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# DIFFERENCE PROPERTIES FOR SOME CLASSES OF FUNCTIONS

**Abstract**

We show the difference property and the double difference property for some classes of real-valued functions.

## Introduction

The paper is continuation of [Ko] and is strictly connected with some results and methods presented in the papers by Laczkovich [L1], [L2], [L3] and Keleti [Ke].

Let $\mathbb{G}$ stand for the additive group equal to $\mathbb{R}$ or $\mathbb{T}$ where $\mathbb{T}$ is the circle group $\mathbb{R}/\mathbb{Z}$ (and $\mathbb{Z}$ denotes the additive group of all integers). Functions defined on $\mathbb{T}$ can be treated as functions defined on $\mathbb{R}$ and being periodic with period 1. For a fixed function $f : \mathbb{G} \to \mathbb{R}$ and any $h \in \mathbb{G}$, *the difference function* $\Delta_h f : \mathbb{G} \to \mathbb{R}$ is defined by

$$\Delta_h f(x) = f(x+h) - f(x),$$

and *the double difference function* $Df : \mathbb{G}^2 \to \mathbb{R}$ is defined by

$$Df(x,y) = f(x+y) - f(x) - f(y).$$

Let $\mathcal{F}$ and $\mathcal{F}^{(2)}$ be fixed families of functions from $\mathbb{G}$ to $\mathbb{R}$ and from $\mathbb{G}^2$ to $\mathbb{R}$, respectively. We say that $\mathcal{F}$ (respectively, the pair $(\mathcal{F}, \mathcal{F}^{(2)})$) possesses the *difference property* (respectively, the *double difference property*), if every function $f : \mathbb{G} \to \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ (respectively, $Df \in \mathcal{F}^{(2)}$) for each

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$h \in \mathbb{G}$, is of the form $f = g + A$ where $g \in \mathcal{F}$ and $A$ is an additive function. A class $\mathcal{F}$ is called *translation invariant* if, for any $f \in \mathcal{F}$ and $a, b \in \mathbb{G}$, the function $g(x) := f(x + a) + b$, $x \in \mathbb{G}$, belongs to $\mathcal{F}$. If $A \subset \mathbb{G}$ and $x \in \mathbb{G}$, we write $A + x = \{a + x : a \in A\}$.

We consider ideals of subsets of $\mathbb{G}$ (or $\mathbb{G}^2$). Throughout the paper, we assume that every ideal $\mathcal{I}$ has the following properties:

- $\{x\} \in \mathcal{I}$ for each $x \in \mathbb{G}$,
- each set in $\mathcal{I}$ has empty interior,
- $\mathcal{I}$ is *translation invariant*, i.e. $A + x \in \mathcal{I}$ for any $A \in \mathcal{I}$ and $x \in \mathbb{G}$.

If $\mathcal{I}$ is an ideal of subsets of $\mathbb{G}$, we say that a property holds $\mathcal{I}$-*almost everywhere* on $\mathbb{G}$, or for $\mathcal{I}$-*almost all* $x \in \mathbb{G}$, if it holds for all points $x \in \mathbb{G}$ except for some of them which form a set in $\mathcal{I}$. A pair $(\mathcal{F}, \mathcal{F}^{(2)})$ is called *hereditary* (respectively, $\mathcal{I}$-*hereditary)* if all (respectively, $\mathcal{I}$-almost all) sections $f^y$ are in $\mathcal{F}$ for every $f \in \mathcal{F}^{(2)}$ (where $f^y(x) = f(x, y)$, $x \in \mathbb{G}$). (Note that the present definition of a hereditary pair is different from that given in [Ko].) We shall consider the following ideals of subsets of $\mathbb{G}$:

- $\mathcal{N}$ = the family of Lebesgue null sets,
- $\mathcal{M}$ = the family of meager sets,
- $\mathcal{M}_0$ = the family of nowhere dense sets,
- $\mathcal{I}_0$ = the family of countable sets.

Let $\mathcal{N}^{(2)}$ stand for the $\sigma$-ideal of Lebesgue null sets in $\mathbb{G}^2$. The symbols $\mathcal{M}^{(2)}$, $\mathcal{M}_0^{(2)}$ and $\mathcal{I}_0^{(2)}$ have the analogous meanings. If $\mathcal{I}$ is an ideal, we denote

$$\mathcal{I}^\star = \{A\colon (\exists B \in \mathcal{I},\ \text{of type } F_\sigma)\ A \subset B\}.$$

Then $\mathcal{I}^\star$ forms an ideal contained in $\mathcal{I}$.

## 1 $\mathcal{I}$-essentially continuous functions and Sierpiński sets

Assuming CH Sierpiński [S] constructed a set $E \subset \mathbb{R}$ such that $E \notin \mathcal{N}$, $\mathbb{R} \setminus E \notin \mathcal{N}$ and $(E + h) \setminus E \in \mathcal{N}$ for each $h \in \mathbb{R}$. Erdős (see [dB1, p. 195]) observed that the characteristic function $\chi_E$ of $E$ witnesses the lack of the difference property for the family $L_0$ of all Lebesgue measurable functions on $\mathbb{R}$. Laczkovich [L3] proved that the nonexistence of a Sierpiński set is equivalent to the difference property for $L_0$. He studied the following condition for an invariant ideal $\mathcal{I}$ of sets in an Abelian group $X$:

> there exists a set $E \subset X$ such that $E \notin \mathcal{I}$, $X \setminus E \notin \mathcal{I}$ and $(E+h) \setminus E \in \mathcal{I}$ for every $h \in X$.

In our paper this condition will be used for $X = \mathbb{R}$ and it will be denoted by $(S_{\mathcal{I}})$.

Let $C$ stand for the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For a fixed ideal $\mathcal{I}$ of subsets of $\mathbb{R}$ we denote

$$C_{\mathcal{I}} = \{f \in \mathbb{R}^{\mathbb{R}} : (\exists g \in C)\{x : f(x) \neq g(x)\} \in \mathcal{I}\}.$$

Functions in $C_{\mathcal{I}}$ will be called *$\mathcal{I}$-essentially continuous.* We are going to prove that if $\mathcal{I} \subset \mathcal{N}$ then $\neg(S_{\mathcal{I}})$ is equivalent to the difference property for $C_{\mathcal{I}}$.

We need some auxiliary facts.

**Proposition 1.1.** *[Ke, Thm 2.9] If $f \in L_0$ and $\Delta_h f \in C_{\mathcal{N}}$ for each $h \in \mathbb{R}$, then $f \in C_{\mathcal{N}}$.*

**Proposition 1.2.** *Let $\mathcal{I}$, $\mathcal{J}$ be ideals of subsets of $\mathbb{R}$ with $\mathcal{I} \subset \mathcal{J}$, and let $\mathcal{A}$ be a $\sigma$-algebra of sets such that $\mathcal{I} \subset \mathcal{A}$. The following assertions hold:*

*(a) If $f \in C_{\mathcal{I}}$ and $\{x : f(x) \neq g(x)\} \in \mathcal{J}$ for some $g \in C$, then $\{x : f(x) \neq g(x)\} \in \mathcal{I}$.*

*(b) Assume $\neg(S_{\mathcal{I}})$. If $f \in C_{\mathcal{J}}$ and $\Delta_h f \in C_{\mathcal{I}}$ for each $h \in \mathbb{R}$ then $f \in C_{\mathcal{I}}$.*

*(c) Assume $\neg(S_{\mathcal{I}})$. If $f : \mathbb{R} \to \mathbb{R}$ and $\{x : \Delta_h f(x) \neq 0\} \in \mathcal{I}$ for each $h \in \mathbb{R}$ then $f$ is $\mathcal{A}$-measurable.*

Proof. (a) This is a simple application of the fact that two continuous functions coinciding on a dense set are equal.

(b) Let $f \in C_{\mathcal{J}}$ and $\Delta_h f \in C_{\mathcal{I}}$ for each $h \in \mathbb{R}$. Since $f \in C_{\mathcal{J}}$ there is a $g \in C$ such that $Z := \{x : f(x) \neq g(x)\} \in \mathcal{J}$. If we put $p = f - g$, then $Z = \{x : p(x) \neq 0\}$. In the case $Z \in \mathcal{I}$ we have $f \in C_{\mathcal{I}}$, so assume that $Z \notin \mathcal{I}$. Since $Z \in \mathcal{J}$, we get $\mathbb{R} \setminus Z \notin \mathcal{I}$. Hence by $\neg(S_{\mathcal{I}})$ we have $(Z - h_0) \setminus Z \notin \mathcal{I}$ for some $h_0 \in \mathbb{R}$. On the other hand, $(Z - h_0) \setminus Z \subset \{x : \Delta_{h_0} p(x) \neq 0\}$, so, to obtain a contradiction, let us prove that the last set is in $\mathcal{I}$. By the definition of $Z$ we get $\{x : \Delta_{h_0} p(x) \neq 0\} \subset Z \cup (Z - h_0) \in \mathcal{J}$. Moreover $\Delta_{h_0} p = \Delta_{h_0} f - \Delta_{h_0} g \in C_{\mathcal{I}}$, so from (a) it follows that $\{x : \Delta_{h_0} p(x) \neq 0\} \in \mathcal{I}$.

(c) (See [L3, Thm 7].) Suppose that $f$ is not $\mathcal{A}$-measurable. Thus there is a $c \in \mathbb{R}$ with $E := \{x : f(x) > c\} \notin \mathcal{A}$. From $\mathcal{I} \subset \mathcal{A}$ it follows that $E \notin \mathcal{I}$ and $\mathbb{R} \setminus E \notin \mathcal{I}$. Let $h \in \mathbb{R}$. Observe that $(E - h) \setminus E \subset \{x : \Delta_h f(x) \neq 0\} \in \mathcal{I}$. Hence $(S_{\mathcal{I}})$ holds true, contrary to our assumption. $\square$

**Theorem 1.3.** *Let $\mathcal{I}$ be an ideal of sets in $\mathbb{R}$ such that $\mathcal{I} \subset \mathcal{N}$. Then the condition $\neg(S_{\mathcal{I}})$ is equivalent to the difference property for $C_{\mathcal{I}}$.*

PROOF. (I) The demonstration that $(S_{\mathcal{I}})$ excludes the difference property for $C_{\mathcal{I}}$ goes back to the idea of Erdős. Namely, if $(S_{\mathcal{I}})$ holds true, pick an $E \notin \mathcal{I}$ with $\mathbb{R} \setminus E \notin \mathcal{I}$ and $(E+h) \setminus E \in \mathcal{I}$ for each $h \in \mathbb{R}$. Hence for $f = \chi_E$ we have $f \notin C_{\mathcal{I}}$ and $\Delta_h f \in C_{\mathcal{I}}$ for each $h \in \mathbb{R}$. Suppose that $C_{\mathcal{I}}$ has the difference property. Then $f = g + A$ where $g \in C_{\mathcal{I}}$ and $A$ is additive. Hence $A = f - g$ is additive and bounded on a set of positive measure, so (by Ostrowski's theorem [Os]) $A$ is continuous and consequently, $f \in C_{\mathcal{I}}$.

(II) Now assume $\neg(S_{\mathcal{I}})$. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\Delta_h f \in C_{\mathcal{I}}$ for each $h \in \mathbb{R}$. By [Ke, Thm 2.13] the function $f$ admits a decomposition $f = g + A + \varphi$ where $g \in C_{\mathcal{N}}$, $A$ is additive and $Z_h := \{x : \Delta_h \varphi(x) \neq 0\} \in \mathcal{N}$ for each $h \in \mathbb{R}$. Since $g \in C_{\mathcal{N}}$, there exists an $r \in C$ with $B := \{x : g(x) \neq r(x)\} \in \mathcal{N}$. Let $p = g - r$. Then $p \in C_{\mathcal{N}}$ and $B = \{x : p(x) \neq 0\}$. Observe that $\Delta_h p(x) = 0$ for any $x, h \in \mathbb{R}$ with $x \notin B \cup (B - h)$. Consequently,

$$(\forall h, x \in \mathbb{R})\ (x \notin B \cup (B-h) \cup Z_h \ \Rightarrow\ \Delta_h(p+\varphi)(x) = 0). \tag{1}$$

On the other hand, $p + \varphi = g - r + \varphi = f - A - r$. Hence $\Delta_h(p+\varphi) = \Delta_h f - A(h) - \Delta_h r$ for each $h \in \mathbb{R}$. Thus from the assumption that $\Delta_h f \in C_{\mathcal{I}}$ for each $h \in \mathbb{R}$, and from the continuity of $r$, it follows that $\Delta_h(p+\varphi) \in C_{\mathcal{I}}$ for each $h \in \mathbb{R}$. This together with (1) and Proposition 1.2(a) implies that $\{x : \Delta_h(p+\varphi)(x) \neq 0\} \in \mathcal{I}$. Now, from Proposition 1.2(c) we infer that $p + \varphi \in L_0$. Consequently $\varphi \in L_0$. Since $\Delta_h \varphi \in C_{\mathcal{N}}$ for each $h \in \mathbb{R}$, by Proposition 1.1 we get $\varphi \in C_{\mathcal{N}}$. Now $g + \varphi \in C_{\mathcal{N}}$ and $\Delta_h(g+\varphi) = \Delta_h(f - A) = \Delta_h f - A(h)$ is in $C_{\mathcal{I}}$ for each $h \in \mathbb{R}$, which by Proposition 1.2(b) means that $g + \varphi \in C_{\mathcal{I}}$. $\square$

By the theorem of Trzeciakiewicz [T], we have $(S_{\mathcal{I}_0}) \iff$ CH. (See also [L3, Remark 2, p.668].) Thus we obtain

**Corollary 1.4.** $\neg$CH *is equivalent to the difference property for* $C_{\mathcal{I}_0}$.

**Remarks.** 1. For $\mathcal{I} = \mathcal{N}$, the condition $(S_{\mathcal{I}})$ is independent of ZFC [L3]. Hence, by Theorem 1.3, the difference property for $C_{\mathcal{N}}$ is independent of ZFC. We expect similar results for $\mathcal{I} = \mathcal{N}^\star$, and for $\mathcal{I}$ equal to the $\sigma$-ideal of $\sigma$-porous sets. To have this, one needs models of ZFC in which $\neg(S_{\mathcal{I}})$ is false. From [L3, Thm 2] it follows that $\neg(S_{\mathcal{I}})$ is implied by $\mathrm{cov}(\mathcal{I}) > \mathrm{non}^\star(\mathcal{I})$, thus it suffices to find models in which $\mathrm{cov}(\mathcal{I}) > \mathrm{non}^\star(\mathcal{I})$ holds. Note that models with $\mathrm{cov}(\mathcal{I}) > \mathrm{non}(\mathcal{I})$ for $\mathcal{I} = \mathcal{N}^\star$ and $\mathcal{I} = \sigma$-porous sets were found in [BJ, 2.6] and [R, Thms 1 and 6], respectively. However, this is not enuogh since unfortunately $\mathrm{non}(\mathcal{I}) \leq \mathrm{non}^\star(\mathcal{I})$ [L3, Thm 2]. For the definitions of $\mathrm{cov}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$ and $\mathrm{non}^\star(\mathcal{I})$, see [L3].

2. Let us consider $\mathcal{M}$ instead of $\mathcal{N}$ in Theorem 1.3. Part (I) of the proof still works since we can use Mehdi's theorem [M] instead of Ostrowski's theorem. Part (II) works provided any $f$, with $\Delta_h f \in C_{\mathcal{M}}$ for each $h \in \mathbb{R}$,

admits a decomposition $f = g + A + \varphi$ where $g \in C_{\mathcal{M}}$, $A$ is additive and $\{x : \Delta_h \varphi(x) \neq 0\} \in \mathcal{M}$. However, we do not know whether the last property holds true.

3. Assume CH. Observe that, if $\mathcal{I}$ is a $\sigma$-ideal such that $\mathcal{I} \subset \mathcal{N}$ or $\mathcal{I} \subset \mathcal{M}$, then $C_{\mathcal{I}}$ does not have the difference property. It suffices to use the Erdős type argument based on the Sierpiński set [S] and its category analog. In fact, the proofs for $\mathcal{I} = \mathcal{N}$ and $\mathcal{I} = \mathcal{M}$ are contained in [Ke, Thm 2.11] and [BKW, Thm 2.2]. A general case is similar.

## 2 Some Classes of Functions With the Difference Property

A real-valued function on $\mathbb{R}$ is called *pointwise discontinuous* if its set of continuity points is dense or, equivalently, its set of discontinuity points is meager. Laczkovich in [L2] proved that the family of pointwise discontinuous functions on $\mathbb{R}$ has the difference property. From the following lemma we shall derive that some important subclasses of this family also possess the difference property.

**Lemma 2.1.** *Let $\mathcal{F}$ be equal to the family of all pointwise discontinuous functions on $\mathbb{R}$ and let $\mathcal{G}$ be a subfamily of $\mathcal{F}$ invariant under addition of constants. If*

$$\forall f \in \mathcal{F} \; ( (\forall h \in \mathbb{R} \quad \Delta_h f \in \mathcal{G}) \;\Rightarrow\; f \in \mathcal{G} ) \tag{2}$$

*then $\mathcal{G}$ has the difference property.*

Proof. (Cf. [Ke, Lemma 1.1]). Assume that $f\colon \mathbb{R} \to \mathbb{R}$ and $\Delta_h f \in \mathcal{G}$ for each $h \in \mathbb{R}$. Since $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{F}$ has the difference property, we have $f = g + A$ where $g \in \mathcal{F}$ and $A$ is additive. Thus $\Delta_h f = \Delta_h g + A(h)$ which implies that $\Delta_h g \in \mathcal{G}$ (for each $h \in \mathbb{R}$). Hence, by (2), we get $g \in \mathcal{G}$. □

First consider the family of functions continuous $\mathcal{I}$-almost everywhere where $\mathcal{I}$ is a given ideal. Since the set of discontinuity points of any function is of type $F_\sigma$, the functions continuous $\mathcal{I}$-almost everywhere coincide with those continuous $\mathcal{I}^\star$-almost everywhere. Since (by our preliminary claim) each set in $\mathcal{I}$ has empty interior, we have $\mathcal{I}^\star \subset \mathcal{M}$. Thus it follows that each function continuous $\mathcal{I}$-almost everywhere is continuous $\mathcal{M}$-almost everywhere or, in other words, it is pointwise discontinuous.

**Lemma 2.2.** *For $f\colon \mathbb{R} \to \mathbb{R}$ let $\omega(f, x)$ denote the oscillation of $f$ at a point $x \in \mathbb{R}$. For an arbitrary $h \in \mathbb{R}$, if $\Delta_h f$ is continuous at a point $x_0$ then $\Delta_h \omega(f, x_0) = 0$.*

PROOF. We have $\omega(f,x) = \overline{f}(x) - \underline{f}(x)$ where $\overline{f}(x) = \max\{f(x), \limsup_{t\to x} f(t)\}$, $\underline{f}(x) = \min\{f(x), \liminf_{t\to x} f(t)\}$ for $x \in \mathbb{R}$. Since $f(x+h) = \Delta_h f(x) + f(x)$ for each $x \in \mathbb{R}$ and $\lim_{x\to x_0} \Delta_h f(x) = \Delta_h f(x_0)$, we have $\overline{f}(x_0+h) = \Delta_h f(x_0) + \overline{f}(x_0)$ and $\underline{f}(x_0+h) = \Delta_h f(x_0) + \underline{f}(x_0)$. Hence $\omega(f, x_0+h) = \overline{f}(x_0+h) - \underline{f}(x_0+h) = \overline{f}(x_0) - \underline{f}(x_0)$ and thus $\Delta_h \omega(f, x_0) = 0$. □

**Theorem 2.3.** *Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{R}$. If $\neg(S_{\mathcal{I}^\star})$ then the family of all functions continuous $\mathcal{I}$-almost everywhere on $\mathbb{R}$ has the difference property.*

PROOF. We shall use Lemma 2.1 with $\mathcal{G}$ equal to the family of functions continuous $\mathcal{I}$-almost everywhere. So, we shall check condition (2). Let $f$ be pointwise discontinuous, and let $\Delta_h f$ be continuous $\mathcal{I}$-almost everywhere, for each $h \in \mathbb{R}$. Fix an $h \in \mathbb{R}$. By Lemma 2.2 we have $\Delta_h \omega(f,x) = 0$ at each point $x$ of continuity of $\Delta_h f$. Consequently, $\Delta_h \omega(f,\cdot)$ is equal $\mathcal{I}^\star$-almost everywhere to a continuous (zero) function. On the other hand, $\{x \in \mathbb{R} : \omega(f,x) \neq 0\} \in \mathcal{M}$ since $f$ is pointwise discontinuous. From $\neg(S_{\mathcal{I}^\star})$, inclusion $\mathcal{I}^\star \subset \mathcal{M}$ and Proposition 1.2(b) it follows that $\omega(f,\cdot) \in C_{\mathcal{I}^\star}$. Moreover, by Proposition 1.2(a) we have $\{x \in \mathbb{R} : \omega(f,x) \neq 0\} \in \mathcal{I}^\star$ which means that $f$ is continuous $\mathcal{I}$-almost everywhere. □

**Corollary 2.4.** *The family of all functions continuous $\mathcal{M}_0$-almost everywhere on $\mathbb{R}$ has the difference property.*

PROOF. We use the fact that $\neg(S_{\mathcal{M}_0})$ is true. (See [L2, Remark 7].) □

We shall show that, in some cases, the assumption $\neg(S_{\mathcal{I}^*})$ in Theorem 2.3 is superfluous.

**Theorem 2.5.** *The family of all functions continuous $\mathcal{I}_0$-almost everywhere on $\mathbb{R}$ has the difference property.*

PROOF. In a former version of the paper, the above statement was derived under $\neg$CH from Theorem 2.3 and the equivalence $(S_{\mathcal{I}_0}) \iff$ CH. Recently, I. Recław has communicated us the following ZFC proof. Apply Lemma 2.1 with $\mathcal{G}$ equal to the set of all functions continuous $\mathcal{I}_0$-almost everywhere. We need to check condition (2), so suppose it is false. Thus there is an $f \in \mathcal{F}$ such that $\Delta_h f \in \mathcal{G}$ for each $h \in \mathbb{R}$ and the set $F$ of discontinuity points of $f$ is uncountable. Pick a perfect set $P \subset F$ and a countable set $D \subset P$ dense in $P$. Since $F$ is meager, there is an $h \in \mathbb{R}$ such that $(F-h) \cap D = \emptyset$. Then $(\mathbb{R} \setminus (F-h)) \cap P$ is uncountable (as a dense $G_\delta$ set in $P$) and $f(x+h) - f(x)$ is discontinuous at each point of $(\mathbb{R} \setminus (F-h)) \cap P$ because $f(x+h)$ is continuous at each point of this set and $f(x)$ is discontinuous. Contradiction. □

**Theorem 2.6.** *The family of all functions continuous $\mathcal{N}$-almost everywhere on $\mathbb{R}$ has the difference property.*

Proof. We check condition (2) of Lemma 2.1 where $\mathcal{G}$ stands for the set of functions continuous $\mathcal{N}$-almost everywhere. Let $f$ be pointwise discontinuous, and let $\Delta_h f$ be continuous $\mathcal{N}$-almost everywhere for each $h \in \mathbb{R}$. As in the proof of Theorem 2.3, we infer from Lemma 2.2 that $\Delta_h \omega(f, \cdot) \in C_{\mathcal{N}}$ for each $h \in \mathbb{R}$. Since $\omega(f, \cdot)$ is upper-semicontinuous (hence measurable), therefore by Proposition 1.1, it belongs to $C_{\mathcal{N}}$. Thus there is a continuous function $g$ such that $\{x \in \mathbb{R} : \omega(f, x) = g(x)\}$ is of full measure. This set is dense of type $G_\delta$, so it is comeager. On the other hand, $\{x \in \mathbb{R} : \omega(f, x) = 0\}$ is comeager since $f$ is pointwise discontinuous. It implies that $g = 0$ everywhere which means that $f$ is continuous $\mathcal{N}$-almost everywhere. □

**Remark.** It is well known that the functions continuous $\mathcal{N}$-almost everywhere bounded on a given compact interval are exactly the Riemann integrable functions. Note that the difference property for the family of Riemann integrable functions was shown by de Bruijn [dB2]. His method of proof is different. We do not know how to derive Theorem 2.6 from de Bruijn's result.

In [L1, Thm 8] it is proved that the family of all approximately continuous functions on $\mathbb{R}$ has the difference property. A category analog of approximately continuous functions was introduced in [PWW], and those functions will be called *category approximately continuous*. We are going to give two different proofs of the difference property for category approximately continuous functions.

**Proposition 2.7.** *Let $H \subset \mathbb{R}$ and $H \notin \mathcal{M}$ (respectively, $H \notin \mathcal{N}$). If a function $f : \mathbb{R} \to \mathbb{R}$ has the Baire property (is measurable) and $\Delta_h f$ is category approximately continuous (approximately continuous) for every $h \in H$ then so is $f$.*

Proof. Let $B$ stand for the set of category approximate continuity (respectively, the approximate continuity) points of $f$. It is known that $B$ is comeager (respectively, of full measure). (See e.g. [CLO, Thms 1.3.2, 2.5.6].) Thus for any $x_0 \in \mathbb{R}$ we have $B \cap (x_0 + H) \neq \emptyset$ and so, there exists an $h_0 \in H$ such that $x_0 + h_0 \in B$. From $f(x) = f(x + h_0) - \Delta_{h_0} f(x)$ and from the assumptions it follows that $x_0 \in B$. □

**Theorem 2.8.** *The family of all category approximately continuous functions on $\mathbb{R}$ has the difference property.*

Proof. (I) Consider the statement of Lemma 2.1 with $\mathcal{G}$ equal to the set of all category approximately continuous functions (they are in Baire class 1 and consequently, they are pointwise discontinuous). Obviously pointwise discontinuous functions have the Baire property. Thus Proposition 2.7 yields the condition (2) in Lemma 2.1.

(II) We give a category analogue of the argument in [L1, Thm 8]. Since $\Delta_h f$ is category approximately continuous for every $h$, therefore the function $Df$ is separately category approximately continuous and by [BLW] it is of Baire class 2. Then by [L1, Thm 7] we have $f = g + A$ where $g$ is of Baire 2 and $A$ is additive. Pick a point $x_0$ at which $g$ is category approximately continuous (the set of such points is comeager). Then $g(x+h) = g(x) + \Delta_h f(x) - A(x)$ implies that $g$ is category approximately continuous at $x_0 + h$. Since $h$ is arbitrary, $g$ is category approximately continuous everywhere. □

**Remark.** The first of the above arguments can be used in the measure case, too. Namely, the Baire class 1 has the difference property (that has been derived by Laczkovich [L4] from his main result of [L2]). We use this class as $\mathcal{F}$ in the statement of Lemma 2.1. The role of $\mathcal{G}$ is played by the approximately continuous functions. (Obviously such a version of Lemma 2.1 works with the same proof.)

Recall [Km] that a function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *quasi-continuous* at a point $x_0 \in \mathbb{R}$ if, for any open neighbourhoods $U$ of $x_0$, and $V$ of $f(x_0)$, there exists a nonempty open set $G \subset U$ such that $f[G] \subset V$. A function is called quasi-continuous on $\mathbb{R}$ if it is quasi-continuous at each point of $\mathbb{R}$. Quasi-continuous functions of two variables are defined analogously.

**Theorem 2.9.** *The family of all quasi-continuous functions on $\mathbb{R}$ has the difference property.*

Proof. It is known that every quasi-continuous function is pointwise discontinuous. To get the assertion we use Lemma 2.1. In fact, we shall prove that, for an $H \subset \mathbb{R}$ with $H \notin \mathcal{M}$, if a function $f\colon \mathbb{R} \to \mathbb{R}$ is pointwise discontinuous and $\Delta_h f$ is quasi-continuous for each $h \in H$ then $f$ is quasi-continuous. Thus, let $H$ and $f$ be as above. Denote by $E$ the set of continuity points of $f$. Then $E$ is comeager. Let $x_0 \in \mathbb{R}$. There exists an $h \in H$ such that $x_0 + h \in E$. Hence $f(x) = f(x+h) - \Delta_h f(x)$ is quasi-continuous at $x_0$ as a sum of a function continuous at $x_0$ and a function quasi-continuous at $x_0$. □

## 3 Some Classes of Functions With the Double Difference Property

From the definitions given in Introduction we immediately derive the following lemma

**Lemma 3.1.** *Let $\mathcal{F}$ and $\mathcal{F}^{(2)}$ be fixed families of functions from $\mathbb{G}$ to $\mathbb{R}$ and from $\mathbb{G}^2$ to $\mathbb{R}$, respectively. If $(\mathcal{F}, \mathcal{F}^{(2)})$ is hereditary and $\mathcal{F}$ has the difference property, then $(\mathcal{F}, \mathcal{F}^{(2)})$ has the double difference property.*

Another variant of this lemma was proved in [Ko, Prop.1]:

**Lemma 3.2.** *Let $\mathcal{F}$ and $\mathcal{F}^{(2)}$ be fixed families of functions from $\mathbb{G}$ to $\mathbb{R}$ and from $\mathbb{G}^2$ to $\mathbb{R}$, respectively. Assume that $\mathcal{F}$ constitutes a translation invariant additive group and $\mathcal{I}$ is a $\sigma$-ideal of subsets of $\mathbb{G}$. If $(\mathcal{F}, \mathcal{F}^{(2)})$ is $\mathcal{I}$-hereditary and $\mathcal{F}$ has the difference property, then $(\mathcal{F}, \mathcal{F}^{(2)})$ has the double difference property.*

**Remark.** Lemma 3.2 remains true if $\mathcal{I}$ is an ideal (not necessarily a $\sigma$-ideal) since the same proof given in [Ko] works.

**Theorem 3.3.** *Let $\mathcal{J} \in \{\mathcal{M}, \mathcal{N}, \mathcal{M}_0, \mathcal{I}_0\}$. If $\mathcal{F}$ (respectively, $\mathcal{F}^{(2)}$) stands for the family of all functions from $\mathbb{G}$ (respectively, $\mathbb{G}^2$) to $\mathbb{R}$ that are continuous $\mathcal{J}$-almost (respectively, $\mathcal{J}^{(2)}$-almost) everywhere, then $(\mathcal{F}, \mathcal{F}^{(2)})$ has the double difference property.*

Proof. For $\mathcal{J} = \mathcal{M}$, the theorem was proved in [Ko, Thm 1]. For $\mathcal{J} = \mathcal{N}$, we use Theorem 2.6, Lemma 3.2 and the Fubini theorem. For $\mathcal{J} = \mathcal{M}_0$, observe that the respective pair $(\mathcal{F}, \mathcal{F}^{(2)})$ is $\mathcal{M}$-hereditary, by a version of the Kuratowski-Ulam theorem [O, Thm 15.1]. So, Corollary 2.4 and Lemma 3.2 yield the assertion. Similarly, for $\mathcal{J} = \mathcal{I}_0$ we use Theorem 2.5 and Lemma 3.2. $\square$

In the sequel, the family of category approximately continuous functions on $\mathbb{R}$ will be denoted by $CAC$. Similarly as in the measure case (see [GNN]), there are two standard variants of the notion of a category density point for plane sets. They were introduced and described in [CW] and [BLW]. We call them an *ordinary category density point* and a *strong category density point*. The both notions generate, in a usual way, topologies that are named the *ordinary category density topology* and the *strong category density topology* in the plane. (See [CW].) In turn, if we consider any of these topologies in the domain, and the natural topology on $\mathbb{R}$ – in the range, the respective continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ are called *ordinarily category approximately continuous* and *strongly category approximately continuous* functions of two variables. The family of these last functions will be denoted by $SCAC$. From [BLW, Thm 1.4] it follows that the pair $(CAC, SCAC)$ is hereditary.

The following notion of a function from $[0,1]^2$ to $\mathbb{R}$ with finite variation was introduced by Idczak in [I]. Namely, $f\colon [0,1]^2 \to \mathbb{R}$ is said to be of *finite variation* if the functions $f(\cdot, 0)$, $f(0, \cdot)$ are of finite variation and the associated interval function $F_f$ defined by

$$F_f(P) = f(\bar{x}, \bar{y}) - f(\bar{x}, y) - f(x, \bar{y}) + f(x, y)$$

for $P = [x, \bar{x}] \times [y, \bar{y}] \subset [0,1]^2$, has a finite variation. It was observed in [I] that, for every function $f\colon [0,1]^2 \to \mathbb{R}$ of finite variation and for any $x, y \in [0,1]$, the functions $f(x, \cdot), f(\cdot, y)$ are of finite variation. The above notions and properties can easily be adapted to the case when $f\colon \mathbb{T}^2 \to \mathbb{R}$. Thus, if $BV(\mathbb{T}^2)$ (respectively, $BV(\mathbb{T})$) denotes the family of real-valued functions with finite variation on $\mathbb{T}^2$ (respectively, on $\mathbb{T}$) then the pair $(BV(\mathbb{T}), BV(\mathbb{T}^2))$ is hereditary.

By Lemma 3.1, from the above facts, Theorem 2.8 and the result of de Bruijn [dB1] that $BV(\mathbb{T})$ has the difference property, we obtain:

**Theorem 3.4.** *The pairs* $(CAC, SCAC)$ *and* $(BV(\mathbb{T}), BV(\mathbb{T}^2))$ *have the double difference property.*

**Remark.** Let $QC$ and $QC^{(2)}$ denote the families of all quasi-continuous functions on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. The newest result of [KoM] states that the pair $(QC, QC^{(2)})$ is $\mathcal{M}$-hereditary. Hence from Lemma 3.1 and Theorem 2.9 it follows that this pair has the double difference property.

In the sequel we shall use the following result from [Ko, Thm 2]:

**Lemma 3.5.** *Let* $\mathcal{F}$ *and* $\mathcal{F}^{(2)}$ *be families of functions from* $\mathbb{G}$ *to* $\mathbb{R}$ *and from* $\mathbb{G}^2$ *to* $\mathbb{R}$*, respectively. Assume that* $\mathcal{F}$ *is an additive group of functions such that every additive function from* $\mathcal{F}$ *is linear. Let* $\mathcal{G}$ *be a subgroup of* $\mathcal{F}$ *containing all linear functions and let* $\mathcal{G}^{(2)} \subset \mathcal{F}^{(2)}$*. If* $(\mathcal{F}, \mathcal{F}^{(2)})$ *has the double difference property, then the following conditions are equivalent:*

- *(a)* $\forall f \in \mathcal{F}\ (\, Df \in \mathcal{G}^{(2)} \Rightarrow f \in \mathcal{G}\,)$,
- *(b)* $(\mathcal{G}, \mathcal{G}^{(2)})$ *has the double difference property.*

Let $C(\mathbb{G}^i), UC(\mathbb{G}^i)$ and $Lip(\mathbb{G}^i)$ denote, respectively, the families of continuous, uniformly continuous and Lipschitz functions from $\mathbb{G}^i$ to $\mathbb{R}$ (where $i = 1, 2$). It is known that for $\mathbb{G} = \mathbb{T}$ the classes $UC(\mathbb{G})(= C(\mathbb{G}))$ and $Lip(\mathbb{G})$ have the difference property. (See [dB1], [BBL] and [Ke].) In the case $\mathbb{G} = \mathbb{R}$, the analogs of these results are false which can be easily shown by the use of the function $f(x) = x^2, x \in \mathbb{R}$. We are going to prove that the pairs $(UC(\mathbb{R}), UC(\mathbb{R}^2))$ and $(Lip(\mathbb{R}), Lip(\mathbb{R}^2))$ have the double difference property. Our method of proof is based on Lemma 3.5.

**Proposition 3.6.** *Assume that an* $f\colon \mathbb{R} \to \mathbb{R}$ *is continuous at* $0$*. If* $Df \in UC(\mathbb{R}^2)$ *then* $f \in UC(\mathbb{R})$*.*

Proof. Let $\varepsilon > 0$. There exist $\delta_1, \delta_2 > 0$ such that $|f(x) - f(0)| < \varepsilon/2$ for each $x \in \mathbb{R}$ with $|x| < \delta_1$, and $|Df(p) - Df(q)| < \varepsilon/2$ for any $p, q \in \mathbb{R}^2$ with the Euclidean norm $||p - q|| < \delta_2$. Put $\delta = \min\{\delta_1, \delta_2/\sqrt{2}\}$. Then, for any $x, y \in \mathbb{R}$ with $|x-y| < \delta$, we obtain $|f(x)-f(y)| = |Df(x-y, y) - Df(0, x) + f(x-y) - f(0)| \leq |Df(x-y, y) - Df(0, x)| + |f(x-y) - f(0)| < \varepsilon$. □

**Proposition 3.7.** *If $f \in UC(\mathbb{R})$ and $Df \in Lip(\mathbb{R}^2)$ then $f \in Lip(\mathbb{R})$.*

Proof. The idea of the proof comes from [BBL, Thm 2, proof of (ii)⇒(i)]. By assumption there exist $L, \delta > 0$ such that $|Df(x,h) - Df(y,h)| \leq L|x-y|$ for any $x, y, h \in \mathbb{R}$, and $|f(x) - f(y)| \leq 1$ whenever $|x-y| < \delta$. Fix an $h_0 \in (0, \delta)$. Thus $|\Delta_{h_0} f(x)| \leq 1$ for each $x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$. Consider the integral

$$I_{xy} = \int_0^{h_0} (f(y+h) - f(x+h))dh.$$

One can easily check that we have $I_{xy} = \int_x^y \Delta_{h_0} f(h)dh$. Thus $|I_{xy}| \leq |x-y|$. Now we have $|f(x)-f(y)| = |(1/h_0)\int_0^{h_0}(f(x)-f(y))dh| = |(1/h_0)\int_0^{h_0}(Df(x,h)-Df(y,h) + f(y+h) - f(x+h))dh| \leq (1/h_0)\int_0^{h_0}|Df(x,h) - Df(y,h)|dh + (1/h_0)|I_{xy}| < L|x-y| + (1/h_0)|x-y| = (L + (1/h_0))|x-y|$. □

**Theorem 3.8.** *The pairs $(UC(\mathbb{R}), UC(\mathbb{R}^2))$ and $(Lip(\mathbb{R}), Lip(\mathbb{R}^2))$ have the double difference property.*

Proof. We know that the pair $(C(\mathbb{R}), C(\mathbb{R}^2))$ has the double difference property, by [dB1] and Lemma 3.1. Proposition 3.6 shows that condition (a) in Lemma 3.5 is true with $\mathcal{F} = C(\mathbb{R}), \mathcal{F}^{(2)} = C(\mathbb{R}^2), \mathcal{G} = UC(\mathbb{R})$ and $\mathcal{G}^{(2)} = UC(\mathbb{R}^2)$. So, condition (b) of Lemma 3.5 yields the first assertion of our theorem. Similarly we deduce the second assertion from Proposition 3.7 and from the first assertion. □

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