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COORDINATE FUNCTIONS OF SPACE FILLING CURVES

Abstract

Given a continuous function $X(t)$ mapping $[0,1]$ continuously onto $[0,1]$, several properties are given which this function must fulfill if there is to be another continuous function $Y(t)$ so that $(X(t),Y(t))$ takes the unit interval onto the unit square.

Many of the references to the literature on space-filling curves can be found in [1] where the nondifferentiability and dimension of preimages and graphs of the historical mappings are given. Here, by coordinate functions will be meant continuous functions $X(t)$ and $Y(t)$ which are assumed to take $[0,1]$ onto $[0,1]$ so that $F(t)=(X(t),Y(t))$ takes $[0,1]$ onto the unit square. Presented here are some properties of such X and Y which will perhaps suggest a characterization. The problem of finding a characterization should involve showing how to construct a function $X(t)$ given $Y(t)$ satisfying certain properties so that the pair form a space-filling curve.

It is convenient to consider $Y(t)$ as given and visualize $X(t)$ as 'sliding' the x-coordinate in such a way that all the points in the square are covered by $(X(t),Y(t))$. With this in mind one can first observe the well known fact that for each $y\in[0,1]$, one has $Y^{-1}(y)$ uncountable. Otherwise, it would not be possible for $(X(t),Y(t))$ to cover the line segment $I_y=\{(x,y):x\in[0,1]\}$. Since $Y(t)$ is continuous, each $Y^{-1}(y)$ is a closed set and consists of at most countably many line segments along with a (possibly empty) nowhere dense perfect set and an at most countable set of points. However, it is not possible to have the line segments cover $Y^{-1}(y)$ by themselves. In fact, a coordinate function for a space-filling curve must have a nowhere dense perfect set in each $Y^{-1}(y)$ which is mapped by F onto $I_y=\{(x,y):x\in[0.1]\}$. This will follow from Theorem 1.

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Theorem 1. *If $F(t) = (X(t), Y(t))$ takes $[0,1]$ continuously onto $[0,1] \times [0,1]$ then, for each $y \in [0,1]$, if B_y is the interior of $Y^{-1}(y)$ and P_y is the perfect part of $Y^{-1}(y) \backslash B_y$ then $X(P_y) = [0,1]$.*

Proof. Fix $y \in [0,1]$. Let P_y and B_y be as above and let $A_y = Y^{-1}(y) \backslash (P_y \cup B_y)$. Then A_y is at most countable. Suppose, if possible that $x \in X(B_y) \backslash X(A_y \cup P_y)$. Let $y_n \to y$ with y_n not equal to y so that $x = X(t_n)$ and $y_n = Y(t_n)$; that is, $(x, y_n) = F(t_n)$. By choosing, if necessary, a sub-sequence of $\{t_n\}$, the sequence $\{t_n\}$ can be assumed to converge, say to t_o. Since t_o is not in B_y, because this set is open and no t_n belongs to it, $X(t_o)$ does not equal x, a contradiction. Thus each x is the image of a point of $A_y \cup P_y$ and since A_y is at most countable and $X(P_y)$ is closed, each x must be the image of a point of P_y. (This is because, if $x' \in X(A_y)$, we then have that there are $x'_n \to x'$ with $x'_n \in X(P_y)$ and $x'_n = X(t'_n)$ with $t'_n \in P_y$; a sub-sequence of the t'_n converges, say to t'_o, and $t'_o \in P_y$ so that $X(t'_o) = \lim X(t'_n) = x'$.)

It is somewhat intuitive that for each $F(t) = (X(t), Y(t))$, a space filling curve, if h is a homeomorphism from $[0,1]$ onto $[0,1]$, then $F(h(t))$ and $(h(X(t)), h(Y(t)))$ are space filling. Checking the proof indicates that a considerably more general result holds.

Theorem 2. *Given $X(t)$ and $Y(t)$ where $F(t) = (X(t), Y(t))$ is a space-filling curve and any H taking $[0,1]$ onto $[0,1]$, $F(H(t))$ takes $[0,1]$ onto the square as does $(H(X(t)), H(Y(t)))$. Thus if H is continuous, these maps are also space-filling curves.*

Proof. Given $(a,b) \in [0,1] \times [0,1]$, if $X(s) = a$ and $Y(s) = b$ and $H(t) = s$, then $F(H(t)) = (a,b)$. Similarly, if $X(t) \in H^{-1}(a)$ and $Y(t) \in H^{-1}(b)$ (and there are such t since $(X(t), Y(t))$ is onto the square) then $H(X(t)) = a$ and $H(Y(t)) = b$ and $(a,b) = (H(X(t)), H(Y(t)))$.

Thus, starting with basic well known space filling curves, compositions give rise to different types of coordinate functions and curves.

However, it is not in general possible to distort the graph of a coordinate function and be guaranteed that it is still one. For example, given the standard Peano curve, (for an illustration, see [1] p. 36) the graph of $X(t)$ is contained in

$$[0,1/3] \times [0,1/3] \cup [1/3,2/3] \times [1/3,2/3] \cup [2/3,1] \times [2/3,1].$$

If this middle part of the square is distorted into a parallelogram with vertices at $(1/3,1/3), (1/3,5/9), (2/3,4/9)$, and $(2/3,2/3)$ by contracting in a linear fashion each line segment above the points in the interval $[1/3,2/3]$ and letting the function be determined by the corresponding point in the parallelogram, the resulting function will no longer be a coordinate function. This is because $X^{-1}(x)$ will not have a diameter bounded away from 0, even though each $X^{-1}(x)$ will be a perfect set. Similar considerations show that for each natural number n the inverse image of a point under a coordinate function cannot be the union of n sets the sum of whose diameters are not bounded away from 0.

Theorem 3. *If $F(t) = (X(t), Y(t))$ is continuous from $[0,1]$ onto the square, then for any fixed natural number n and for every $y \in [0,1]$ if $P_y \cap Y^{-1}(y) = \bigcup_{k=1}^{n} A_{k,y}$, then $\{\max_k(diam A_{k,y}) : y \in [0,1]\}$ is bounded away from 0.*

PROOF. If not, there are X and Y and y_i and a natural number n so $P_{y_i} \cap Y^{-1}(y_i) = \bigcup_{k=1}^{n} A_{k,y_i}$ and $\lim_i \max_k(diam A_{k,y_i}) = 0$. Then choose $a_{i,k}$ and $b_{i,k}$ so $A_{k,y_i} \subset [a_{i,k}, b_{i,k}]$ and $diam A_{k,y_i} = b_{i,k} - a_{i,k}$. By selecting sub-sequences of the y_i one may assume there is $y_o = \lim_i y_i$ and $a_k = \lim_i a_{i,k}$ so that $a_k = \lim_i b_{i,k}$ also. Then X must take one of each of $[a_{i,k}, b_{i,k}]$ for each i onto an interval of length $1/n$. But then one of the numbers a_k will have a sub-sequence of intervals $[a_{i,k'}, b_{i,k'}]$ each mapped by X onto an interval of length at least $1/n$. This contradicts the fact that X is continuous and must have $\lim_i X(a_{i,k'}) = \lim_i X(b_{i,k'})$.

This last result makes it difficult to produce an $X(t)$ given a $Y(t)$. To have a space-filling curve, it is not necessary to have the sets $Y^{-1}(y)$ large. It is not difficult to construct a space filling curve using Theorem 2 where each $Y^{-1}(y)$ and each $X^{-1}(x)$ are of Hausdorff dimension 0. Space-filling curves do not have to be efficient and can spend much of the time t not filling any area at all while the coordinate functions look somewhat like they should.

One further note: Given a function $Y(t)$ and $y \in (0,1)$ the set M_y consisting of those $t \in Y^{-1}(y)$ which are relative maxima for Y is an open subset of $Y^{-1}(y)$ and the set $P_y \backslash M_y$ must have the property stated in Theorem 1; namely, $X(P_y \backslash M_y) = [0,1]$. Similarly, this must hold for the set of relative minima N_y in each $Y^{-1}(y)$ with $y \in (0,1)$. This follows from the same argument used in Theorem 1 with $\{y_n\}$ decreasing to y, (respectively, increasing to y). Finally, these sets, $P_y \backslash M_y$ for $y \in (0,1)$ and $P_y \backslash N_y$ for $y \in (0,1)$ must have the property described in Theorem 3.

References

[1] H. Sagan, *Space-filling Curves*, Springer-Verlag, New York, 1994.