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A NOTE ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES BY BOSANQUET-LINFOOT METHOD

Abstract

In this note we show that a theorem due to L.I. Holder on the absolute summability of Fourier series by the Bosanquet-Linfoot method $|\alpha, \beta|$, can be improved upon. Our theorem then provides a better refinement to the classical theorem of Bosanquet on summability $|C, \gamma|$ of Fourier series.

1 Introduction

With intentions to make refinements on results familiar on Cesàro summability of Fourier series, Bosanquet and Linfoot [4, 5] introduced the concept of a new summability method that they termed as the method (α, β) :

Definition 1. Let $\alpha > 0$, β a real number, or $\alpha = 0$, and $\beta \geq 0$. A series $\sum u_n$ is said to be summable (α, β) to a sum s, if

$$\lim_{\omega\to\infty} \sum_{n<\omega} B \left(1 - \frac{n}{\omega}\right)^{\alpha} \left(\log \frac{C}{1 - \frac{n}{\omega}}\right)^{-\beta} u_n = s,$$

for all sufficiently large values of C and $B = (\log C)^{\beta}$. The method $(0, 0)$ is thus defined as the convergence.

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The absolute summability method $|\alpha, \beta|$ was introduced much later by Boyer and Holder [6]. Indeed the series $\sum u_n$ is said to be absolutely summable (α, β), or just summable $|\alpha, \beta|$, if

$$\int_A^\infty \left| \frac{d}{d\omega} \left(\sum_{n<\omega} B \left(1 - \frac{n}{\omega}\right)^\alpha \left(\log \frac{C}{1 - \frac{n}{\omega}}\right)^{-\beta} u_n \right) \right| d\omega < \infty,$$

where $A > 0$ and the parameters α, β and the constants B and C are as for the method (α, β). The method $|0, 0|$ is again taken to define the absolute convergence.

We may note that for $\beta = 0$ the method (α, β) is the same as the Riesz method (R, n, α). Similarly, for $\beta = 0$ the method $|\alpha, 0|$ is the Riesz method $|R, n, \alpha|$. As these Riesz methods are equivalent to the corresponding Cesàso methods of 'order' α, results for summability (α, β) and summability $|\alpha, \beta|$ relate to results on Cesàso summability.

For the method $|\alpha, \beta|$, Boyer and Holder (loc.cit.) gave the following consistency results:

$|\alpha, \beta| \subset |\alpha', \beta'|$ where either (i) $\alpha' > \alpha$ (and any β and β'), or (ii) $\alpha' = \alpha$ and $\beta' \geq \beta$.
(As usual, over here by '$|T|$' we also mean the vector space of series summable by the method $|T|$.)

In particular, then $|\alpha, \beta| \subset |C, \alpha'|$, for $\alpha' > \alpha$ and β any real number.

2 Notation

Let $f \in L(-\pi, \pi)$ and be a periodic function of period 2π and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_0^\infty A_n(x).$$

Let

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

and let

$$\Phi_0(t) = \phi(t)$$

and for $\alpha > 0$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du$$
$$\phi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t), \quad \alpha \geq 0.$$

The study of $|\alpha,\beta|$-summability of Fourier series was taken up by Holder[9] and Patra[13] rather simultaneously. Holder gave the following theorem:

Theorem A. [9]. *Let $\alpha \geq 0$ and suppose $\phi_\alpha(t) \in BV(0,\pi)$. Then the Fourier series $\sum A_n(x)$ is summable $|\alpha,\beta|$, where (i) $\beta > 1$ if $\alpha = 0$ and (ii) $\beta > 2$ if $\alpha > 0$.*

The case $\alpha = 0$, $\beta > 1$ was given independently by Patra [13].

In view of the inclusion relations between the method $|\alpha,\beta|$ and the Cesàso method $|C,\gamma|$, this theorem improves upon the classical result of Bosanquet:

Theorem B. [1, 2]. *Let $\gamma > \alpha \geq 0$. Then $\phi_\alpha(t)BV(0,\pi) \Rightarrow \sum A_n(x) \in |C,\gamma|$.*

We note that in Theorem A the parameter β has two different ranges associated with different values of α : (i) $\beta > 1$ if $\alpha = 0$ and (ii) $\beta > 2$ for $\alpha > 0$. A natural question arises whether it is possible to have '$\beta > 1$' for all α, $\alpha \geq 0$, in Theorem A. Here we show that the answer to this question is in affirmative. However, we need to recall some further concepts for this.

3 Nevanlinna Summability

Nevanlinna [12] and Moursund [10, 11] studied a series-to-function transformation, which we call as the method $N(q_\delta)$, with intentions to improve upon some known results on Cesàro summability of Fourier series and differentiated Fourier series. Ray and Samal [15], Samal [16] and Dikshit [8] have studied the corresponding absolute summability method $|N(q_\delta)|$:

Definition 2. Let $\sum u_n$ be a given series and let $F(\omega) = \sum\limits_{n<\omega} u_n$. Let $q_\delta = q_\delta(t)$ be defined for $0 \leq t < 1$. The $N(q_\delta)$-transform $N(F, q_\delta)$ of F is defined by

$$N(F, q_\delta)(\omega) = \int_0^1 q_\delta(t) F(\omega t) dt.$$

The series $\sum u_n$ is said to be summable by the method $N(q_\delta)$ to a sum s if

$$\lim_{\omega\to\infty} N(F, q_\delta)(\omega) = s.$$

$\sum u_n \in |N(q_\delta)|$, if for some $A > 0$,

$$\int_A^\infty \left| \frac{d}{d\omega} N(F, q_\delta)(\omega) \right| < \infty.$$

The parameter δ, $\delta \geq 0$, controls the comparative strength of the methods $N(q_\delta)$ and $|N(q_\delta)|$.

The $|N(q_\delta)|$-summability of Fourier series has been done rather extensively quite recently in [8].

For the regularity of the method, and for other requirements, the kernel function q_δ for the method $|N(q_\delta)|$ is supposed to satisfy the following hypotheses:

(1) $\displaystyle\int_0^1 q_\delta(t)dt = 1.$

(2) In the case $0 \le \delta < 1$, $q_\delta(t)$ is increasing for $0 < t < 1$.

(3) In the case $\delta \ge 1$, let $p = [\delta]$, the integral point of δ. It is taken that q_δ satisfies the following (3a)-(3d):

(3a) $q_\delta(t)$ is non-increasing for $0 < t < 1$,

(3b) $\left(\dfrac{d}{dt}\right)^{p-1} q_\delta(t) \in AC[0,1]$

(3c) $\left(\dfrac{d}{dt}\right)^{k} q_\delta(t)\Bigg]_{t=1} = 0,\ k = 0, 1, 2, \cdots, (p-1)$

(3d) $(-1)^p \left(\dfrac{d}{dt}\right)^{p} q_\delta(t) \ge 0$, and is increasing.

Also, for $\delta \ge 0$, $p = [\delta]$,

(4) $\dfrac{Q_\delta(t)}{t^{\delta-p+1}} \in L(0,1)$, where $Q_\delta(t) = \displaystyle\int_{1-t}^1 q_\delta^{(p)}(x)dx.$

In [8] the following theorem on the absolute Nevanlinna summability of Fourier series has been proved:

Theorem C. [8]. *Let $\alpha \ge 0$ and let the function q_α satisfy the conditions (1)-(4) with $\delta = \alpha$. If $\phi_\alpha(t) \in BV(0,\pi)$ then $\sum A_n(x) \in |N(q_\alpha)|$.*

The case $\alpha = 0$ is due to Ray and Samal [15].

4 The Main Result

We give the following theorem:

Theorem 4.1.. *Let $\alpha \ge 0$. If $\phi_\alpha(t) \in BV(0,\pi)$ then the series $\sum A_n(x)$ is summable $|\alpha,\beta|$, for all $\beta > 1$.*

Remark 4.2.. 1. Recall that (i) $|\alpha,\beta| \subset |\alpha',\beta'|$, for $\alpha' > \alpha$ and any β and β', and that (ii) $|C,\alpha| = |\alpha,0|$. In view of these relations we may note that the theorem given above presents a refinement to Theorem B of Bosanquet. It may be remarked that most of the familiar theorems given for various other absolute summability methods tend to produce extensions of Theorem B, rather than bring improvements on it.

2. Very recently Chandra and Karanjgaokar[7] have given an alternate proof for the case $\alpha = 1$ of the theorem.

Proof of Theorem 4.1.. We deduce the theorem from Theorem C.

Let $G_{\alpha,\beta}(u) = Bu^{\alpha}(\log \frac{C}{u})^{-\beta}$, for $0 < u \le 1$ and let $G_{\alpha,\beta}(0) = 0$. Then in order that $\sum u_n$ be summable $|\alpha,\beta|$, one needs to show that

$$\int_A^{\infty} \left| \sum_{n<\infty} G'_{\alpha,\beta}\left(1 - \frac{n}{\omega}\right) n u_n \right| \frac{d\omega}{\omega^2} < \infty,$$

for some $A > 0$. However, $\sum u_n \in |N(q_{\alpha})|$, if and only if,

$$\int_A^{\infty} \left| \sum_{n<\omega} q_{\alpha}\left(\frac{n}{\omega}\right) n u_n \right| \frac{d\omega}{\omega^2} < \infty,$$

for some $A > 0$. Thus a proof for the theorem is done if we have that $q_{\alpha}(t) = G'_{\alpha,\beta}(1-t), \quad 0 \le t < 1$ and $q_\alpha(1) = G'_{\alpha,\beta}(0) = 0$ meets the hypotheses on q_{α} in Theorem C. However, we note that for $0 < t < 1$,

(i) $G'_{0,\beta}(1-t) = \frac{B}{(1-t)}\beta(\log \frac{C}{1-t})^{-\beta-1}$,

(ii) for $\alpha > 0$, $G^{(k)}_{\alpha,\beta}(1-t) = B(1-t)^{\alpha-k}(\log \frac{C}{(1-t)})^{-\beta} \sum_{l=0}^{k} C_l (\log \frac{C}{(1-t)})^{-l}$, where $k = 0, 1, 2, \cdots, h$, $h = [\alpha]$ and $C_0, C_1, \cdots, C_h$ are some constants, and

(iii) $Q_\alpha(t) = G^{[h]}_{\alpha,\beta}(t)$.

As necessary, the value of the functions at either of the end points of $[0,1]$ is defined by the corresponding one-sided limit. Thus with $\beta > 1$ all the requirements in (1)-(4) on q_{δ} are met with $\delta = \alpha$ and

$$q_{\alpha}(t) = G'_{\alpha,\beta}(1-t), \quad 0 \le t < 1.$$

This complete a proof of the theorem. □

Remark 4.3.. 1. Bosanquet and Kestelman [3] proved that in the case $\alpha = 1$ of Theorem B of Bosanquet, the summability $|C, \gamma|$, $\gamma > 1$, may not, in general, be replaced by summability $|C, 1|$. One believes that here again, in the case $\alpha = 1$, the summability $|\alpha, \beta|$, $\alpha = 1$, $\beta > 1$, in general, may not be replaced by the summability $|\alpha, \beta|$, $\alpha = 1$, $\beta = 1$. That is, the summability $|\alpha, \beta|$, $\alpha = 1$, $\beta = 1$, of Fourier series is not a local property of the generating function of the series (cf. Chandra and Karanjgaokar [7]).

2. One may note that results on summability factors for Fourier series for the summability $|\alpha, \beta|$, as given by Patra [14] may also now be deduced as very special cases of Theorem 2.4 on Nevanlinna summability as proved in [8].

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