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AN EXAMPLE ILLUSTRATING $P^g(K) \neq P^g_0(K)$ FOR COMPACT SETS OF FINITE PREMEASURE

Abstract

We construct a doubling gauge function g and a compact set $L \subset \mathbb{R}$ for which $\mathcal{P}^g(L) < \mathcal{P}^g_0(L) < \infty$.

D. J. Feng, S. Hua and Z. Y. Wen proved in [1] that for every compact set $K \subset \mathbb{R}^n$ and for every $0 \leq s \leq n$,

$$\mathcal{P}^s_0(K) < \infty \Rightarrow \mathcal{P}^s_0(K) = \mathcal{P}^s(K),$$

where $\mathcal{P}^s$ and $\mathcal{P}^s_0$ denote the s-dimensional packing measure and premeasure, respectively. (The definition and the basic properties of packing measures and premeasures see e.g. in [2].) One can check that their proof works for every gauge function g and the corresponding packing measure and premeasure $\mathcal{P}^g$, $\mathcal{P}^g_0$, provided that for every positive ε there are positive δ and t_0, such that

$$\frac{g((1+\delta)t)}{g(t)} < 1 + \varepsilon \quad \forall t < t_0.$$

Especially, if $g(t) = t^s L(t)$ where L is slowly varying in the sense of Karamata; that is, $\lim_{t \to 0} \dfrac{L(ct)}{L(t)} = 1$ for every $c > 0$ (see [3]), then

$$\mathcal{P}^g_0(K) < \infty \Rightarrow \mathcal{P}^g_0(K) = \mathcal{P}^g(K) \qquad (*)$$

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for every compact set K. These are the gauge functions which naturally arise in dynamics and stochastic processes. R. D. Mauldin asked whether $(*)$ remains true for any gauge function g.

In this paper we show that $(*)$ is false for general gauge functions g and for the packing measure and premeasure $\mathcal{P}^g, \mathcal{P}^g_0$. We prove that it is not even true for doubling measures. We prove the following theorem.

Theorem 1. *There exists a doubling gauge function g, and compact sets $K \subset L \subset \mathbb{R}$, for which*

$$\mathcal{P}^g_0(K) < 1 \leq \mathcal{P}^g_0(L) < \infty, \tag{$**$}$$

and $L \setminus K$ is countable.

The following is an immediate corollary of Theorem 1.

Theorem 2. *There exists a doubling gauge function g and a compact set $L \subset \mathbb{R}$, for which $\mathcal{P}^g(L) < \mathcal{P}^g_0(L) < \infty$.*

We will use the notations

$$a_n = 2^n, \quad b_n = 4a_n + 2, \quad c_n = \prod_{m=1}^{n} b_m, \quad d_n = 80^{-n^3}.$$

For every $n \in \mathbb{N}$ we define a set of c_n pairwise disjoint intervals

$$\mathcal{I}^n = \{I^n_j = [x^n_j, y^n_j] : 1 \leq j \leq c_n\}$$

of length d_n, as follows. We choose an interval I^0 of length 1 arbitrarily. If $\mathcal{I}^{n-1}$ has been defined, then for every $1 \leq j \leq c_{n-1}$ we choose the b_n subintervals

$$[x^{n-1}_j + 6d_n, x^{n-1}_j + 7d_n], \quad [y^{n-1}_j - 7d_n, y^{n-1}_j - 6d_n],$$

$$[x^{n-1}_j + i \cdot \frac{d_{n-1}}{2a_n} + 4d_n, x^{n-1}_j + i \cdot \frac{d_{n-1}}{2a_n} + 5d_n] \quad (0 \leq i \leq 2a_n - 1),$$

$$[y^{n-1}_j - i \cdot \frac{d_{n-1}}{2a_n} - 5d_n, y^{n-1}_j - i \cdot \frac{d_{n-1}}{2a_n} - 4d_n] \quad (1 \leq i \leq 2a_n).$$

These are pairwise disjoint subintervals, since $12d_n < d_{n-1}/2a_n$ for every $n \geq 1$.

Let

$$K = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{c_n} I^n_j, \quad L = K \cup \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{c_{n-1}} \bigcup_{i=0}^{2a_n} \{x^{n-1}_j + i \cdot \frac{d_{n-1}}{2a_n}\}.$$

Then both K and L are compact, and L is the union of the Cantor set K and countable many points. We put

$$e_n = \frac{d_{n-1}}{2a_n} - 7d_n, \quad f_n = \frac{d_{n-1}}{2a_n} - 8d_n, \quad g_n = 10d_n.$$

It is easy to check that $e_n > f_n > g_n > e_{n+1}$. We define

$$g(t) = \begin{cases} \frac{1}{2a_n c_{n-1}} & \text{if } g_{n-1} \geq t/2 \geq e_n \\ \frac{1}{10a_n c_{n-1}} & \text{if } t/2 = f_n \end{cases}$$

and we extend g to the intervals $[g_n, f_n]$ and $[f_n, e_n]$ linearly. Then we obtain a gauge function, the only thing we need to check is that $g(f_n) > g(e_{n+1})$. We will prove $(**)$. We will also prove that g is doubling.

PROOF THAT $\mathcal{P}_0^g(K) < 1$.

Let μ be the (unique) probability measure of support K, for which $\mu(I_j^n) = 1/c_n$ for every n, j. Let I be an arbitrary interval whose midpoint belongs to K, and for which $|I| < d_1 = 1/80$. Let n be the first index for which I intersects only one of the intervals of $\mathcal{I}^{n-1}$, but at least 2 of the intervals of $\mathcal{I}^n$.

Since the distance between the intervals I_j^n, $I_{j'}^n$ is at least d_n for every $j \neq j'$, the midpoint of I belongs to an interval I_j^n, and I intersects at least two intervals of $\mathcal{I}^n$, we have $|I| \geq 2d_n$. Then from $|I| < d_1$, $n \geq 2$ follows. The length of I_j^n is d_n; so $I_j^n \subset I$. Thus

$$\mu(I) \geq 1/c_n. \tag{1}$$

On the other hand, it is easy to see from the construction that for every $1 \leq k, 1 \leq j \leq c_k$, and for every $x \in I_j^k$ there is an index $j' \neq j$ and a point $y \in I_{j'}^k$ for which $|x-y| < 9d_k < g_k$. Therefore, since I intersects only one of the intervals of $\mathcal{I}^{n-1}$, we have $|I| < 2g_{n-1}$ and

$$g(|I|) \leq g(2g_{n-1}) = \frac{1}{2a_n c_{n-1}}. \tag{2}$$

If $|I| \leq 2f_n$, then

$$g(|I|) \leq g(2f_n) = \frac{1}{10a_n c_{n-1}} \leq \frac{1}{2b_n c_{n-1}} = \frac{1}{2c_n}. \tag{3}$$

From (1) and (3)

$$g(|I|) \leq \frac{1}{2} \cdot \mu(I)$$

follows. On the other hand, if $|I| > 2f_n$, then it is also easy to see from the construction that I covers at least 3 of the intervals of $\mathcal{I}^n$. Thus

$$\mu(I) \geq \frac{3}{c_n} = \frac{3}{b_n c_{n-1}}. \tag{4}$$

Since $n \geq 2, a_n \geq 4$ and hence from (2) and (4) we obtain

$$g(|I|) \leq \frac{b_n}{6a_n} \cdot \mu(I) = \frac{4a_n + 2}{6a_n} \cdot \mu(I) \leq \frac{3}{4} \cdot \mu(I).$$

So for every interval I for which $I < 1/80$ and whose midpoint belongs to K we have $g(|I|) < 3/4 \cdot \mu(I)$. Thus $\mathcal{P}^g_\varepsilon(K) \leq 3/4$ for every $\varepsilon < 1/80$. From this we obtain $\mathcal{P}^g_0(K) \leq 3/4 < 1$. □

PROOF THAT $1 \leq \mathcal{P}^g_0(L)$.

For every interval I^{n-1}_j, the points

$$x^{n-1}_{ji} = x^{n-1}_j + 2i \cdot d_{n-1}/2a_n \quad 1 \leq i \leq a_n - 1$$

belong to L and the intervals $I^{n-1}_{ji} = (x^{n-1}_{ji} - e_n, x^{n-1}_{ji} + e_n)$ are pairwise disjoint subintervals of I^{n-1}_j. It is also easy to see that each interval I^{n-1}_{ji} covers 2 of the intervals of $\mathcal{I}^n$ and disjoint from all the other intervals of $\mathcal{I}^n$. We have $\mu(I^{n-1}_{ji}) = 2/c_n$. Thus for every $n \geq 1$ we have

$$\sum_{i=1}^{a_n-1} \mu(I^{n-1}_{ji}) = \frac{2a_n - 2}{c_n} = \frac{2a_n - 2}{(4a_n + 2)c_{n-1}} \geq \frac{2}{10c_{n-1}} = \frac{2}{10} \cdot \mu(I^{n-1}_j). \tag{5}$$

We also have

$$g(|I^{n-1}_{ji}|) = g(2e_n) = \frac{1}{2a_n c_{n-1}} > \frac{2}{b_n c_{n-1}} = \frac{2}{c_n} = \mu(I^{n-1}_{ji}). \tag{6}$$

We fix an $m \geq 1$ and define

$$\mathcal{I}_m = \{I^{m-1}_{ji} : 1 \leq j \leq c_{m-1}, 1 \leq i \leq a_m - 1\},$$

and if $\mathcal{I}_m, \mathcal{I}_{m+1}, \ldots, \mathcal{I}_n$ have been defined for an $n \geq m$, then we put

$$\mathcal{I}_{n+1} = \{I^n_{ji} : 1 \leq j \leq c_n, 1 \leq i \leq a_{n+1} - 1, I^n_j \not\subset \bigcup_{\ell=m}^{n} \cup \mathcal{I}_\ell\}.$$

Then $\bigcup_{\ell=m}^{\infty} \cup\mathcal{I}_\ell$ is a $2e_m$-packing of L. It is easy to see from (5) by induction that $\mu(L \setminus \bigcup_{\ell=m}^{m+k-1} \cup\mathcal{I}_\ell) \leq (8/10)^k$. Thus $\mu(\bigcup_{\ell=m}^{\infty} \cup\mathcal{I}_\ell) = 1$. Therefore, from (6) we obtain $\mathcal{P}^g_{2e_m}(L) \geq 1$ for every $m \geq 1$ and thus $\mathcal{P}^g_0(L) \geq 1$. □

Proof that $\mathcal{P}^g_0(L) < \infty$.

Let I be an arbitrary interval whose midpoint belongs to L, and for which $|I| < d_1 = 1/80$. Let the midpoint of I be x. If $x \in K$, then we know $g(I) < 3/4 \cdot \mu(I)$ from the proof of $\mathcal{P}^g_0(K) < 1$. If $x \notin K$, then

$$x = x_j^{m-1} + i \cdot \frac{d_{m-1}}{2a_m}$$

for some m, j, i.

If $|I| \leq 10d_m = g_m$, then $|I|/2 \leq 5d_m \in [e_{m+1}, g_m]$. Thus

$$g(|I|) \leq \frac{1}{2a_{m+1}c_m} = \frac{1}{4a_m c_m} = \frac{1}{4a_m b_m c_{m-1}}. \tag{7}$$

If $10d_m < |I|$, then I covers at least 2 of the intervals of $\mathcal{I}^m$ and of course x belongs to an interval of $\mathcal{I}^{m-1}$ and does not belong to $\mathcal{I}^m$.

As before, let n be the smallest index for which I intersects only one of the intervals of $\mathcal{I}^{n-1}$, but at least 2 of the intervals of $\mathcal{I}^n$. We have seen in the proof of $\mathcal{P}^g_0(K) < 1$, that if the midpoint of I belongs to $\mathcal{I}^n$, then $g(|I|) < 3/4 \cdot \mu(I)$. If the midpoint of I does not belong to $\mathcal{I}^n$, then $m-1 < n$. On the other hand, I intersects 2 intervals of $\mathcal{I}^m$. Thus $n \leq m$. So in this case $n = m$. We have

$$\mu(I) > 2/c_n, \tag{8}$$

and (since x belongs to $\mathcal{I}^{n-1}$ and I intersects only one of the intervals of $\mathcal{I}^{n-1}$) we obtain $|I| < 2g_{n-1}$. From (8)

$$g(|I|) \leq g(2g_{n-1}) = \frac{1}{2a_n c_{n-1}} = \frac{4a_n + 2}{2a_n c_n} \leq \frac{2a_n + 1}{2a_n} \cdot \mu(I) < 2 \cdot \mu(I).$$

So for every I, either $g(|I|) < 2\mu(I)$ or $x = x_j^{m-1} + i \cdot \frac{d_{m-1}}{2a_m}$ and $g(|I|)$ can be estimated by (7). But

$$\sum_{m=1}^{\infty} \sum_{j=1}^{c_{m-1}} \sum_{i=0}^{2a_m} \frac{1}{4a_m b_m c_{m-1}} < \sum_{m=1}^{\infty} \frac{1}{b_m} < 1.$$

This proves $\mathcal{P}^g_0(L) < 3 < \infty$. □

PROOF OF DOUBLING.

It is enough to prove that there exists a constant C, such that if t is small enough, then $g(2t)/g(t) < C$. We put $\tilde{g}(u) = g(u/2)$, and prove $\frac{\tilde{g}(2u)}{\tilde{g}(u)} < C$ for every u small enough. We fix a small u, let n be the index for which $u \in [e_{n+1}, e_n]$. It is easy to check that $2e_n < g_{n-1}$. Thus $2u < g_{n-1}$. We know that $\tilde{g}$ is constant $5\tilde{g}(f_n)$ on $[e_n, g_{n-1}]$.

If u is small enough, then n is large enough. It is easy to see that

$$\frac{\tilde{g}(e_{n+1})}{e_{n+1}} = \frac{\tilde{g}(g_n)}{e_{n+1}} > \frac{\tilde{g}(g_n)}{g_n},$$

and for suitable large n

$$\frac{\tilde{g}(g_n)}{g_n} > \frac{\tilde{g}(f_n)}{f_n},$$

that is, the function $\tilde{g}(x)/x$ monotone decreases on $[e_{n+1}, f_n]$. Thus if u and $2u \in [e_{n+1}, f_n]$, then $\tilde{g}(u)/u > \tilde{g}(2u)/2u$; that is, $\tilde{g}(2u)/\tilde{g}(u) < 2$. If $u \in [e_{n+1}, f_n]$ and $2u > f_n$, then $\tilde{g}(2u) = 5\tilde{g}(f_n)$, and $\tilde{g}(f_n)/f_n < \tilde{g}(u)/u$ where $f_n < 2u$. Thus $\tilde{g}(2u)/\tilde{g}(u) < 10$. Finally, if $u > f_n$, then it is immediate that $\tilde{g}(2u)/\tilde{g}(u) = 5\tilde{g}(f_n)/\tilde{g}(u) \leq 5$. □

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