstock integral in $\mathbb{R}^m$. In this paper, we formulate an alternative version, again in $\mathbb{R}^m$, in the language of Henstock. The difference between Lu's version and ours lies in the following: Lu considered singular points to be those at which the primitive function is not differentiable and showed that the set of singular

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points is of Γ-measure zero. In our case, we consider the singular points x to be those satisfying $|F(I) - f(x)|I|| \geq \varepsilon |I|$ for some intervals I, and hence depending on ε, where F is the primitive function of f. In short, we consider a larger set of singular points than that of Lu.

2 Preliminaries

A closed and bounded interval E in $\mathbb{R}^m$ refers to a rectangle in $\mathbb{R}^m$, that is

$$E = \{(x_1, x_2, ...x_m) : a_i \leq x_i \leq b_i \quad \text{for} \quad i = 1, 2, ...m\}.$$

The measure of a set X, denoted by $|X|$, is the Lebesgue measure and in particular, the measure of E is its volume $|E| = \prod_{i=1}^{m} (b_i - a_i)$. A division $\Delta = \{(x, I)\}$ of E is any finite set of point-interval pairs with $x = (x_1, x_2, ...x_m)$ a vertex of the rectangle I, and the intervals nonoverlapping such that $\bigcup_{\Delta} I = E$. In contrast, a partial division D of E is a finite set of point-interval pairs described similarly except that $\bigcup_{D} I \subseteq E$. A partial division $D = \{(x, I)\}$ is said to be tagged in X or X-tagged if $x \in X$.

As in the case of $\mathbb{R}$, the gauge δ is defined similarly except that the one-dimensional open intervals are changed to open balls. That is, $\{(x, I)\}$ is δ-fine if for each (x, I), I is contained in the open ball whose center is x and whose radius is $\delta(x)$.

A function f is said to be Kurzweil-Henstock integrable on E if there exists $A \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists a gauge $\delta : E \longrightarrow (0, 1)$ such that $|(\Delta) \sum f(x) |I| - A| < \varepsilon$ whenever Δ is a δ-fine division of E. The number A is called the integral of f over E and it is written as $A = \int_E f$. Moreover, f is Kurzweil-Henstock integrable on any subinterval I of E. This gives rise to an additive function of interval F such that for any subinterval $I \subset E$, $F(I) = \int_I f$ and for any collection of nonoverlapping subintervals $\{I_i : 1 \leq i \leq n\}$, $F\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} F(I_i)$. The function F is called the primitive of f and is uniquely determined.Henstock's lemma is also true, that is, for every $\varepsilon > 0$ there exists a gauge $\delta : E \rightarrow (0, 1)$ such that $(\Delta) \sum |f(x)|I| - F(I)| < \varepsilon$ whenever Δ is any δ-fine division of E. This means that the above inequality is true even for any δ-fine partial division D.

The function F is said to be differentiable at $x \in E$ with derivative value $f(x)$ if $\lim_{|I| \to 0} \frac{F(I)}{|I|} = f(x)$ where I is a nondegenerate closed interval in E and x is a vertex of I.

The functions f_n are equiintegrable with corresponding integrals $A_n \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a gauge $\delta : E \rightarrow (0, 1)$ independent of n such that for all n $|(\Delta)\sum f_n(x)|I| - A_n| < \varepsilon$ whenever Δ is a δ-fine division of E. Furthermore, if the functions F_n are the primitives of f_n then for every $\varepsilon > 0$ there exists a gauge $\delta : E \to (0,1)$ independent of n such that for all n

$$(D)\sum |f_n(x)|I| - F_n(I)| < \varepsilon$$

whenever D is a δ-fine partial division of E.

3 A Fundamental Theorem

For a given pair of functions f and F on E, let

$$\Gamma_\varepsilon = \{(x,I) : |F(I) - f(x)|I|| \geq \varepsilon |I|\}.$$

From the collection of all δ-fine point-interval pairs $(x,I) \in \Gamma_\varepsilon$, a subset X of E may be obtained, that is

$$X(\varepsilon,\delta) = \{x \in E : \text{there is a } \delta\text{-fine } (x,I) \in \Gamma_\varepsilon\}.$$

Here, the collection of all δ-fine point-interval pairs $(x,I) \in \Gamma_\varepsilon$ is a covering for $X(\varepsilon,\delta)$. The set $X(\varepsilon,\delta)$ can be considered as a set of singularities in some sense. The following theorem says that if f is Kurzweil-Henstock integrable on E with primitive F then for each $\varepsilon > 0$ there exists a gauge δ on E such that the interval E is divided into $E \backslash X$ and the set of singularities X. This formulation differs from that in [6] where the set of singularities is fixed for all $\varepsilon > 0$.

Theorem 1. *Let f and F be functions defined on an interval E in $\mathbb{R}^m$. Then f is Kurzweil-Henstock integrable with primitive F if and only if for every $\varepsilon > 0$ there exists a gauge $\delta : E \to (0,1)$ such that $(D)\sum |F(I)| < \varepsilon$ and $(D)\sum |f(x)|I|| < \varepsilon$ whenever $D = \{(x,I)\}$ is a δ-fine partial division in Γ_ε.*

PROOF. The following shows that with the given inequalities above, it is true that f is Kurzweil-Henstock integrable with primitive F.

If $\Delta = \{(x,I)\}$ is a δ-fine division of E, then

$$(D)\sum |F(I) - f(x)|I|| \leq (D\backslash\Gamma_\varepsilon)\sum |F(I) - f(x)|I|| \\ + (D\cap\Gamma_\varepsilon)\sum |F(I)| + (D\cap\Gamma_\varepsilon)\sum |f(x)|I|| < \varepsilon(|E| + 2).$$

The next part shows that if f is Kurzweil-Henstock integrable with primitive F then the inequalities given in the above theorem are satisfied.

Let $E_k = \{x \in E : k-1 \leq |f(x)| < k\}$. Since f is Kurzweil-Henstock integrable with primitive F, it is true that for every $\varepsilon > 0$ there exists a gauge $\delta_k : E \to (0,1)$ such that $(\Delta_k) \sum |F(I) - f(x)|I|| < \frac{\varepsilon^2}{k2^{k+1}}$. whenever Δ_k is δ_k-fine division of E. A gauge δ may be chosen such that $\delta(x) \leq \delta_k(x)$ if $x \in E_k$. Therefore for any δ-fine partial division $D \subset \Gamma_\varepsilon$,

$$(D) \sum |f(x)|I|| < \sum_{n=1}^{\infty} \frac{k}{\varepsilon} (D) \sum_{x \in E_k} |F(I) - f(x)|I|| < \frac{\varepsilon}{2}.$$

Furthermore, δ may be appropriately chosen so that

$$(D) \sum |F(I)| \leq (D) \sum |F(I) - f(x)|I|| + (D) \sum |f(x)|I|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is complete. □

Remark. *In the statement of Theorem 1, the inequality involving the function f may be restated as one which does not involve f. This restatement gives rise to Theorem 3 which is actually equivalent to Theorem 1. The equivalence is proved by showing that the inequality $(D) \sum |I| < \varepsilon$ leads to $(D) \sum |f(x)|I|| < \varepsilon$ which is precisely the inequality appearing in Theorem 1.*

Theorem 2. *Let f and F be functions defined on an interval E in $\mathbb{R}^m$. Then f is Kurzweil-Henstock integrable with primitive F if and only if for every $\varepsilon > 0$ there exists a gauge $\delta : E \to (0,1)$ such that $(D) \sum |F(I)| < \varepsilon$ and $(D) \sum |I| < \varepsilon$. whenever $D = \{(x, I)\}$ is a δ-fine partial division in Γ_ε.*

PROOF. If f is Kurzweil-Henstock integrable with primitive F then the two inequalities stated above are satisfied. The proof of this statement can be seen in the proof of the previous theorem.

Now, let E_k be as defined previously. This part of the proof shows that the two inequalities on f and F in Theorem 1 are satisfied and f is Kurzweil-Henstock integrable with primitive F.

Let $\varepsilon > 0$. Then for every $\varepsilon_k > 0$ there exists $\delta_k(x) > 0$ such that $(D_k) \sum |F(I)| < \varepsilon_k$ and $(D_k) \sum |I| < \varepsilon_k$ whenever D_k is a δ_k-fine partial division in Γ_{ε_k} where $\varepsilon_k = \frac{\varepsilon}{k2^{k+1}}$ and

$$\Gamma_{\varepsilon_k} = \{(x, I) : |F(I) - f(x)|I|| \geq \varepsilon_k |I|\}.$$

Let δ be so that $\delta(x) \leq \delta_k(x)$, $x \in E_k$. Then, for any δ-fine partial division $D \subset \Gamma_\varepsilon \subset \bigcap_{k=1}^{\infty} \Gamma_{\varepsilon_k}$,

$$(\Delta) \sum |f(x)|I|| < \sum_{k=1}^{\infty} k (D) \sum_{x \in E_k} |I| = \sum_{k=1}^{\infty} k \frac{\varepsilon}{k2^{k+1}} < \varepsilon.$$

The proof is complete. □

4 Convergence Theorems

Let $\{f_n\}$ and $\{F_n\}$ be sequences of functions defined on an interval E in $\mathbb{R}^m$. Also let

$$\Gamma_{\varepsilon,n} = \{(x, I) : |F_n(I) - f_n(x)|I|| \geq \varepsilon |I|\}.$$

The functions f_n are said to satisfy the condition for uniform integrability 1 (UI_1) with primitives F_n if for any $\varepsilon > 0$ there exists a gauge δ independent of n such that

$$(D)\sum |F_n(I)| < \varepsilon \quad \text{and} \quad (D)\sum |f_n(x)|I|| < \varepsilon.$$

whenever D is a δ-fine partial division in $\Gamma_{\varepsilon,n}$. It is true that if the functions f_n satisfy the UI_1 condition then they are equiintegrable on E. The proof can be seen in that of Theorem 1.

The functions f_n are said to satisfy the condition for uniform integrability 2 (UI_2) with primitives F_n if for any $\varepsilon > 0$ there exists a gauge δ independent of n such that

$$(D)\sum |F_n(I)| < \varepsilon \;\; \text{and} \;\; (D)\sum |I| < \varepsilon.$$

whenever D is a δ-fine partial division in $\Gamma_{\varepsilon,n}$. If the functions f_n satisfy the UI_1 condition then it is also true that they satisfy the UI_2 condition. This can be easily shown by noting that UI_1 on the functions f_n implies equiiintegrability of f_n which in turn implies UI_2 on the functions f_n.

In the following theorem, $f_n \to f$ almost everywhere means $f_n(x) \to f(x)$ for each $x \in E \backslash X$ where X is of Lebesgue measure zero or simply measure zero.

Lemma 3. *Suppose the functions f_n are equiintegrable on the interval $E \subset \mathbb{R}^m$ with $f_n \to f$ almost everywhere. Let the primitives F_n of f_n satisfy the uniform Strong Lusin condition $USL(E)$: For every set $X \subset E$ of measure zero, for every $\varepsilon > 0$ there exists a gauge δ independent of n such that $(D)\sum |F_n(I)| < \varepsilon$ for all n whenever D is a δ-fine, X-tagged partial division of E. Then f is Kurzweil-Henstock integrable on E and $\int_E f = \lim_{n\to\infty} \int_E f_n$.*

PROOF. Since the functions f_n are equiintegrable, it is true that for every $\varepsilon > 0$ there exists a gauge δ independent of n such that for all n

$$(\Delta)\sum |f_n(x)|I| - F_n(I)| < \frac{\varepsilon}{2}$$

whenever Δ is a δ-fine division of E. By the $USL\,(E)$ condition on the primitives F_n , for any set $X \subset E$ of measure zero, and for any $\varepsilon > 0$ there is a gauge α independent of n such that $(D)\sum |F_n\,(I)| < \frac{\varepsilon}{2}$ for all n whenever D is an α-fine, X-tagged partial division of E. The gauge δ may be chosen such that $\delta\,(x) \le \alpha\,(x)$ for each $x \in E$.

Let

$$X = \{x \in E : f_n\,(x) \nrightarrow f\,(x)\}\,,$$
$$f_n^* = f_n \chi_{E\setminus X} \quad \text{and} \quad f^* = f\chi_{E\setminus X}.$$

where $\chi_{E\setminus X}$ is the characteristic function of $E\setminus X$. The functions f_n^* are also equiintegrable on E since

$$(\Delta)\sum |f_n^*\,(x)\,|I| - F_n\,(I)| = (\Delta)\sum_{x\in E\setminus X} |f_n\,(x)\,|I| - F_n\,(I)| + (\Delta)\sum_{x\in X} |F_n\,(I)| < \varepsilon.$$

The rest of the proof is now patterned after that of Theorem 3.7.5 in [5]. □

Theorem 4. *Suppose the functions f_n satisfy the UI_1 condition while their primitives F_n satisfy the $USL\,(E)$ condition. If $f_n \to f$ almost everywhere then f is Kurzweil-Henstock integrable on E and $\int_E f = \lim_{n\to\infty} \int_E f_n$.*

PROOF. If the functions f_n satisfy the UI_1 condition then they are equiintegrable on E. The desired result follows immediately from the preceding lemma. The proof is complete. □

A set $X \subset E$ is of variation zero if for every $\varepsilon > 0$ there exists a gauge δ such that $(D)\sum |I| < \varepsilon$ whenever $D = \{(x, I)\}$ is a δ-fine partial division tagged in X. This definition can be found, for example, in [3], [4] or [5]. In $\mathbb{R}$, measure zero and variation zero are equivalent. The proof uses the Vitali covering theorem. In $\mathbb{R}^m, m > 1$, the equivalence still holds but to prove that variation zero implies measure zero, the Vitali cover chosen must consist of cubes.

Remark. *Let X be the set of points at which the primitive function F is not differentiable. We can not prove that X is of variation zero. What Henstock and Lu have proved is that X is of inner variation zero in the sense that we consider only partial divisions D inside Γ_ε, and not all δ-fine partial divisions tagged in X as in the definition of variation zero.*

Lemma 5. *Let $g : E \to \mathbb{R}$. Then for any subset $X \subset E$ of variation zero, for every $\varepsilon > 0$ there is a gauge δ such that $(D)\sum |g(x)|I|| < \varepsilon$ whenever D is a δ-fine partial division tagged in X.*

PROOF. Since X is of variation zero then for every $\varepsilon > 0$, for every $k \in \mathbb{N}$, there exists a gauge δ_k such that $(D)\sum |I| < \frac{\varepsilon}{k2^k}$ whenever D is a δ_k-fine partial division tagged in X. Let E_k be as in the proof of Theorem 1. Also let δ be obtained by diagonalizing on the functions δ_k as in the proof of Theorem 1 or Theorem 3. Then

$$(D)\sum |g(x)|I|| < \sum_{k=1}^{\infty} k\,(D)\sum_{x\in E_k} |I| < \varepsilon$$

whenever D is a δ-fine partial division tagged in X. The proof is complete. □

In this paper, $f = g$ variationally almost everywhere($v.a.e.$) on an interval $E \subset \mathbb{R}^m$ if $f(x) = g(x)$ on $E\backslash X$ where X is of variation zero.

Theorem 6. *Let f be Henstock-Kurzweil integrable on an interval $E \subset \mathbb{R}^m$ with primitive F. If $f = g$ v.a.e then g is Henstock-Kurzweil integrable on E with primitive F.*

PROOF. Suppose f is Kurzweil-Henstock integrable with primitive F. Then for every $\varepsilon > 0$ there exists a gauge δ such that

$$(\Delta)\sum |F(I) - f(x)|I|| < \varepsilon$$

whenever Δ is a δ-fine division of E. Let X be a subset of E such that $g(x) = f(x)$ on $E\backslash X$ and X is of variation zero. By the preceding lemma, the gauge δ above can be chosen appropriately such that for every δ-fine, X-tagged partial division D,

$$(D)\sum |g(x)|I|| < \varepsilon \quad \text{and} \quad (D)\sum |f(x)|I|| < \varepsilon.$$

It then follows that

$$\begin{aligned}(\Delta)\sum |F(I) - g(x)|I|| \le{}& (\Delta)\sum_{x\in E\backslash X} |F(I) - f(x)|I|| \\ &+ (\Delta)\sum_{x\in X} |F(I) - f(x)|I|| + (\Delta)\sum_{x\in X} |f(x)|I|| \\ &+ (\Delta)\sum_{x\in X} |g(x)|I|| < 3\varepsilon.\end{aligned}$$

The proof is complete. □

A function F satisfies the variational Strong Lusin condition $(SL_V(E))$ if for every subset X of E of variation zero, for every $\varepsilon > 0$ there exists a gauge δ such that $(D)\sum |F(I)| < \varepsilon$ whenever D is a δ-fine, X-tagged partial division of E. The functions F_n satisfy the uniform variational Strong Lusin condition $(USL_V(E))$ if each of the functions F_n satisfies the $SL_V(E)$ condition with the gauge δ independent of n.

Theorem 7. *Suppose the functions f_n satisfy the UI_1 condition while their primitives F_n satisfy the $USL_V(E)$ condition. If $f_n \to f$ v.a.e. then f is Kurzweil-Henstock integrable on E and $\int_E f = \lim_{n\to\infty} \int_E f_n$.*

PROOF. If $f_n \to f$ v.a.e. then the set $X = \{x \in E : f_n(x) \nrightarrow f(x)\}$ must be of variation zero. As in the proof of Lemma 4, the functions f_n^* and f^* can be similarly defined. The proof is now similar to that of Theorem 5. □

Theorem 8. *The following two statements are equivalent:*

(1) *The functions f_n satisfy the UI_1 condition and the corresponding primitives satisfy the $USL(E)$ condition with $f_n \to f$ a.e.*
(2) *The functions f_n are equiintegrable on E and the corresponding primitives satisfy the $USL(E)$ condition with $f_n \to f$ a.e.*

Each of the above statements imply the following:
(3) *The functions f_n satisfy the UI_2 condition and the corresponding primitives satisfy the $USL(E)$ condition with $f_n \to f$ a.e.*

PROOF. It is easy to see that (1) ⇒ (2) ⇒ (3). Now by the equiintegrability of the functions f_n on E, it follows that for every $\varepsilon \in (0,1)$ for every $k \in \mathbb{N}$ there exists a gauge δ_k independent of n such that for all n

$$(\Delta)\sum |F_n(I) - f_n(x)|I|| < \frac{\varepsilon^2}{3(k2^{k+1})}$$

whenever Δ is a δ_k-fine division of E. Let X be as in the proof of Lemma 4 and

$$C_k = \{x \in E\backslash X : k-1 \le |f_n(x)| < k, \quad n = 1,2...\}.$$

By the $USL(E)$ condition on the primitives F_n, for a given $\varepsilon > 0$ there is a gauge α independent of n such that for all n $(D)\sum |F_n(I)| < \frac{\varepsilon}{6}$ whenever D is an α-fine partial division tagged in X. A gauge δ can be chosen by diagonalization on the functions δ_k such that $\delta(x) \le \alpha(x)$. Now, for a δ-fine partial division D in $\Gamma_{\varepsilon,n}$,

$$(D)\sum |f_n(x)|I|| \le \sum_{k=1}^{\infty} \frac{k}{\varepsilon} (D) \sum_{x\in C_k} |F_n(I) - f_n(x)|I||$$

$$+ (D) \sum_{x\in X} |F_n(I) - f_n(x)|I|| + (D) \sum_{x\in X} |F_n(I)|$$
$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}$$

and

$$(D) \sum |F_n(I)| \le (D) \sum |F_n(I) - f_n(x)|I|| + (D) \sum |f_n(x)|I||$$
$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{2} < \varepsilon.$$

Therefore the functions f_n satisfy the UI_1 condition. The proof is complete. □

Theorem 9. *Suppose the functions f_n satisfy the UI_2 condition and the corresponding primitives satisfy the $USL(E)$ condition with $f_n \to f$ a.e. Then f is Kurzweil-Henstock integrable on E and $\int_E f = \lim_{n\to\infty} \int_E f_n$.*

Proof. Let X, f_n^*, and f^* be as defined in the proof of Lemma 4 while C_k as defined in the previous proof. As in the proof of Theorem 3, for every $\varepsilon > 0$ there exists a gauge δ obtained diagonally such that

$$(D) \sum |f_n^*(x)|I|| < \varepsilon$$

whenever D is a δ-fine, $E\backslash X$-tagged partial division in $\Gamma_{\varepsilon,n}^*$ where

$$\Gamma_{\varepsilon,n}^* = \{(x, I) : |F_n(I) - f_n^*(x)|I|| \ge \varepsilon |I|\}, \quad \Gamma_{\varepsilon,n}^* \ne \Gamma_{\varepsilon,n}.$$

The UI_2 condition also implies that for every $\varepsilon > 0$ there exists a gauge α independent of n such that

$$(D) \sum |F_n(I)| < \frac{\varepsilon}{2} \quad \text{and} \quad (D) \sum |I| < \frac{\varepsilon}{2}$$

whenever D is an α-fine partial division in $\Gamma_{\frac{\varepsilon}{2},n}^*$. And the $USL(E)$ condition says that, with the same ε as above there is a gauge β independent of n such that for all n

$$(D) \sum |F_n(I)| < \frac{\varepsilon}{2}$$

whenever D is β-fine partial division tagged in X. Certainly the gauge δ can be chosen such that

$$\delta(x) \le \min(\alpha(x), \beta(x)).$$

Finally, for any δ-fine partial division D in $\Gamma_{\varepsilon,n}^* \subset \Gamma_{\frac{\varepsilon}{2},n}^*$

$$(D) \sum |F_n(I)| = (D) \sum_{x\in E\backslash X} |F_n(I)| + (D) \sum_{x\in X} |F_n(I)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and

$$(D) \sum |f_n^*(x)|I|| < \varepsilon.$$

So the functions f_n^* satisfy the UI_1 condition. The proof is complete. □

The three main ideas in the Kurzweil-Henstock theory are the δ-fine divisions, variation and the decomposability of δ. Henstock has shown that a theory of integration can be developed based on these ideas and the resulting theory is not covered by measure theory. It is a general belief that measure theory is possible due to the countable additivity of measure. The Kurzweil-Henstock theory demonstrated that the same results can be proved using only finite δ-fine divisions. An axiomatic approach to the Kurzweil -Henstock integral is possible. It is called the division space [4]. Like measure space, a division space is a family of point-interval pairs satisfying a set of postulates. A Henstock-like integral can be defined on the division space. We could have formulated our definitions and theorems in terms of variation only without reference to measure. If so, then our results can be extended easily to the division space.

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