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OUTER MEASURES GENERATED BY A COUNTABLY ADDITIVE MEASURE ON A RING OF SETS

Abstract

Let $\mathcal{R}$ be any ring of subsets of a set X which is not an algebra and let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$. Suppose that μ is a countably additive measure on $\mathcal{R}$ and that μ^* is the outer measure generated by $(\mu, \mathcal{R})$. If X is a countable union of sets in $\mathcal{R}$, then there is a unique countably additive measure ν on $\mathcal{A}$ which extends μ, and the outer measure generated by $(\nu, \mathcal{A})$ coincides with μ^*. If X is not a countable union of sets in $\mathcal{R}$, then there exists a family $\{\mu_p : 0 \leq p \leq \infty\}$ of countably additive measures on $\mathcal{A}$ such that each μ_p agrees with μ on $\mathcal{R}$. For $0 \leq p \leq \infty$, let μ_p^* denote the outer measure generated by $(\mu_p, \mathcal{A})$. Then we have $\mu_0^* \leq \mu_p^* \leq \mu_q^* \leq \mu_\infty^* = \mu^*$ for $0 < p < q < \infty$. Moreover, if $\mathcal{M}$ and $\mathcal{M}_p$, respectively, denotes the σ-algebra of μ^*-measurable and μ_p^*-measurable sets, then $\mathcal{M}_p = \mathcal{M}_1 \subset \mathcal{M}_0 = \mathcal{M}_\infty = \mathcal{M}$ for all positive real numbers p. As examples, we give countably additive measures on rings for which $\mathcal{M} = \mathcal{M}_1$ and $\mathcal{M} \neq \mathcal{M}_1$, respectively. By the outer measures generated by μ we shall mean the outer measures μ^* and μ_p^* $(0 \leq p \leq \infty)$.

1 Preliminaries

Throughout the paper, X denotes a fixed but arbitrary nonempty set unless otherwise stated, and $\mathcal{P}(X)$ denotes the power set of X. For each subset E of X, let E^c denote the complement of E (relative to X), i.e., $E^c = X - E$. By definition, a ring of subsets of X or simply a ring in X is a nonempty family of subsets of X which is closed under the formation of unions and differences, and

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an algebra of subsets of X is a ring in X containing X. A ring in X which is closed under the formation of countable unions is called a σ-ring, and a σ-ring in X containing X is called a σ-algebra. Let $\mathcal{R}$ be any ring in X, let $\mathcal{R}^c$ be the family of all complements of sets in $\mathcal{R}$, and let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$, that is, the smallest algebra containing $\mathcal{R}$. Trivially $\mathcal{A} = \mathcal{R} = \mathcal{R}^c$ if $X \in \mathcal{R}$, and $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$ if $X \notin \mathcal{R}$. Let $\mathcal{R}_\sigma$ and $\mathcal{A}_\sigma$ denote the family of all countable unions of sets in $\mathcal{R}$ and $\mathcal{A}$, respectively.

A non-negative extended real-valued set function μ defined on a ring $\mathcal{R}$ in X is called a measure or a finitely additive measure on $\mathcal{R}$ if $\mu(\emptyset) = 0$ and $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ for every finite collection $\{A_1, \cdots, A_n\}$ of pairwise disjoint sets in $\mathcal{R}$. A measure μ on $\mathcal{R}$ is called a countably additive measure if $\mu(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$ for every pairwise disjoint sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathcal{R}$ whose union is also in $\mathcal{R}$.

Let μ be a countably additive measure on a ring $\mathcal{R}$ in X. For each $E \subset X$, define $\mu^*(E)$ by

$$\mu^*(E) = \inf \{ \textstyle\sum_{n=1}^\infty \mu(A_n) : E \subset \cup_{n=1}^\infty A_n, A_n \in \mathcal{R} \ \ (n = 1, 2, \cdots)\}$$

if E can be covered by a set in $\mathcal{R}_\sigma$ and otherwise define $\mu^*(E) = \infty$ following the convention that $\inf \emptyset = \infty$. Then μ^* is a (Carathéodory) regular outer measure on X which agrees with μ on $\mathcal{R}$. The outer measure μ^* constructed in this way is called the outer measure generated by $(\mu, \mathcal{R})$ (see, e.g., [2, pp.163-165],[3, pp.32-33],[4, pp.36, 41-44]).

For ease of our argument we prove the following simple lemmas.

Lemma 1. *Let $\mathcal{R}$ be any ring in X with $X \notin \mathcal{R}$ and $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$.*

(i) If $A \in \mathcal{R}$ and $B \in \mathcal{R}^c$, then $A \cup B \in \mathcal{R}^c$ and $A \cap B \in \mathcal{R}$.

(ii) If $A, B \in \mathcal{R}^c$, then $A \cup B, A \cap B \in \mathcal{R}^c$.

(iii) If $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $A \in \mathcal{R}$ or $B \in \mathcal{R}$.

(iv) If $\{A_n\}_{n=1}^\infty$ is a pairwise disjoint sequence of sets in $\mathcal{A}$, then there is at most one set $A_n \in \mathcal{R}^c$.

PROOF. Assertions (i) and (ii) follow from de Morgan's laws. To prove (iii), suppose that A and B are disjoint sets in $\mathcal{R}^c$. Since $A \subset B^c \in \mathcal{R}$, we have $A = A \cap B^c \in \mathcal{R}$ by (i). This contradiction proves (iii). Assertion (iv) follows from (iii). □

Lemma 2. *Let $\mathcal{R}$ be any ring in X. Then the following assertions are equivalent:*

(i) $X \in \mathcal{R}_\sigma$;

(ii) $\mathcal{R}^c \subset \mathcal{R}_\sigma \cap (\mathcal{R}_\sigma)^c$;

(iii) $\mathcal{R}_\sigma \cap (\mathcal{R}_\sigma)^c \neq \emptyset$.

Proof. Suppose that $X \in \mathcal{R}_\sigma$. Assume first that $X \in \mathcal{R}$. Then $\mathcal{R}$ is an algebra so $\mathcal{R} = \mathcal{R}^c$. Since $\mathcal{R} \subset \mathcal{R}_\sigma$, we have $\mathcal{R} = \mathcal{R}^c \subset (\mathcal{R}_\sigma)^c$ so that $\mathcal{R}^c = \mathcal{R} \subset \mathcal{R}_\sigma \cap (\mathcal{R}_\sigma)^c$. Next assume that $X \notin \mathcal{R}$. Then $\mathcal{R} \cap \mathcal{R}^c = \emptyset$. Let $\{X_n\}_{n=1}^\infty$ be any sequence of sets in $\mathcal{R}$ such that $X = \cup_{n=1}^\infty X_n$. For each $A \in \mathcal{R}^c$, we have $A \cap X_n \in \mathcal{R}$ for all n by Lemma 1 (i), so $A = \cup_{n=1}^\infty A \cap X_n \in \mathcal{R}_\sigma$, and hence $\mathcal{R}^c \subset \mathcal{R}_\sigma$. We have $\mathcal{R}^c \subset (\mathcal{R}_\sigma)^c$, since $\mathcal{R} \subset \mathcal{R}_\sigma$. Thus (i) implies (ii). Plainly (ii) implies (iii). Now suppose that (iii) holds, and let $A \in \mathcal{R}_\sigma \cap (\mathcal{R}_\sigma)^c$. We have $A, A^c \in \mathcal{R}_\sigma$, so $X = A \cup A^c \in \mathcal{R}_\sigma$. Thus (iii) implies (i). □

The next lemma follows immediately from Lemma 2.

Lemma 3. *Let $\mathcal{R}$ be any ring in X. Then $X \notin \mathcal{R}_\sigma$ if and only if $\mathcal{R}_\sigma \cap (\mathcal{R}_\sigma)^c = \emptyset$.*

Lemma 4. *Let $\mathcal{R}$ be any ring in X and let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$.*

(i) If $X \in \mathcal{R}_\sigma - \mathcal{R}$, then $A_\sigma = \mathcal{R}_\sigma$.

(ii) If $X \notin \mathcal{R}_\sigma$, then, for each $A \in \mathcal{A}_\sigma$, one and only one of the following alternatives holds: $A \in \mathcal{R}_\sigma$ or A is of the form $E \cup F$, where $E \in \mathcal{R}_\sigma$, $F \in \mathcal{R}^c$, and $E \cap F = \emptyset$.

Proof. For (i), suppose that $X \in \mathcal{R}_\sigma - \mathcal{R}$. Since $X \notin \mathcal{R}$, we have $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$ so $\mathcal{R}_\sigma \subset \mathcal{A}_\sigma$. Since $X \in \mathcal{R}_\sigma$, we infer from Lemma 2 that $\mathcal{R}^c \subset \mathcal{R}_\sigma$ so $\mathcal{A} \subset \mathcal{R}_\sigma$. Consequently, $\mathcal{A}_\sigma \subset \mathcal{R}_\sigma$ and hence $\mathcal{A}_\sigma = \mathcal{R}_\sigma$. Thus (i) is established. To prove (ii), suppose that $X \notin \mathcal{R}_\sigma$. Obviously $X \notin \mathcal{R}$, so $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$. Assume that A is an arbitrary set in $\mathcal{A}_\sigma$. Then there is a pairwise disjoint sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathcal{A}$ such that $A = \cup_{n=1}^\infty A_n$. If $A_n \in \mathcal{R}$ for all n, then $A \in \mathcal{R}_\sigma$. Otherwise, by Lemma 1 (iv) there is exactly one set $A_i \in \mathcal{R}^c$ such that $A_n \in \mathcal{R}$ for all $n \neq i$, so that letting $E = \cup_{n \neq i} A_n$ and $F = A_i$ we have $A = E \cup F$, where $E \in \mathcal{R}_\sigma$, $F \in \mathcal{R}^c$, and $E \cap F = \emptyset$. If there were a set $A \in \mathcal{R}_\sigma$ of the form $E \cup F$, where $E \in \mathcal{R}_\sigma, F \in \mathcal{R}^c$, and $E \cap F = \emptyset$, then $F = F \cap A \in \mathcal{R}_\sigma$, so $F \in \mathcal{R}_\sigma \cap \mathcal{R}^c$. This is a contradiction by Lemma 3. Thus (ii) is established. □

Assume that $X \notin \mathcal{R}_\sigma$ and that $E \in \mathcal{R}_\sigma, F \in \mathcal{R}^c$, and $E \cap F \neq \emptyset$. We have that $E \cup F \in \mathcal{A}_\sigma - \mathcal{R}_\sigma$ and $E - F \in \mathcal{R}_\sigma$, so $E \cup F = (E - F) \cup F$. Note that $F - E \in (\mathcal{R}^c)_\delta = (\mathcal{R}_\sigma)^c$ and $F - E$ need not be in $\mathcal{R}^c$. Thus we have that every set in $\mathcal{A}_\sigma - \mathcal{R}_\sigma$ is represented in at least one way as the union of disjoint sets from $\mathcal{R}_\sigma$ and $\mathcal{R}^c$, respectively, and that such a representation of a set in $\mathcal{A}_\sigma - \mathcal{R}_\sigma$ need not be unique, for $X = E \cup (X - E)$ for all $E \in \mathcal{R}$.

Example 1. Let X be a nonempty countable set, let $\mathcal{R}$ be the ring of all finite subsets of X, and let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$. Trivially

$\mathcal{R} = \mathcal{A} = \mathcal{P}(X)$ if X is finite. If X is infinite, then $\mathcal{R}_\sigma$ is the σ-ring of all countable subsets of X such that $X \in \mathcal{R}_\sigma - \mathcal{R}$, so that $\mathcal{R}_\sigma = \mathcal{A}_\sigma = \mathcal{P}(X)$.

Example 2. Let X be any uncountable set and let $\mathcal{R}$ be the ring of all finite subsets of X. Then $\mathcal{R}_\sigma$ is the σ-ring of all countable subsets of X such that $X \notin \mathcal{R}_\sigma$ or, equivalently, $\mathcal{R}_\sigma \cap (\mathcal{R}_\sigma)^c = \emptyset$ by Lemma 3. Plainly $\mathcal{R}_\sigma$ is the σ-ring generated by $\mathcal{R}$.
A set $E \subset X$ is called cofinite or cocountable if E^c is finite or countable. Let $\mathcal{A}$ and $\mathcal{B}$ denote the algebra and the σ-algebra generated by $\mathcal{R}$, respectively. Then $\mathcal{A}$ consists of the finite and the cofinite subsets of X, i.e., $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$, and $\mathcal{B}$ consists of the countable and the cocountable subsets of X, i.e., $\mathcal{B} = \mathcal{R}_\sigma \cup (\mathcal{R}_\sigma)^c$. Assert that $\mathcal{A}_\sigma = \mathcal{R}_\sigma \cup \mathcal{R}^c$. Suppose that A is any set in $\mathcal{A}_\sigma - \mathcal{R}_\sigma$ and let $A = E \cup F$, where $E \in \mathcal{R}_\sigma$, $F \in \mathcal{R}^c$ and $E \cap F = \emptyset$. Since $E \subset F^c \in \mathcal{R}$, we have $E \in \mathcal{R}$, so $E \cup F \in \mathcal{R}^c$ by Lemma 1 (i). Consequently, $\mathcal{A}_\sigma - \mathcal{R}_\sigma = \mathcal{R}^c$ and hence the assertion follows from Lemma 4 (ii). We see readily that $\mathcal{A} \subsetneqq \mathcal{B} \subsetneqq \mathcal{P}(X)$.

The next lemma is a version of Lemma 3.4.1 in [1, p.76].

Lemma 5. *(cf. [5, Problem 9, p.258]) Let $\mathcal{R}$ be an arbitrary ring in X which is not an algebra, let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$, and let μ be any measure on $\mathcal{R}$.*

(i) Define μ_0 on $\mathcal{A}$ by $\mu_0(E) = \sup\{\mu(A) : A \subset E, A \in \mathcal{R}\}$ for all E in $\mathcal{A}$. Then μ_0 is a measure on $\mathcal{A}$ such that $\mu_0(E) = \mu(E)$ if $E \in \mathcal{R}$.

(ii) For $0 < p \leq \infty$, define μ_p on $\mathcal{A}$ by $\mu_p(E) = \mu(E)$ if $E \in \mathcal{R}$ and $\mu_p(E) = \mu_0(E) + p$ if $E \in \mathcal{R}^c$. Then μ_p is a measure on $\mathcal{A}$.

(iii) Every measure ν on $\mathcal{A}$ such that $\nu(E) = \mu(E)$ for all E in $\mathcal{R}$ is of the form μ_p for some $p \in [0, \infty]$.

The measures μ_p are called the measures induced by the measure μ and parameters $p \in [0, \infty]$.

PROOF. We have $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$. To prove (i), suppose first that $E \in \mathcal{R}$. By the definition of μ_0, we have $\mu(E) \leq \mu_0(E)$. For any $A \in \mathcal{R}$ with $A \subset E$, we have $\mu(A) \leq \mu(E)$. Taking the supremum of $\mu(A)$ over all such A we obtain $\mu_0(E) \leq \mu(E)$ and hence $\mu_0(E) = \mu(E)$. To prove the additivity of μ_0, suppose that E and F are arbitrary sets in $\mathcal{A}$ such that $E \cap F = \emptyset$. If $E, F \in \mathcal{R}$, then there is nothing to prove. Otherwise, by Lemma 1 (iii) we can assume that $E \in \mathcal{R}$ and $F \in \mathcal{R}^c$, so $E \cup F \in \mathcal{R}^c$ by Lemma 1 (i). For any $A \in \mathcal{R}$ with $A \subset E \cup F$, we have that $A = (A \cap E) \cup (A \cap F)$, where $A \cap E, A \cap F \in \mathcal{R}$, so

$$\mu(A) = \mu(A \cap E) + \mu(A \cap F) \leq \mu(E) + \mu_0(F) = \mu_0(E) + \mu_0(F).$$

Taking the supremum of $\mu(A)$ over all $A \in \mathcal{R}$ with $A \subset E \cup F$ we obtain $\mu_0(E \cup F) \le \mu_0(E) + \mu_0(F)$. To prove the reverse inequality, suppose that $B \in \mathcal{R}$ and $B \subset F$. Since $E \cup F \supset E \cup B \in \mathcal{R}$, we have

$$\mu(E) + \mu(B) = \mu(E \cup B) \le \mu_0(E \cup F).$$

Taking the supremum of $\mu(B)$ over all such B we obtain $\mu_0(E) + \mu_0(F) \le \mu_0(E \cup F)$, and hence the additivity of μ_0 follows. By induction μ_0 is finitely additive on $\mathcal{A}$, so (i) is established. To prove (ii), assume that $0 < p \le \infty$ and that $E \in \mathcal{R}, F \in \mathcal{R}^c$, and $E \cap F = \emptyset$. Then $E \cup F \in \mathcal{R}^c$. By the definition of μ_p, together with (i), we have

$$\mu_p(E \cup F) = \mu_0(E \cup F) + p = \mu_0(E) + \mu_0(F) + p = \mu_p(E) + \mu_p(F).$$

By an argument given in the proof of (i) we show that μ_p is additive on $\mathcal{A}$ so it is also finitely additive on $\mathcal{A}$. Thus (ii) is established. To prove (iii), suppose that ν is any measure on $\mathcal{A}$ which agrees with μ on $\mathcal{R}$, and let E be an arbitrary set in $\mathcal{R}^c$. For any $A \in \mathcal{R}$ with $A \subset E$, we have $\mu(A) = \nu(A) \le \nu(E)$, so $\mu_0(E) \le \nu(E)$. Therefore, $\mu_0 \le \nu$ on $\mathcal{R}^c$. If $\nu = \mu_0$ on $\mathcal{R}^c$, we are done. Suppose that $\mu_0 \ne \nu$ on $\mathcal{R}^c$. Then there is a set $F \in \mathcal{R}^c$ such that $\mu_0(F) < \nu(F)$, so $0 \le \mu_0(F) < \infty$. Define $p = \nu(F) - \mu_0(F)$. We have $0 < p \le \infty$. Assert that $\nu(E) = \mu_p(E)$. For this, assume first that $0 < p < \infty$ or, equivalently, $0 < \nu(F) < \infty$. We have that

$$\nu(F) = \nu(E \cap F) + \mu(F - E) < \infty \text{ and } \mu_0(F) = \mu_0(E \cap F) + \mu_0(F - E) < \infty.$$

Since $E - F, F - E \in \mathcal{R}$, we also have that $\nu(E - F) = \mu_0(E - F)$ and $\nu(F - E) = \mu_0(F - E)$. Then we have $p = \nu(E \cap F) - \mu_0(E \cap F)$. Consequently,

$$\mu_p(E) = \mu_0(E) + p = \mu_0(E - F) + \mu_0(E \cap F) + \nu(E \cap F) - \mu_0(E \cap F) = \nu(E).$$

Next assume that $p = \infty$ or, equivalently, $\nu(F) = \infty$. Since $\nu(F) = \nu(E \cap F) + \nu(F - E)$ and $\nu(F - E) = \mu_0(F - E) \le \mu_0(F) < \infty$, we obtain $\nu(E \cap F) = \infty$, so $\nu(E) = \infty$. By the definition of μ_∞, we have $\mu_\infty(E) = \infty$. Thus the assertion holds and hence (iii) is established. □

Lemma 6. *Let $\mathcal{R}$ be any ring in X such that $X \notin \mathcal{R}$ and let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$. Suppose that μ is a countably additive measure on $\mathcal{R}$ and that $\{\mu_p\}$ are the measures on $\mathcal{A}$ that are induced by μ and parameters $p \in [0, \infty]$. Then:*

(i) *μ_0 is a countably additive measure on $\mathcal{A}$ which extends μ.*
(ii) *If $X \in \mathcal{R}_\sigma$, then μ_0 is a unique countably additive measure on*

$\mathcal{A}$ which extends μ.

(iii) If $X \notin \mathcal{R}_\sigma$, then each μ_p with $0 < p \leq \infty$ is a countably additive measure on $\mathcal{A}$ which extends μ.

(iv) If $X \notin \mathcal{R}_\sigma$, then every countably additive measure ν on $\mathcal{A}$ which extends μ is of the form μ_p for some $p \in [0, \infty]$.

PROOF. For (i), suppose that $\{E_n\}_{n=1}^\infty$ is a pairwise disjoint sequence of sets in $\mathcal{A}$ such that $E = \cup_{n=1}^\infty E_n \in \mathcal{A}$. Assume first that $E \in \mathcal{R}$. By Lemma 1 (i) we have $E_n \in \mathcal{R}$ for all n, so that by the countable additivity of μ, together with Lemma 5 (i),

$$\mu_0(E) = \mu(E) = \textstyle\sum_{n=1}^\infty \mu(E_n) = \sum_{n=1}^\infty \mu_0(E_n).$$

Next assume that $E \in \mathcal{R}^c$ and that $A \in \mathcal{R}$ and $A \subset E$. Since $A = \cup_{n=1}^\infty A \cap E_n$, where $A \cap E_n \in \mathcal{R}$ for all n by Lemma 1 (i), we obtain again by the countable additivity of μ that $\mu(A) = \sum_{n=1}^\infty \mu(A \cap E_n) \leq \sum_{n=1}^\infty \mu_0(E_n)$. Taking the supremum of $\mu(A)$ over all such A we obtain $\mu_0(E) \leq \sum_{n=1}^\infty \mu_0(E_n)$. Since μ_0 is finitely additive on $\mathcal{A}$ by Lemma 5 (i), we have, for any positive integer n, $\sum_{i=1}^n \mu_0(E_i) = \mu_0(\cup_{i=1}^n E_i) \leq \mu_0(E)$. Letting $n \longrightarrow \infty$ we obtain the reverse inequality so that (i) holds.

For (ii), suppose that $X \in \mathcal{R}_\sigma$ and that ν is any countably additive measure on $\mathcal{A}$ which agrees with μ on $\mathcal{R}$. By part (i), ν agrees with μ_0 on $\mathcal{R}$. Assert that ν agrees with μ_0 on $\mathcal{R}^c$. For this, suppose that A is any set in $\mathcal{R}^c$. Since $X \in \mathcal{R}_\sigma$, we have $\mathcal{R}^c \subset \mathcal{R}_\sigma$ by Lemma 2, so that there is a pairwise disjoint sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathcal{R}$ such that $A = \cup_{n=1}^\infty A_n$. By the countable additivity of ν and μ_0, we obtain $\nu(A) = \sum_{n=1}^\infty \nu(A_n) = \sum_{n=1}^\infty \mu_0(A_n) = \mu_0(A)$. Thus the assertion is established and hence (ii) holds.

For (iii), suppose that $X \notin \mathcal{R}_\sigma$ and that $0 < p \leq \infty$. Trivially μ_p agrees with μ on $\mathcal{R}$ by the definition of μ_p. Let $\{E_n\}_{n=1}^\infty$ be any pairwise disjoint sequence of sets in $\mathcal{A}$ such that $E = \cup_{n=1}^\infty E_n \in \mathcal{A}$. If $E \in \mathcal{R}$, then $E_n \in \mathcal{R}$ for all n, so

$$\mu_p(E) = \mu(E) = \textstyle\sum_{n=1}^\infty \mu(E_n) = \sum_{n=1}^\infty \mu_p(E_n),$$

since μ is countably additive on $\mathcal{R}$. Suppose that $E \in \mathcal{R}^c$. It follows from Lemma 1 (iv), together with Lemma 3, that there is a unique set $E_i \in \mathcal{R}^c$ such that $E_n \in \mathcal{R}$ for all $n \neq i$. Consequently, we obtain from part (i) and the definition of μ_p that

$$\begin{aligned} \mu_p(E) &= \mu_0(E) + p = \textstyle\sum_{n=1}^\infty \mu_0(E_n) + p = \mu_0(E_i) + p + \sum_{n \neq i} \mu(E_n) \\ &= \mu_p(E_i) + \textstyle\sum_{n \neq i} \mu_p(E_n) = \sum_{n=1}^\infty \mu_p(E_n). \end{aligned}$$

Thus (iii) holds. Assertion (iv) follows from Lemma 5 (iii), together with parts (i) and (iii). $\square$

Suppose that $\mathcal{R}$ is a σ-ring in X which is not a σ-algebra and that $\mathcal{A}$ is the σ-algebra generated by $\mathcal{R}$. Let μ be any countably additive measure on $\mathcal{R}$. By parts (i), (iii) and (iv) of Lemma 6, the measures $\mu_p \quad (0 \leq p \leq \infty)$ are the only countably additive measures on $\mathcal{A}$ which extend μ (see [5, Problem 9, p.258]).

Let $\mathcal{R}$ be any ring in X. It is easy to give an alternate, but equivalent, definition of $\mathcal{R}_\sigma$: define $\mathcal{R}_\sigma$ as the family of unions of increasing sequences of sets in $\mathcal{R}$. Notice that $\mathcal{R}_\sigma$ is the smallest family of subsets of X containing $\mathcal{R}$ and closed under the formation of finite intersections and countable unions. Let μ be a countably additive measure on $\mathcal{R}$. Define $\bar{\mu}$ on $\mathcal{R}_\sigma$ as follows : for each $A \in \mathcal{R}_\sigma$, let $\bar{\mu}(A) = \lim_n \mu(A_n)$, where $\{A_n\}_{n=1}^\infty$ is any increasing sequence of sets in $\mathcal{R}$ such that $A = \cup_{n=1}^\infty A_n$. We show readily that $\bar{\mu}$ is defined unambiguously on $\mathcal{R}_\sigma$. Furthermore, $\bar{\mu}$ is a unique monotone increasing, countably additive non-negative extended real-valued set function on $\mathcal{R}_\sigma$ which agrees with μ on $\mathcal{R}$. We also have that $\bar{\mu}(A) = \sup\{\mu(B) : B \subset A, \quad B \in \mathcal{R}\}$ for all A in $\mathcal{R}_\sigma$. For the outer measure μ^* generated by $(\mu, \mathcal{R})$, we obtain that $\mu^*(E) = \inf\{\bar{\mu}(A) : E \subset A \in \mathcal{R}_\sigma\}$ if E can be covered by a set in $\mathcal{R}_\sigma$ and $\mu^*(E) = \infty$ otherwise.

Lemma 7. *Let $\mathcal{R}$ be any ring in X such that $X \in \mathcal{R}_\sigma - \mathcal{R}$ and let $\mathcal{A}$ be the algebra generated by $\mathcal{R}$. Suppose that μ is a countably additive measure on $\mathcal{R}$ and that μ_0 is the unique countably additive measure on $\mathcal{A}$ extending μ. Then $\mathcal{R}_\sigma = \mathcal{A}_\sigma$ and $\bar{\mu} = \overline{(\mu_0)}$ on $\mathcal{R}_\sigma$.*

Proof. The first equality follows from Lemma 4 (i). The set functions $\bar{\mu}$ and $\overline{(\mu_0)}$ denotes, respectively, the extension of μ to $\mathcal{R}_\sigma$ and μ_0 to $\mathcal{A}_\sigma$. Thus both $\bar{\mu}$ and $\overline{(\mu_0)}$ are defined on $\mathcal{R}_\sigma = A_\sigma$. Suppose that A is an arbitrary set in $\mathcal{R}_\sigma$ and that $\{A_n\}_{n=1}^\infty$ is an increasing sequence of sets in $\mathcal{R}$ such that $A = \cup_{n=1}^\infty A_n$. By the definitions of $\bar{\mu}$ and $\overline{(\mu_0)}$ we obtain that $\bar{\mu}(A) = \lim_n \mu(A_n) = \lim_n \mu_0(A_n) = \overline{(\mu_0)}(A)$. Thus the lemma is established. $\square$

Theorem 1. *Let $X, \mathcal{R}, \mathcal{A}, \mu$ and μ_0 be as in Lemma 7. Then $\mu^*(E) = \mu_0^*(E)$ for all $E \subset X$.*

Proof. Suppose that E is an arbitrary subset of X. Then E has a cover in $\mathcal{R}_\sigma$, since $E \subset X \in \mathcal{A}_\sigma = \mathcal{R}_\sigma$ by the first part of Lemma 7. We obtain from the second part of Lemma 7 that

$$\mu^*(E) = \inf\{\bar{\mu}(A) : E \subset A \in \mathcal{R}_\sigma\} = \inf\{\overline{(\mu_0)}(A) : E \subset A \in \mathcal{A}_\sigma\} = \mu_0^*(E).$$

Thus the theorem is established. □

Example 3. Let X denote the real line $\mathbb{R}$ and let $\mathcal{R}$ denote the ring of all unions of finite collections of pairwise disjoint intervals $[a, b)$, where $-\infty < a \leq b < \infty$. Obviously $X \in \mathcal{R}_\sigma - \mathcal{R}$. For any finite collection $\{[a_i, b_i), i = 1, \cdots, n\}$ of pairwise disjoint intervals in $\mathcal{R}$, define $\mu(\cup_{i=1}^n [a_i, b_i)) = \sum_{i=1}^n (b_i - a_i)$. Then μ is a countably additive measure on $\mathcal{R}$. Let $\mathcal{A} = \mathcal{R} \cup \mathcal{R}^c$ and let μ_0 denote the unique countably additive measure on $\mathcal{A}$ which extends μ. We have $\mu_0(E) = \infty$ for all $E \in \mathcal{R}^c$. By Theorem 1 we have $\mu^* = \mu_0^*$. The outer measure $\mu^*$is called Lebesgue outer measure on $\mathbb{R}$.

For any subset E of X, $cardE$ denotes the cardinal number of E.

Example 4. Let X denote the set $\mathbb{N}$ of all positive integers and let $\mathcal{R}$ denote the ring of all finite subsets of X. Then $X \notin \mathcal{R}$. The algebra $\mathcal{A}$ generated by $\mathcal{R}$ consists of the finite and the cofinite subsets of X so $\mathcal{A} \subsetneqq \mathcal{R}_\sigma = \mathcal{A}_\sigma = \mathcal{P}(X)$. Suppose that μ is any measure on $\mathcal{R}$. Trivially μ is countably additive on $\mathcal{R}$. Let μ_0 and $\bar{\mu}$ denote the countably additive measure extending μ to $\mathcal{A}$ and $\mathcal{P}(X)$, respectively. We have that $\mu_0(E) = \sum_{n \in E} \mu(\{n\})$ for all $E \in \mathcal{A}$ and $\bar{\mu}(E) = \sum_{n \in E} \mu(\{n\})$ for all $E \subset X$. Then $\bar{\mu}$ is a countably additive measure on $\mathcal{P}(X)$ such that $\bar{\mu}(E) = \mu^*(E) = \mu_0^*(E)$ for all $E \in \mathcal{P}(X)$.

Next suppose that $\mu(E) = cardE$ for all $E \in \mathcal{R}$. Plainly μ is a measure on $\mathcal{R}$. Then the countably additive measure $\bar{\mu}$ on $\mathcal{P}(X)$ is counting measure, that is, $\bar{\mu}(E) = n$ if E is finite and has n elements and $\bar{\mu}(E) = \infty$ if E is infinite.

2 Main Result and Examples

Throughout this section we shall assume that $\mathcal{R}$ is an arbitrary ring in X such that $X \notin \mathcal{R}_\sigma$, $\mathcal{A}$ is the algebra generated by $\mathcal{R}$, and μ is any countably additive measure on $\mathcal{R}$. Let $\bar{\mu}$ denote the unique extension of μ from $\mathcal{R}$ to $\mathcal{R}_\sigma$ and μ^* the outer measure generated by $(\mu, \mathcal{R})$. Form the countably additive measures μ_p $(0 \leq p \leq \infty)$ on $\mathcal{A}$ that are induced by μ and parameters $p \in [0, \infty]$. For $0 \leq p \leq \infty$, let $\overline{(\mu_p)}$ denote the unique extension of μ_p from $\mathcal{A}$ to $\mathcal{A}_\sigma$, and let μ_p^* denote the outer measure generated by $(\mu_p, \mathcal{A})$. Let $\mathcal{M}$ and $\mathcal{M}_p$ $(0 \leq p \leq \infty)$ denote the σ-algebra of all μ^*-measurable and μ_p^*-measurable subsets of X, respectively.

Lemma 8. *Define $(\bar{\mu})_0$ on $\mathcal{A}_\sigma$ by $(\bar{\mu})_0(E) = \bar{\mu}(E)$ if $E \in \mathcal{R}_\sigma$ and $(\bar{\mu})_0(E \cup F) = \sup\{\bar{\mu}(A) : A \subset E \cup F, A \in \mathcal{R}_\sigma\}$ if $E \in \mathcal{R}_\sigma, F \in \mathcal{R}^c$,and $E \cap F = \emptyset$. Then:*

(i) $(\bar{\mu})_0(E) = \overline{(\mu_0)}(E) = \bar{\mu}(E)$ *if* $E \in \mathcal{R}_\sigma$*;*
(ii) $(\bar{\mu})_0(F) = \overline{(\mu_0)}(F) = \mu_0(F)$ *if* $F \in \mathcal{R}^c$*;*

(iii) $(\bar{\mu})_0(E \cup F) = \bar{\mu}(E) + \mu_0(F)$ *if* $E \in \mathcal{R}_\sigma,\ F \in \mathcal{R}^c$, *and* $E \cap F = \emptyset$;
(iv) $(\bar{\mu})_0$ *is defined unambiguously on* $\mathcal{A}_\sigma - \mathcal{R}_\sigma$;
(v) $(\bar{\mu})_0(A) = \overline{(\mu_0)}(A)$ *for all* $A \in \mathcal{A}_\sigma$.

Proof. For (i), suppose that $E \in \mathcal{R}_\sigma$ and that $\{E_n\}_{n=1}^\infty$ is an increasing sequence of sets in $\mathcal{R}$ such that $E = \cup_{n=1}^\infty E_n$. We have that $(\bar{\mu})_0(E) = \bar{\mu}(E) = \lim_n \mu(E_n) = \lim_n \mu_0(E_n) = \overline{(\mu_0)}(E)$, so that (i) holds.

For (ii), suppose that $F \in \mathcal{R}^c$. By the definitions of $(\bar{\mu})_0$ and $\overline{(\mu_0)}$, we have that $(\bar{\mu})_0(F) \geq \mu_0(F) = \overline{(\mu_0)}(F)$. To prove the reverse inequality, suppose that A is any set in $\mathcal{R}_\sigma$ such that $A \subset F$, and let $\{A_n\}_{n=1}^\infty$ be an increasing sequence of sets in $\mathcal{R}$ such that $A = \cup_{n=1}^\infty A_n$. Since $\mu(A_n) \leq \mu_0(F)$ for all n, we obtain that $\bar{\mu}(A) = \lim_n \mu(A_n) \leq \mu_0(F)$. Taking the supremum of $\bar{\mu}(A)$ over all $A \in \mathcal{R}_\sigma$ with $A \subset F$ we get $(\bar{\mu})_0(F) \leq \mu_0(F)$. Thus (ii) holds.

For (iii), suppose that $E \in \mathcal{R}_\sigma, F \in \mathcal{R}^c$, and $E \cap F = \emptyset$, and let A be any set in $\mathcal{R}_\sigma$ such that $A \subset E \cup F$. Then $A = (A \cap E) \cup (A \cap F)$, where $A \cap E,\ A \cap F \in \mathcal{R}_\sigma$ and $A \cap E \cap F = \emptyset$. Since $\bar{\mu}$ is monotone increasing and countably additive on $\mathcal{R}_\sigma$, we obtain from (ii) that

$$\bar{\mu}(A) = \bar{\mu}(A \cap E) + \bar{\mu}(A \cap F) \leq \bar{\mu}(E) + (\bar{\mu})_0(F) = \bar{\mu}(E) + \mu_0(F).$$

Taking the supremum of $\bar{\mu}(A)$ over all such A we have that $(\bar{\mu})_0(E \cup F) \leq \bar{\mu}(E) + \mu_0(F)$. To prove the reverse inequality, suppose that $B \subset F$ and $B \in \mathcal{R}$. Since $E \cup B \in \mathcal{R}_\sigma$ and $E \cup B \subset E \cup F$, we have that

$$\bar{\mu}(E) + \mu(B) = \bar{\mu}(E) + \bar{\mu}(B) = \bar{\mu}(E \cup B) \leq (\bar{\mu})_0(E \cup F).$$

Taking the supremum of $\mu(B)$ over all $B \in \mathcal{R}$ with $B \subset F$ we have that $\bar{\mu}(E) + \mu_0(F) \leq (\bar{\mu})_0(E \cup F)$. Thus (iii) holds.

For (iv), suppose that A is any set in $\mathcal{A}_\sigma - \mathcal{R}_\sigma$ and that $A = E_1 \cup F_1 = E_2 \cup F_2$, where $E_i \in \mathcal{R}_\sigma,\ F_i \in \mathcal{R}^c$, and $E_i \cap F_i = \emptyset$ for $i = 1, 2$. Since E_1 is the union of two disjoint sets $E_1 \cap E_2$ and $E_1 \cap F_2$ that are in $\mathcal{R}_\sigma$, we obtain that $\bar{\mu}(E_1) = \bar{\mu}(E_1 \cap E_2) + \bar{\mu}(E_1 \cap F_2)$. Since $F_1 = (F_1 \cap E_2) \cup (F_1 \cap F_2)$, where $F_1 \cap E_2 \in \mathcal{R}_\sigma$, $F_1 \cap F_2 \in \mathcal{R}^c$ and $F_1 \cap E_2 \cap F_2 = \emptyset$, we have from (ii) and (iii) that $\mu_0(F_1) = (\bar{\mu})_0(F_1) = \bar{\mu}(F_1 \cap E_2) + \mu_0(F_1 \cap F_2)$. Consequently, again by (iii) we obtain that

$$\begin{aligned}(\bar{\mu})_0(E_1 \cup F_1) &= \bar{\mu}(E_1) + \mu_0(F_1) \\ &= \bar{\mu}(E_1 \cap E_2) + \bar{\mu}(E_1 \cap F_2) + \bar{\mu}(F_1 \cap E_2) + \mu_0(F_1 \cap F_2).\end{aligned}$$

Interchanging E_1 and F_1 by E_2 and F_2, respectively, in the above equations we see at once that $(\bar{\mu})_0(E_1 \cup F_1) = (\bar{\mu})_0(E_2 \cup F_2)$. Thus (iv) holds.

For (v), suppose that A is any set in $\mathcal{A}_\sigma - \mathcal{R}_\sigma$ of the form $E \cup F$, where

$E \in \mathcal{R}_\sigma$, $F \in \mathcal{R}^c$, and $E \cap F = \emptyset$. Let $\{E_n\}_{n=1}^\infty$ be an increasing sequence of sets in $\mathcal{R}$ with $E = \cup_{n=1}^\infty E_n$. Since $\{E_n \cup F\}_{n=1}^\infty$ is an increasing sequence of sets in $\mathcal{A}$ with $A = \cup_{n=1}^\infty (E_n \cup F)$, we obtain that

$$\begin{aligned}\overline{(\mu_0)}(A) &= \lim_n \mu_0(E_n \cup F) \\ &= \lim_n \mu_0(E_n) + \mu_0(F) = \lim_n \mu(E_n) + \mu_0(F) = \bar{\mu}(E) + \mu_0(F),\end{aligned}$$

so by (iii), $\overline{(\mu_0)}(A) = (\bar{\mu})_0(A)$. Now Assertion (v) follows from (i) and Lemma 4 (ii). $\square$

The simple proof of the next lemma is omitted.

Lemma 9. *For $0 < p \le \infty$, define $(\bar{\mu})_p$ on $\mathcal{A}_\sigma$ by $(\bar{\mu})_p(A) = \bar{\mu}(A)$ if $A \in \mathcal{R}_\sigma$ and $(\bar{\mu})_p(A) = (\bar{\mu})_0(A) + p$ if $A \in \mathcal{A}_\sigma - \mathcal{R}_\sigma$. Then :*

(i) $(\bar{\mu})_p(E) = \overline{(\mu_p)}(E) = \bar{\mu}(E)$ *if* $E \in \mathcal{R}_\sigma$;

(ii) $(\bar{\mu})_p(F) = \overline{(\mu_p)}(F) = \mu_p(F)$ *if* $F \in \mathcal{R}^c$;

(iii) $(\bar{\mu})_p(E \cup F) = \overline{(\mu_p)}(E \cup F) = \bar{\mu}(E) + \mu_p(F)$ *if* $E \in \mathcal{R}_\sigma,\ F \in \mathcal{R}^c$, *and* $E \cap F = \emptyset$;

(iv) $(\bar{\mu})_p(A) = \overline{(\mu_p)}(A)$ *for all* $A \in \mathcal{A}_\sigma$.

For $0 \le p \le \infty$, we write $\bar{\mu}_p(A)$ for $(\bar{\mu})_p(A) = \overline{(\mu_p)}(A)$, where $A \in \mathcal{A}_\sigma$.

Lemma 10. *Let E be any subset of X.*

(i) *If E has a cover in $\mathcal{R}_\sigma$, then $\mu_p^*(E) = \mu^*(E)$ for all $p \in [0, \infty]$.*

(ii) *If E has no cover in $\mathcal{R}_\sigma$, then*
$\mu_p^*(E) = \inf\{\bar{\mu}_p(A) : E \subset A \in \mathcal{A}_\sigma - \mathcal{R}_\sigma\}$ for all $p \in [0, \infty]$.

(iii) *If E has no cover in $\mathcal{R}_\sigma$, then*
$\mu_p^*(E) = \mu_0^*(E) + p$ for all $p \in (0, \infty)$, and $\mu_\infty^*(E) = \infty$.

(iv) *$\mu_0^*(E) \le \mu_p^*(E) \le \mu_q^*(E) \le \mu_\infty^*(E) = \mu^*(E)$, where $0 < p < q < \infty$.*

PROOF. To prove (i), suppose that $E \subset C \in \mathcal{R}_\sigma$ and that $p \in [0, \infty]$. Let $\mathcal{C}$ and $\mathcal{D}$ denote the family of all coverings of E from $\mathcal{R}_\sigma$ and $\mathcal{A}_\sigma$, respectively. We have that $C \in \mathcal{C} \subset \mathcal{D}$ so by part (i) of Lemma 8 or 9,

$$\begin{aligned}\mu_p^*(E) &= \inf\{\bar{\mu}_p(D) : D \in \mathcal{D}\} \le \inf\{\bar{\mu}_p(D) : D \in \mathcal{C}\} \\ &= \inf\{\bar{\mu}(D) : D \in \mathcal{C}\} = \mu^*(E).\end{aligned}$$

Trivially the reverse inequality holds if $\mu_p^*(E) = \infty$. To complete the proof of (i), assume first that $0 \le p < \infty$ and $\mu_p^*(E) < \infty$. For any $\epsilon > 0$, there is an $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and $\bar{\mu}_p(A) \leq \mu_p^*(E) + \epsilon$. Since $E \subset A \cap C \in \mathcal{R}_\sigma$, we have from part (i) of Lemma 8 or 9 that $\mu^*(E) \leq \bar{\mu}(A \cap C) = \bar{\mu}_p(A \cap C) \leq \bar{\mu}_p(A) \leq \mu_p^*(E) + \epsilon$, and hence $\mu^*(E) \leq \mu_p^*(E)$, since ϵ is an arbitrary positive real number. Next assume that $\mu_\infty^*(E) < \infty$. Then we have $\mu_\infty^*(E) = \inf\{\bar{\mu}_\infty(A) : E \subset A \in \mathcal{A}_\sigma\} < \infty$. By the definition of μ_∞ we have that $\mu_\infty(F) = \infty$ for all $F \in \mathcal{R}^c$, so that $\bar{\mu}_\infty(A) = \infty$ for all $A \in \mathcal{A}_\sigma - \mathcal{R}_\sigma$ by Lemma 9 (iii). Consequently,

$$\mu_\infty^*(E) = \inf\{\bar{\mu}_\infty(A) : E \subset A \in \mathcal{R}_\sigma\} = \inf\{\bar{\mu}(A) : E \subset A \in \mathcal{R}_\sigma\} = \mu^*(E)$$

by Lemma 4 (ii), together with Lemma 9 (i). Thus (i) is established.

To prove (ii), suppose that E has no cover in $\mathcal{R}_\sigma$. Since $E \subset X \in \mathcal{R}^c \subset \mathcal{A}_\sigma - \mathcal{R}_\sigma$, E has at least one cover from $\mathcal{A}_\sigma - \mathcal{R}_\sigma$ so by Lemma 4 (ii) we establish (ii).

To prove (iii), again suppose that E has no cover in $\mathcal{R}_\sigma$. Assume first that $0 < p < \infty$. Since $\bar{\mu}_p(A) = \bar{\mu}_0(A) + p$ for all $A \in \mathcal{A}_\sigma - \mathcal{R}_\sigma$ by Lemma 9, we obtain from (ii) that $\mu_p^*(E) = \inf\{\bar{\mu}_0(A) + p : E \subset A \in \mathcal{A}_\sigma - \mathcal{R}_\sigma\} = \mu_0^*(E) + p$. As we noted in the proof of (i), we have that $\bar{\mu}_\infty(A) = \infty$ for all $A \in \mathcal{A}_\sigma - \mathcal{R}_\sigma$, so that by (ii), we get $\mu_\infty^*(E) = \infty$. Thus (iii) is established.

If E has no cover in $\mathcal{R}_\sigma$, then by the definition of μ^*, together with (iii), we have $\mu^*(E) = \mu_\infty^*(E) = \infty$. By (i), we have $\mu^*(E) = \mu_\infty^*(E)$. The inequalities in (iv) now follow from (i) and (iii). Thus (iv) is established. □

We now turn to the relations among the σ-algebras $\mathcal{M}$ and $\mathcal{M}_p$ with $0 \leq p \leq \infty$. Since $\mu^* = \mu_\infty^*$ by Lemma 10 (iv), we obtain $\mathcal{M} = \mathcal{M}_\infty$. The next proposition shows that $\mathcal{M}_p \subset \mathcal{M}$ for all $p \in [0, \infty)$.

Proposition 1. *For $0 \leq p < \infty$, every μ_p^*-measurable subset of X is μ^*-measurable.*

Proof. Suppose that E is an arbitrary μ_p^*-measurable subset of X, and let T be any subset of X with $\mu^*(T) < \infty$. Obviously T has a cover in $\mathcal{R}_\sigma$, so the same is true for $T \cap E$ and $T \cap E^c$, respectively. By Lemma 10 (i) we have that $\mu_p^*(T) = \mu^*(T)$, $\mu_p^*(T \cap E) = \mu^*(T \cap E)$, and $\mu_p^*(T \cap E^c) = \mu^*(T \cap E^c)$. Consequently, by the μ_p^*-measurability of E we obtain that

$$\mu^*(T) = \mu_p^*(T) = \mu_p^*(T \cap E) + \mu_p^*(T \cap E^c) = \mu^*(T \cap E) + \mu^*(T \cap E^c),$$

and hence E is μ^*-measurable. □

Proposition 2. *For each $E \subset X$, E is μ^*-measurable if and only if E is μ_0^*-measurable.*

PROOF. Suppose that E is μ^*-measurable and that T is any subset of X with $\mu_0^*(T) < \infty$. First assume that T has a cover in $\mathcal{R}_\sigma$. Then $T \cap E$ and $T \cap E^c$, respectively, has a cover in $\mathcal{R}_\sigma$. Since E is μ^*-measurable, using Lemma 10 (i) we obtain that

$$\mu_0^*(T) = \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c) = \mu_0^*(T \cap E) + \mu_0^*(T \cap E^c),$$

and hence E is μ_0^*-measurable.

Next assume that T has no cover in $\mathcal{R}_\sigma$. By Lemma 10 (ii) there is, for any $\epsilon > 0$, a set $A = F \cup G$, where $F \in \mathcal{R}_\sigma$, $G \in \mathcal{R}^c$ and $F \cap G = \emptyset$, such that $T \subset A$ and $\bar{\mu}_0(A) \leq \mu_0^*(T) + \epsilon$. By Lemma 8 we have that $\bar{\mu}_0(A) = \bar{\mu}(F) + \mu_0(G)$. We show easily that there is a set $H \in \mathcal{R}_\sigma$ such that $H \subset G$ and $\mu_0(G) = \bar{\mu}(H)$. Set $B = F \cup H$. We have that $B \in \mathcal{R}_\sigma$, $B \subset A$ and $\bar{\mu}_0(A) = \bar{\mu}(B)$. Since $A - B = G - H$ is μ_0^*-measurable and since $\mu_0(G) < \infty$, we have that $\mu_0^*(A - B) = \mu_0^*(G - H) = \mu_0(G) - \bar{\mu}(H) = 0$. Since, for any $C \subset X$, μ_0^* is countably additive on the trace of $\mathcal{M}_0$ on C, i.e., $\{S \cap C : S \in \mathcal{M}_0\}$ (see, e.g., [5, Problem 2, p.291]), we have that $\mu_0^*(A \cap C) = \mu_0^*(B \cap C)$. In particular, we have that $\mu_0^*(A \cap E) = \mu_0^*(B \cap E)$ and $\mu_0^*(A \cap E^c) = \mu_0^*(B \cap E^c)$. Consequently, using the μ^*-measurability of B and E, together with Lemma 10 (i), we obtain that

$$\begin{aligned}
\mu_0^*(T) + \epsilon \geq \bar{\mu}_0(A) = \bar{\mu}(B) &= \mu^*(B \cap E) + \mu^*(B \cap E^c) \\
&= \mu_0^*(B \cap E) + \mu_0^*(B \cap E^c) \\
&= \mu_0^*(A \cap E) + \mu_0^*(A \cap E^c) \geq \mu_0^*(T \cap E) + \mu_0^*(T \cap E^c),
\end{aligned}$$

and hence $\mu_0^*(T) \geq \mu_0^*(T \cap E) + \mu_0^*(T \cap E^c)$, since ϵ is an arbitrary positive real number. Thus E is μ_0^*-measurable. By Proposition 1 we establish the proposition. $\square$

Proposition 3. *For any positive real numbers p and q, a subset E of X is μ_p^*-measurable if and only if E is μ_q^*-measurable.*

PROOF. We assume that $p \neq q$. Suppose that E is μ_p^*-measurable and let T be any subset of X. Suppose first that T has a cover in $\mathcal{R}_\sigma$. Since both $T \cap E$ and $T \cap E^c$ have covers in $\mathcal{R}_\sigma$ and since E is also μ^*-measurable by Proposition 1, we have from Lemma 10 (i) that

$$\mu_q^*(T) = \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c) = \mu_q^*(T \cap E) + \mu_q^*(T \cap E^c).$$

Thus E is μ_q^*-measurable. Next suppose that T has no cover in $\mathcal{R}_\sigma$ with $\mu_q^*(T) < \infty$. By Lemma 10 (iii) we have $\mu_0^*(T) < \infty$. By the assumption on

T, at least one of the sets $T \cap E$ and $T \cap E^c$ has no cover in $\mathcal{R}_\sigma$. Suppose that both of these sets have no cover in $\mathcal{R}_\sigma$. Since E is μ_0^*-measurable by Propositions 1 and 2, we obtain from Lemma 10 (iii) that

$$\begin{aligned}\mu_0^*(T) + p = \mu_p^*(T) &= \mu_p^*(T \cap E) + \mu_p^*(T \cap E^c) \\ &= \mu_0^*(T \cap E) + \mu_0^*(T \cap E^c) + 2p = \mu_0^*(T) + 2p,\end{aligned}$$

so $p = 0$. This is a contradiction. Therefore, we can assume that $T \cap E$ has a cover in $\mathcal{R}_\sigma$ and $T \cap E^c$ has no cover in $\mathcal{R}_\sigma$. Consequently, we have from parts (i) and (iii) of Lemma 10 that

$$\mu_q^*(T) = \mu_0^*(T) + q = \mu_0^*(T \cap E) + \mu_0^*(T \cap E^c) + q = \mu_q^*(T \cap E) + \mu_q^*(T \cap E^c).$$

Thus E is μ_q^*-measurable. Interchanging p and q in the preceding result we establish the proposition. □

Now we formulate the main result of this paper.

Theorem 2. *$\mathcal{M}_p = \mathcal{M}_1 \subset \mathcal{M}_0 = \mathcal{M}_\infty = \mathcal{M}$ for all $p \in (0, \infty)$.*

Proof. By Proposition 3 we have that $\mathcal{M}_p = \mathcal{M}_1$ for all positive real numbers p. We infer from Propositions 1 and 2 that $\mathcal{M}_1 \subset \mathcal{M} = \mathcal{M}_0$. As we noted earlier, we have that $\mathcal{M} = \mathcal{M}_\infty$. □

We need not have $\mathcal{M}_1 = \mathcal{M}$ (see Example 5 or 6 below).

Example 5. Let $X = \{1, 2\}$, $\mathcal{R} = \{\emptyset\}$, and $\mathcal{A} = \{\emptyset, X\}$. Plainly $\mathcal{R}$ is a σ-ring in X with $X \notin \mathcal{R}$, and $\mathcal{A}$ is the σ-algebra generated by $\mathcal{R}$. Define $\mu(\emptyset) = 0$. Then μ is a countably additive measure on $\mathcal{R}$. We have at once that $\mu^*(\emptyset) = 0$ and $\mu^*(E) = \infty$ if E is a nonempty subset of X, so μ^* is also a countably additive measure on $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, X\}$. Thus $\mathcal{M} = \mathcal{P}(X)$. Since $\mu_0(\emptyset) = \mu_0(X) = 0$, we obtain that $\mu_0^*(E) = 0$ for all $E \subset X$. By Lemmas 5 and 10 we have that, for all $p \in (0, \infty]$, $\mu_p(\emptyset) = 0$ and $\mu_p(X) = p$, so $\mu_p^*(\emptyset) = 0$ and $\mu_p^*(E) = p$ if $E = \{1\}, \{2\}$ or X. Notice that $\mu^* = \mu_\infty^*$. We assert that $\mathcal{M}_p = \mathcal{A}$ for all $p \in (0, \infty)$. To prove the assertion, suppose that p is any positive real number. Since $p = \mu_p^*(X) < \mu_p^*(\{1\}) + \mu_p^*(\{2\}) = 2p$, both $\{1\}$ and $\{2\}$ are not μ_p^*-measurable and hence the assertion is established. Thus $\mathcal{M}_1 \neq \mathcal{M}$.

In the following examples, let X denote any uncountable set, let $\mathcal{R}$ denote the ring of all finite subsets of X, and let $\mathcal{A}$ and $\mathcal{B}$ denote, respectively, the algebra and the σ-algebra generated by $\mathcal{R}$ as in Example 2. For any measure μ on $\mathcal{R}$, the measure μ_p $(0 \leq p \leq \infty)$ on $\mathcal{A}$ induced by μ and parameter p is countably additive, since μ is countably additive.

Example 6. Define $\mu(E) = 0$ for all E in $\mathcal{R}$. Plainly μ is a countably additive measure on $\mathcal{R}$. Since $\bar{\mu}(E) = 0$ for all E in $\mathcal{R}_\sigma$, we obtain that $\mu^*(E) = 0$ or ∞ according as E is countable or not. Then μ^* is also a countably additive measure on $\mathcal{P}(X)$ so that $\mathcal{M} = \mathcal{P}(X)$. We next compute μ_0^*. Since $\mu_0(E) = 0$ for all A in $\mathcal{A}$, we obtain that $\bar{\mu}_0(E) = 0$ for all E in $\mathcal{A}_\sigma = \mathcal{R}_\sigma \cup \mathcal{R}^c$ (see Example 2), and hence $\mu_0^*(E) = 0$ for all $E \subset X$. By Lemma 10 (iii) we have that $\mu_1^*(E) = 0$ or 1 according as E is countable or not. Assert that $\mathcal{M}_1 = \mathcal{B}$. Since $\mathcal{B} = \mathcal{R}_\sigma \cup (\mathcal{R}_\sigma)^c$ by Example 2, we obtain $\mathcal{B} \subset \mathcal{M}_1$. To prove the opposite inclusion, suppose that A is any subset of X such that both A and A^c are uncountable. Then $1 = \mu_1^*(X) < \mu_1^*(A) + \mu_1^*(A^c) = 2$ so A is not μ_1^*-measurable. Thus the assertion holds. Consequently, $\mathcal{M}_1 \neq \mathcal{M}$.

Example 7. Define μ on $\mathcal{R}$ by $\mu(E) = card\ E$ for all E in $\mathcal{R}$. Then μ is a countably additive measure on $\mathcal{R}$. We obtain that $\bar{\mu}(E) = card\ E$ if E is finite and $\bar{\mu}(E) = \infty$ if E is countably infinite, so that $\mu^*(E) = card\ E$ if E is finite and $\mu^*(E) = \infty$ if E is infinite. Then μ^* is a countably additive measure on $\mathcal{P}(X)$ so $\mathcal{M} = \mathcal{P}(X)$. On the other hand, we have that $\mu_0(E) = card\ E$ if E is finite and $\mu_0(E) = \infty$ if E is cofinite, and that $\bar{\mu}_0(E) = card\ E$ if E is finite and $\bar{\mu}_0(E) = \infty$ if E is countably infinite or cofinite. Consequently, $\mu_0^* = \mu^*$. By part (iv) of Lemma 10 we obtain that for all $p \in [0, \infty]$, $\mu_p^* = \mu^*$ so $\mathcal{M}_p = \mathcal{M}$.

Example 8. Define μ on $\mathcal{R}$ by $\mu(\emptyset) = 0$ and $\mu(E) = \infty$ if $E \in \mathcal{R}$ and $E \neq \emptyset$. Then μ is a countably additive measure on $\mathcal{R}$. It follows easily that $\mu^*(\emptyset) = 0$ and $\mu^*(E) = \infty$ if $E \neq \emptyset$, and that μ^* is a countably additive measure on $\mathcal{P}(X)$, so $\mathcal{M} = \mathcal{P}(X)$. As in Example 7, we obtain that for all $p \in [0, \infty]$, $\mu_p^* = \mu^*$ so $\mathcal{M}_p = \mathcal{M}$.

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References

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges,* Academic Press, London,1983.

[2] P. Billingsley, *Probability and Measure, 2nd ed.,* Wiley, New York, 1986.

[3] J. L. Doob, *Measure Theory,* Springer–Verlag, New York,1991.

[4] M. M. Rao, *Measure Theory and Integration,* Wiley, New York, 1987.

[5] H. L. Royden, *Real Analysis, 3rd ed.,* Macmillan, New York, 1988.