Real Analysis Exchange
Vol. (), , pp. 593–602

Mariusz Strześniewski, Instytut Matematyki, Uniwersytet Gdanski, Wita Stwosza, 57, 87-952 Gdańsk, Poland.

ON $\mathcal{I}$-ASYMMETRY

Abstract

Sets of approximative asymmetry in the sense of category are introduced. The following theorem is proved.

If $f : \mathbb{R} \to \mathbb{R}$ is a function, then the set of $\mathcal{I}$-asymmetry points of f is of the type $F_{\sigma\delta\sigma}$ and is σ-well-porous.

This illustrates the difference between measure and category. We give an example of a function with the set of $\mathcal{I}$-asymmetry points of the cardinality of the continuum.

In this paper we shall consider basic properties of the category analogue of one-sided density points. This notion is based upon the notion of $\mathcal{I}$-density point which was introduced in [14].

The well-know Young's "Rome Theorem" is the most important theorem giving a relationship between two one-sided cluster (in some sense) sets. Belowska, Kulbacka, Matysiak, Goffman, Jaskuła, Kempisty, Lipiński, Światkoski and Zajiček have collected some relations about "Essential asymmetry". In this paper we shall consider category analogues of these relations.

Throughout this paper B denotes the family of all subsets of $\mathbb{R}$, the real line, having the Baire property, $\mathcal{I}$ denotes the σ-ideal of all meager sets in $\mathbb{R}$. For $a \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ we put $aA = \{ax : x \in A\}$ and $A - a = \{x - a : x \in A\}$.

We say that a sequence of functions f_n converges with respect to $\mathcal{I}$ to the function f if for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ and $A \in \mathcal{I}$ such that $f_{n_{m_p}}(x)$ converges to $f(x)$ for all $x \in \mathbb{R} \setminus A$.

According to [14], 0 is a $\mathcal{I}$-density point from the right of a set $A \in B$ if and only if the sequence of characteristic functions $\chi_{nA\cap[0,1]}$ converges with respect to $\mathcal{I}$ to $\chi_{[0,1]}$. We write then $\mathcal{I}\text{-}d_+(0, A) = 1$. A point $x_0 \in \mathbb{R}$ is a right $\mathcal{I}$-density point of a set $A \in B$ (written $\mathcal{I}\text{-}d_+(x_0, A) = 1$) if and only if $\mathcal{I}\text{-}d_+(0, A - x_0) = 1$. If $\mathcal{I}\text{-}d_+(x_0, \mathbb{R} \setminus A) = 1$, then say that x_0 is a right $\mathcal{I}$-dispersion point of a set A and we write $\mathcal{I}\text{-}d_+(x_0, A) = 0$. It is easy to see that $\mathcal{I}\text{-}d_+(0, A) = 0$ if and only if the sequence $\{\chi_{nA\cap[0,1]}\}_{n\in\mathbb{N}}$ converges with

Key Words: cluster set; $\mathcal{I}$-density; porosity; asymmetry set.
Mathematical Reviews subject classification: 26A03 26E99
Received by the editors January 20, 2000

respect to $\mathcal{I}$ to zero. The notion of left $\mathcal{I}$-density point and of left $\mathcal{I}$-dispersion point are defined in an analogous manner.

We shall need the following lemmas.

Lemma 1. ([14]) *Let $A \in B$. Then $\mathcal{I}\text{-}d_+(0, A) = 0$ ($\mathcal{I}\text{-}d_+(0, A) = 1$) if and only if for every sequence $\{r_n\}_{n\in\mathbb{N}}$ of real numbers converging to $+\infty$ the sequence $\{\chi_{r_n A\cap[0,1]}\}_{n\in\mathbb{N}}$ converges with respect to $\mathcal{I}$ to zero (resp. to $\chi_{[0,1]}$).*

Lemma 2. ([11]) *Let G be an open (resp. closed) set. Then $\mathcal{I}\text{-}d_+(0, G) = 0$ (resp. $\mathcal{I}\text{-}d_+(0, G) = 1$) if and only if for every positive integer n there exist a positive integer k and a positive number δ such that for every $h \in (0, \delta)$ and every $i \in \{1, \dots, n\}$ there exists a integer $j \in \{1, \dots, k\}$ such that*

$$\left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right),\ \left(\frac{i-1}{n} + \frac{j}{nk}\right) h\right) \cap G = \emptyset$$

$$\textit{(resp. } \left[\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) h,\ \left(\frac{i-1}{n} + \frac{j}{nk}\right) h\right] \subseteq G\textit{)}$$

Lemmas 1 and 2 have the left version.

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Baire function (for every open set $G \subseteq \mathbb{R}$ the set $f^{-1}(G)$ has the Baire property) and let $x_0 \in \mathbb{R}$. We define the right (resp. left) $\mathcal{I}$-cluster set $\mathcal{I}\text{-}W_+(f, x_0)$ (resp. $\mathcal{I}\text{-}W_-(f, x_0)$) of f at x_0 as the set of all point $y \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that x_0 is not the right (resp. left) $\mathcal{I}$ -dispersion point of $f^{-1}(U)$ for any neighborhood U of y.

Definition 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Baire function. Then the set $\mathcal{I}\text{-}A(f)$ of all points $x \in \mathbb{R}$ for which $\mathcal{I}\text{-}W_-(f, x) \neq \mathcal{I}\text{-}W_+(f, x)$ we call the set of points of $\mathcal{I}$-asymmetry of f.

Definition 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Baire function. Then the set $\mathcal{I}\text{-}A_\emptyset(f)$ of all points $x \in \mathbb{R}$ for which $\mathcal{I}\text{-}W_-(f, x) \cap \mathcal{I}\text{-}W_+(f, x) = \emptyset$ we call the set of points of strong asymmetry f.

This notion is due to the approximate asymmetry introduced in [19]. We shall use the following theorems concerning the boundary behavior of asymmetry, these theorems we formulate in terms of $\mathcal{I}$-density.

Theorem Z1. ([19] Th.1 p. 200.) *If $f : \mathbb{R} \to \mathbb{R}$ is a Baire function, then $\mathcal{I}\text{-}A(f) = \bigcup_{n=1}^{\infty} \mathcal{I}\text{-}A(M_n) \cap \mathcal{I}\text{-}A(L_n)$, where $M_n,\ L_n \in B$ for all $n \in \mathbb{N}$ and if $M \in B$, then $\mathcal{I}\text{-}A(M)$ denotes the set of of all $x \in \mathbb{R}$ for which $\mathcal{I}\text{-}d_+(x, M) = 0$ and $\mathcal{I}\text{-}d_-(x, M) \neq 0$ or $\mathcal{I}\text{-}d_+(x, M) \neq 0$ and $\mathcal{I}\text{-}d_-(x, M) = 0$.*

Theorem Z2. ([19] Th. 8 p. 209). *If $f:\mathbb{R}\to\mathbb{R}$ is a Baire function, then $\mathcal{I}\text{-}A_\emptyset(f)\subseteq\bigcup_{n=1}^{\infty}\mathcal{I}\text{-}A_\emptyset(M_n)$, where $M_n\in B$ for all $n\in\mathbb{N}$ and if $M\in B$, then $\mathcal{I}\text{-}A_\emptyset(M)$ denotes the set of all $x\in\mathbb{R}$ for which $\mathcal{I}\text{-}d_+(x,M)=0$ and $\mathcal{I}\text{-}d_-(x,M)=1$.*

The assumption of Th. 8 of [19] is that the set of values of f is compact space. We extend the range of f to $\mathbb{R}\cup\{-\infty,+\infty\}$. Th. 8 of [19] has a stronger conclusion. In our paper we need only inclusion.

Theorem 1. *There exists a Baire function $f:\mathbb{R}\to\mathbb{R}$ such that $\mathcal{I}\text{-}A(f)$ has the cardinality of the continuum.*

PROOF. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers from the interval $(0,1/4)$ such that $\lim_{n\to\infty}\varepsilon_n=0$. We define the following contiguous intervals P and noncontiguous intervals T.

$$\begin{aligned}
T_0&=[0,\varepsilon_1],\ T_1=[1-\varepsilon_1,1],\ P_{-1}=(\varepsilon_1,1-\varepsilon_1)\\
T_{0,0}&=[0,\varepsilon_1\varepsilon_2],\ T_{0,1}=[\varepsilon_1-\varepsilon_1\varepsilon_2,\varepsilon_1],\ P_0=(\varepsilon_1\varepsilon_2,\varepsilon_1-\varepsilon_1\varepsilon_2)\\
T_{1,0}&=[1-\varepsilon_1,1-\varepsilon_1+\varepsilon_1\varepsilon_2],\ T_{1,1}=[1-\varepsilon_1,\varepsilon_2,1]\\
P_1&=(1-\varepsilon_1+\varepsilon_1\varepsilon_2,1-\varepsilon_1\varepsilon_2)\ \text{and so on.}
\end{aligned}$$

If for some integer $n>1$ we have noncontiguous closed intervals $T_{\alpha_1,\dots,\alpha_n}$ and open contiguous intervals $P_{\alpha_1,\dots,\alpha_{n-1}}$ $(\alpha_i\in\{0,1\})$, we construct $T_{\alpha_1,\dots,\alpha_{n+1}}$ and $P_{\alpha_1,\dots,\alpha_n}$ as follows. If $T_{\alpha_1,\dots,\alpha_n}=[a,b]$, then

$$T_{\alpha_1,\dots,\alpha_n,0}=[a,a+\varepsilon_{n+1}(b-a)]\,,T_{\alpha_1,\dots,\alpha_n},1=[b-\varepsilon_{n+1}(b-a),b]\,,$$

and

$$P_{\alpha_1,\dots,\alpha_n}=T_{\alpha_1,\dots,\alpha_n}\setminus\Big(T_{\alpha_1,\dots,\alpha_n,0},\cup T_{\alpha_1,\dots,\alpha_n,1},\Big).$$

The intervals of the form $T_{\alpha_1,\dots,\alpha_n}$ ($P_{\alpha_1,\dots,\alpha_{n-1}}$) we call T-intervals (P-intervals) of order n, for $n\in\mathbb{N}$.

Let $C=[0,1]\setminus\Big(P_{-1}\cup\bigcup_{n=1}^{\infty}\bigcup_{\substack{\alpha_1,\dots,\alpha_n\\ \alpha_i\in\{0,1\}}}P_{\alpha_1,\dots,\alpha_n}\Big)$. Now we construct an open set G such that

$$C\setminus\{1\}\subseteq A_1(G)=\{x:\mathcal{I}\text{-}d_-(x,G)=0\ \text{and}\ \mathcal{I}\text{-}d_+(x,G)\neq 0\}.$$

If $P_{\alpha_1,\dots,\alpha_{n-1}}=(a,b)$ is a P-interval of order n, then let

$$G_{\alpha_1,\dots,\alpha_n}=\big(a,a+|T_{\alpha_1,\dots,\alpha_{n-1},0}|\big)\,,$$

where $|T_{\alpha_1,\dots,\alpha_{n-1},0}|$ denotes the length of the T-interval of the order n, and let $G=\bigcup_{n>1}\bigcup_{\substack{\alpha_1,\dots,\alpha_n\\ \alpha_i\in\{0,1\}}}G_{\alpha_1,\dots,\alpha_{n-1}}$. Let $x\in C\setminus\{1\}$. If x is a left endpoint of some P-interval, then $\mathcal{I}\text{-}d_+(x,G) \neq 0$. If x is not a left endpoint of some P-interval, then let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of P-intervals which "converges from the right" to x and $P_n \cap G = \emptyset$ for all $n \in \mathbb{N}$, and in $(x, \inf P_n)$ there are no P-intervals with length greater than or equal to $|P_n|$. Let $G_x = \cup_{n=1}^{\infty} G_n$, where $G_n = G \cap P_n$ for all $n \in N$. Let $\delta_n = \sup(G_n) - x$.

The sequence of characteristic functions $\{\chi_{\delta_n^{-1}}(G_x - x) \cap [0,1]\}_{n\in\mathbb{N}}$ converges to 1 on $(1/2, 1)$ and has no subsequence which converges to zero on $[0,1] \setminus A$, where $A \in \mathcal{I}$. Thus by Lemma 1 we have that $\mathcal{I}\text{-}d_+(x, G_x) \neq 0$. Now we show that $\mathcal{I}\text{-}d_-(x, G_x) \neq 0$. The case when $x = 0$ or x is right endpoint of some P-interval is obvious. In the other cases let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of P-interval which "converges from the left" to x and every $n \in \mathbb{N}$ in $(\sup P_n, x)$ there are not P-interval with length greater than or equal to $|P_n|$. Let $C^1 = C - x$, $G^1 = G - x$, $P_n^1 = P_n - x = (a_n, b_n)$ for all $n \in \mathbb{N}$. Let $\{k_n\}$ $n \in \mathbb{N}$ be an increasing sequence of positive integers. For every $n \in \mathbb{N}$ let j_n denote an index such that $a_{j_n} < -1/k_n \leq a_{j_n+1}$.

The set $k_n G^1 \cap [-1, 0]$ is contained in the sum of three intervals V_1^n, V_2^n, V_3^n, where $V_1^n = k_n(G^1 \cap P_{j_n}^1)$, $V_2^n = [k_n b_{j_n}, \sup(k_n(G^1 \cap P_{j_n+1}^1))]$ and $V_3^n = [k_n b_{j_n+1}, 0]$. Since $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converges to zero, lengths $|V_1^n \cap [-1,0]|$, $|V_2^n|$, and $|V_3^n|$ converge to zero. We have $-1 \in \left(\inf V_1^n, k_n a_{j_n+1}\right]$ for all $n \in \mathbb{N}$. There exists a sequence of positive integers $\{m_n\}_{n\in\mathbb{N}}$ such that the sequence $\{V_2^{m_n}\}_{n\in\mathbb{N}}$ "converges" to some point $p \in [-1, 0]$ (i.e., the sequence of left endpoints and right endpoints of these intervals converge to p). Thus, if $x \notin \{-1, p\}$, then $\lim_{n\to\infty} \chi_{k_{m_n} G^1 \cap [-1,0]}(x) = 0$.

This completes the proof that $\mathcal{I}\text{-}d_-(x, G) = 0$. Taking f as the characteristic function of G we obtain that $\mathcal{I}\text{-}A(f) \supseteq C \setminus \{1\}$ and this completes the proof of the theorem. $\square$

We remark that in our proof we obtain the convergence except in at most two points. Therefore almost everywhere in the sense of category and of measure.

Before the next theorem we introduce the following definition.

Definition 4. For $x \in \mathbb{R}$ and $A \supseteq \mathbb{R}$ let

$$p(x,A) = \lim_{\delta\to 0^+} \sup \frac{|\gamma(x,A,\delta)|}{\delta} \qquad \underline{p}_+(x,A) = \lim_{\delta\to 0^+} \inf \frac{|\gamma_+(x,A,\delta)|}{\delta}$$

$$\underline{p}_-(x,A) = \lim_{\delta\to 0^+} \inf \frac{|\gamma_-(x,A,\delta)|}{\delta} \qquad \underline{p}(x,A) = \max\left(\underline{p}_-(x,A), \underline{p}_+(x,A)\right)$$

where $\gamma(x,A,\delta)$, (resp. $\gamma_+(x,A,\delta)$, and $\gamma_-(x,A,\delta)$) denotes the longest open interval included in $(x-\delta, x+\delta) \setminus A$ (resp. $(x, x+\delta) \setminus A$, $(x-\delta, x) \setminus A$)

We say that a set $A \subseteq \mathbb{R}$ is porous (well porous) if $p(x, A) > 0$ (resp $\underline{p}(x, A) > 0$) at every $x \in A$. If A is a countable union of porous (resp. well porous) sets, then we say that A is σ-porous (resp. σ-well porous).

Theorem 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a Baire function, then the set $\mathcal{I}$-$A(f)$ is σ-well porous and of the type $F_{\sigma\delta\sigma}$.*

Proof. By Theorem Z1 it is sufficient to prove that $\mathcal{I}$-$A(M)$ is well porous and of the type $F_{\sigma\delta\sigma}$ for every set $M \in B$. Let $M \in B$. Then there exist an open set G and a meager set F such that $M = G\Delta F$. By definition we have that $\mathcal{I}$-$A(M) = \mathcal{I}$-$A(G)$. Let

$$A_1(G) = \{x : \mathcal{I}\text{-}d_+(x, G) = 0 \text{ and } \mathcal{I}\text{-}d_-(x, G) \neq 0\}$$
$$A_2(G) = \{x : \mathcal{I}\text{-}d_+(x, G) \neq 0 \text{ and } \mathcal{I}\text{-}d_-(x, G) = 0\}$$

and let $x \in A_1(G)$. By Lemma 2 there exist a natural number k_x and a real number $\delta_x > 0$ such that for every $h \in (0, \delta_x)$ there exists $i_x^h \in \{1, \ldots, k_x\}$ such that $\left(x + \left(\left(i_x^h - 1\right)/k_x\right) h,\ x + \left(i_x^h/k_x\right) h\right) \cap G = \emptyset$. Because $\mathcal{I}$-$d_-(x, G) \neq 0$ we have that $x \in \overline{G}$. Consequently $\underline{p}_+ (x, \mathcal{I}\text{-}A(G)) \geq \underline{p}_+ \left(x, \overline{G}\right) \geq \dfrac{1}{k_x} > 0$.

Similarly if $x \in A_2(G)$, then $\underline{p}_- \left(x, \overline{G)}\right) > 0$. Since $\mathcal{I}$-$A(G) = A_1(G) \cup A_2(G)$, we must show that $\mathcal{I}$-$A(G)$ is well porous. To prove that $\mathcal{I}$-$A(G)$ is of the type $F_{\sigma\delta\sigma}$ we shown first that $F_1 = \{x : I\text{-}d_+(x, F) = 1\}$ is of the type $F_{\sigma\delta}$ for every closed set F. By the Lemma 2, $x \in F_1$ if and only if

$$\begin{aligned}&\text{for every } n \in \mathbb{N} \text{ there exist a positive integer } k_x \text{ and real number } \delta_x > 0\\ &\text{such that for every } h \in (0, \delta_x) \text{ and every } i \in \{1, \ldots, n\} \text{ there exists}\\ &j \in \{1, \ldots, k_x\} \text{ such that } \Big[\big(\frac{i-1}{n} + \frac{j-1}{nk_x}\big)h, \big(\frac{i-1}{n} + \frac{j}{nk_x}\big)h\Big] \subseteq F - x.\end{aligned} \tag{1}$$

Let $n, k \in \mathbb{N}$, $w \in \mathbb{Q} \cap (0, \infty)$, where $\mathbb{Q}$ denotes the set of rational numbers, and let $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, k\}$, $q \in \mathbb{Q} \cap (0, w)$. Let

$$E_{n,k,w,q,i,j} = \{x : \left[\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) q, \left(\frac{i-1}{n} + \frac{j}{nk}\right) q\right] \subseteq F - x\}$$

and

$$B = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{w\in\mathbb{Q}} \bigcap_{q\in\mathbb{Q}\cap(0,w)} \bigcap_{i\in\{1,\ldots,n\}} \bigcup_{j\in\{1,\ldots,k\}} E_{n,k,w,q,i,j}.$$

We show that $F_1 = B$. Let $x \in F_1$. We can assume that the real number δ_x from (1) is rational number and therefore $x \in B$. Now let $x \in B$ and $n \in \mathbb{N}$.

By the definition of B there exist k and w such that

$$x \in \bigcap_{q\in\mathbb{Q}\cap(0,w)} \bigcap_{i\in\{1,\dots,n\}} \bigcup_{j\in\{1,\dots,k\}} E_{n,k,w,q,i,j}.$$

Let $h \in (0,w)$ and $\{q_m\}_{m\in\mathbb{N}}$ be a sequence of rational numbers such that $\lim_{m\to\infty} q_m = h$ and $q_m \in (0,w)$ for all $m \in \mathbb{N}$. For every $m \in \mathbb{N}$ we denote by j_m a positive integer from the set $\{1,\dots,k\}$ such that

$$\left[\left(\frac{i-1}{n}+\frac{j_m-1}{nk}\right)q_m,\ \left(\frac{i-1}{n}+\frac{j_m}{nk}\right)q_m\right] \subseteq F-x$$

the sequence $\{j_m\}_{m\in\mathbb{N}}$ is a bounded sequence of positive integers. Hence there exists a subsequence $\{j_{m_k}\}_{k\in\mathbb{N}}$ which is constant. We assume that $\{q_m\}_{m\in\mathbb{N}}$ is such one that $j_m = j$ is constant for all $m \in \mathbb{N}$. Let

$$y \in [a,b] = \left[\left(\frac{i-1}{n}+\frac{j-1}{nk}\right)h,\ \left(\frac{i-1}{n}+\frac{j}{nk}\right)h\right]$$

$y = a + t(b-a)$, where $t \in [0,1]$. Let $y_m = a_m + t(b_m - a_m)$, where $a_m = \left(\frac{i-1}{n}+\frac{j-1}{nk}\right)q_m$, $b_m = a_m + \left(\frac{1}{nk}\right)_{g_m}$ for all $m \in \mathbb{N}$. Since $y_m \in F-x$ and $F-x$ is closed, we have that $y \in F-x$. Because $y \in [a,b]$ and $h \in (0,w)$ were arbitrary, we conclude, from Lemma 2 that $x \in F_1$. Because F is closed, we have that $E_{n,k,w,q,i,j}$ is closed. By definition of B we conclude that F_1 is a set of type $F_{\sigma\delta}$. Because

$$\{x : \mathcal{I}\text{-}d_-(x,G)=0\} = \{x : \mathcal{I}\text{-}d_-(x,\mathbb{R}\setminus G)=1\}$$

and

$$\{x : \mathcal{I}\text{-}d_+(x,G)=0\} = \{x : \mathcal{I}\text{-}d_+(x,\mathbb{R}\setminus G)=1\},$$

we have that for an open set G

$$\begin{aligned}
X_1 &= \{x : \mathcal{I}\text{-}d_+(x,G)=0\} \text{ is of the type } F_{\sigma\delta}\\
X_2 &= \{x : \mathcal{I}\text{-}d_+(x,G)\neq 0\} \text{ is of the type } G_{\delta\sigma}\\
Y_1 &= \{x : \mathcal{I}\text{-}d_-(x,G)\neq 0\} \text{ is of the type } G_{\delta\sigma}\\
Y_2 &= \{x : \mathcal{I}\text{-}d_-(x,G)=0\} \text{ is of the type } F_{\sigma\delta}.
\end{aligned}$$

By the equality $\mathcal{I}\text{-}A(G) = (X_1\cap Y_1)\cup(X_2\cap Y_2)$ we infer that the set $\mathcal{I}\text{-}A(G)$ is of the type $F_{\sigma\delta\sigma}$. □

Theorem 3. *Let $B \subseteq \mathbb{R}$. There exists a Baire function $f:\mathbb{R}\to\mathbb{R}$ such that $\mathcal{I}\text{-}A_\emptyset(f) = B$ if and only if the set B is countable.*

Proof. It is easy to see, that if B is a countable set of all points of discontinuity, then f is a Baire function and $B = \mathcal{I}\text{-}A_\emptyset(f)$. For proof of Theorem 3, by the Theorem Z2, it is sufficient to prove that $\mathcal{I}\text{-}A_\emptyset(G)$ is countable for every open set G. Let G be an open set and let $B = \mathcal{I}\text{-}A_\emptyset(G) = \{x : \mathcal{I}\text{-}d_+(x,G) = 0 \text{ and } \mathcal{I}\text{-}d_-(x,G) = 1\}$. If $x \in B$, then by the Lemma 2

$$\begin{aligned}&\text{there exist a positive integer } k_x \text{ and a real number } \delta_x > 0\\ &\text{such that for every } h \in (0,\delta_x) \text{ there exists a positive integer}\\ &i_x^h \in \{1,\dots,k_x\} \text{ such that } \left(\frac{(i_x^h - 1)\,h}{k_x}, \frac{(i_x^h)\,h}{k_x}\right) \cap (G - x) = \emptyset.\end{aligned} \tag{2}$$

For k, j are positive integers let $E_{k,j}$ be the set of all points $x \in B$ for which $k_x = k$ and $\left(\frac{1}{j}\right) < \delta_x$, where k_x, δ_x are numbers from (2). Evidently $B = \bigcup_{k,j} E_{k,j}$. We show that every set $E_{k,j}$ is countable. For this it suffices to prove that if $\{x_n\}_{n\in\mathbb{N}}$ is an increasing sequence in $E_{k,j}$, then $\lim_{n\to\infty} x_n \notin E_{k,j}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \uparrow x$ and $x_n \in E_{k,j}$ for all $n \in \mathbb{N}$. We can assume that $x - x_n < \left(\frac{1}{j}\right)$ for all $n \in \mathbb{N}$. Let $i_{x_n}^h$ be a positive integer of $\{1,\dots,k\}$ with respect to (2) for the point x_n, the set G and $h = x - x_n$. Then there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $i_{x_{k_n}}^h = i$ is constant for all $n \in \mathbb{N}$. We examine a sequence of characteristic functions

$$\{\chi_{(x-x_n)^{-1}}(G - x) \cap [-1,0]\}_{n\in\mathbb{N}}. \tag{3}$$

If a point $y \in \mathbb{N} \cap \left(-\frac{k-(i-1)}{k}, -\frac{k-i}{k}\right)$, then the sequence (3) is convergent at this point to zero. We conclude that the sequence (3) converges on an open interval. Hence there exists no subsequence of (3) which converges $\mathcal{I}$- a.e. to $\chi_{[-1,0]}$. Therefore $x \notin B$ and $x \notin E_{k,j}$. □

We remark that in Theorem 2 we can assume that f is a function from $\mathbb{R}$ into a topological space X, where X is locally compact and has countable basis of open sets. Moreover, if $f : \mathbb{R} \to X$ is Baire function and X is compact with a countable basis of open sets, then $\mathcal{I}\text{-}A_\emptyset(f$ is countable. This is a consequence of results by Zajiček [19].

Supplement. In this part we formulate Theorems 2 and 3 for an arbitrary function using some definition of L. Zajiček [20] of $\mathcal{I}$-density, which is equivalent to our definition if we consider Baire functions.

Definition 5. [property 5, Th., page 59, [20]]. Let $A \subseteq \mathbb{R}$. We say that $\mathcal{I}\text{-}d_+(0,A) = 1$ if and only if for any $0 < c < 1$ there exist $\varepsilon > 0$ and

$\delta > 0$ such that for any $0 < x < \delta$ there exists open interval J such that $J \subset^* A \cap (x - cx, x)$ and $|J| \geq \varepsilon x$, where $B \subset^* D$ denotes $B \setminus D$ is meager. Analogously $\mathcal{I}\text{-}d_-(0, A) = 1$ if $\mathcal{I}\text{-}d_+(0, (-1)A) = 1$.

The order notions of $\mathcal{I}$-density is identical as in begin of our paper.

Lemma 3. *Let $0 < c < 1$, $\varepsilon > 0$, $\delta > 0$ and $E_{c,\varepsilon,\delta} = \{y \in \mathbb{R} :$ for any $0 < x < \delta$ there exist open interval J such that $J \subset^* A \cap (y + x - cx, y + x), |J| \geq \varepsilon x\}$. Then $E_{c,\varepsilon,\delta}$ is closed.*

PROOF. Let $y_n \in E_{c,\varepsilon,\delta}$, $y_n \to y$ and $0 < x < \delta$. For $n \in \mathbb{N}$ let $I_n = (a_n, b_n)$ be such that $I_n \subset^* A \cap (y_n + x - cx, y_n + x)$, $|I_n| \geq \varepsilon x$. We can assume that a_n, b_n converge. Let $J = (\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$. Clearly $|J| \geq \varepsilon x$ and $J \subset ((y + x - cx), y + x)$. If $z \in J \setminus A$, then there is $n \in \mathbb{N}$ such that $z \in I_n \setminus A$. Hence $J \setminus A \subset \bigcup_n I_n \setminus A$, where $I_n \setminus A$ is meager for all $n \in \mathbb{N}$. It means that $J \subset^* A$ and ends the proof. $\square$

Lemma 4. *Let $A \subseteq \mathbb{R}$, let $\varepsilon > 0, \delta \geq 0$, let $B = \{x \in \mathbb{R} : \mathcal{I}\text{-}d_+(x, A) = 1$ and $\mathcal{I}\text{-}d_-(x, A) = 0\}$ and $E_{\varepsilon,\delta} = \{x \in B :$ for any $0 < h < \delta$ there exists open interval I_x^n such that $I_x^h \subset^* A \cap \left(x + \dfrac{h}{2}, x + h\right)$, $\left|I_x^h\right| \geq \varepsilon h\}$. Then $E_{\varepsilon,\delta}$ is countable.*

PROOF. We show that if $x_k \in E_{\varepsilon,\delta}$ and $x_k \uparrow x$, then $x \notin E_{\varepsilon,\delta}$. Let $x_k \in E_{\varepsilon,\delta}$, $x_k \uparrow x$ and $x - x_k < \delta$ for all k. Let $I_{x_k}^h$ be an open interval with respect to definition of $E_{\varepsilon,\delta}$ for $x = x_k$, $h = x - x_k$, and let $I_{x_k}^h = (a_k, b_k)$. We have that $1 > \dfrac{b_k - a_k}{x - a_k} > \dfrac{b_k - a_k}{h/2} \geq \varepsilon > 0$. Let $0 < c < \liminf \dfrac{b_k - a_k}{x - a_k}$. Then we have that for this c the property of definition that $\mathcal{I}\text{-}d_-(x, \mathbb{R} \setminus A) = 1$ is not satisfied. $\square$

These lemmas are essentially important and sufficient to prove our asymmetry theorems for an arbitrary functions.

References

[1] C. L. Belna, *Cluster sets of arbitrary functions*, Real Analysis Exchange, **1** (1978), No 1, 7–20.

[2] L. Belowska, *Resolution d'un probleme de M. Z. Zahorski sur les limites approximatives*, Fund. Math. **48** (1960), 277–286.

[3] E. P. Dolżenko, *The boundary properties of arbitrary functions*, Izv. Acad. Nauk SSSR, Ser. mat., **31** (1967), 3–14.

[4] C. Goffman, *On the approximate limits of real functions*, Acta Sci. Math., **23** (1962), 76–78.

[5] U. Hunter, *Essential cluster sets*, Trans. Amer. Math. Soc., **119** (1965), 350–388.

[6] J. Jaskuła, *On the set of points of the approximative assymetry*, Ph. D. Thesis, University of Łódź, 1971.

[7] J. Jedrzejewski, *On the limit numbers of real functions*, Fund. Math., **83** (1973/74), 269–281.

[8] S. Kempisty, *Sur les functions approximativement discontinues*, Fund. Math., **6** (1924), 6–8.

[9] M. Kulbacka, *Sur l'ensemble des points de l'asymetrie approximative*, Acta Sci. Math. Szeged, **21** (1960), 90–93.

[10] J. S. Lipiński, *Sur la discontinuite approximative et le derivee approximative*, Colloq. Math. **10** (1963), 103–109.

[11] E. Łazarow, *On the Baire class of I-approximate derivatives*, Proc. Amer. Math. Soc., **100** (1987), 669–674.

[12] E. Łazarow, W. Wilczyński, *I-approximate derivatives*, Rad. Mat., **5** (1989), 15–27.

[13] A. Matysiak, *Sur les limites approximatives*, Fund. Math., **48** (1960), 363–366.

[14] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *A category analogue of the density topology*, Fund. Math., **125** (1985), 167–173.

[15] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *Remarks on I-density and I-approximately continuous functions*, Comment. Fund. Math. Univ. Carolinae, **26** (1985), 553–564.

[16] T. Świątkowski, *On some generalization of the notion of assymetry of functions*, Colloq. Math., **17** (1967), 77–91.

[17] B. S. Thomson, *Real Functions*, Lect. Notes in Math. **1170** (1980), Springer Verlag.

[18] L. Zajiček, *Sets of ∂-porosity (q)*, Čas. pro pest. mat., **101** (1976), 350–359.

[19] L. Zajiček, *On cluster sets of arbitrary functions*, Fund. Math., **83** (1974), 197–217.

[20] L. Zajiček, *Alternative definitions of I-density topology*, Acta Univ. Carolinae, **28** (1987), 57–61.

[21] M. Strześniewski, *A note on assymetry sets*, Real Analysis Exchange, **14** (1988-89), 469–473.