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POROUS AND BOUNDARY SETS IN DARBOUX-LIKE FUNCTION SPACES

Abstract

We show which subspaces of Darboux-like real function spaces are porous or boundary sets with the metric of uniform convergence.

We are interested in the size of one function subspace inside another function space. Porosity is one such measure of this. In a metric space (X, d), $B(x, r)$ denotes the open ball centered at x with radius $r > 0$. For $M \subset X$, $x \in X$, and $r > 0$, we let $\gamma(x, r, M)$ denote the supremum of the set of all $s > 0$ for which there exists $z \in X$ such that $B(z, s) \subset B(x, r) \setminus M$. M is *porous at x* if $p(x, M) = \limsup_{r \to 0+} \dfrac{\gamma(x, r, M)}{r} > 0$. M is *porous* in X if M is porous at each $x \in \overline{M}$. (Others define porosity for only $x \in M$.) For example, in $X = \mathbb{R}$, $M_1 = \left\{ \pm \dfrac{1}{2^n} : n = 1, 2, 3, \dots \right\}$ is porous at 0 because $p(0, M_1) = \dfrac{1}{4}$, but $M_2 = \left\{ \pm \dfrac{1}{n} : n = 1, 2, 3, \dots \right\}$ is not porous at 0 because $p(0, M_2) = 0$. A porous set M has to be a boundary set in X; i.e., $\overline{X \setminus M} = X$. In this paper, we determine the porosity of subspaces of Darboux-like function spaces. The following functions $f : \mathbb{R} \to \mathbb{R}$ belong to these abbreviated classes of functions. Historically, for Baire class 1 functions $f : \mathbb{R} \to \mathbb{R}$, these classes (except for C, $WCIVP$, and PB) become equal.

1. C - the class of continuous functions.
2. PC - $f : \mathbb{R} \to \mathbb{R}$ is a peripherally continuous function if for each $x \in \mathbb{R}$ and for all open neighborhoods U of x and V of $f(x)$, there exists an open neighborhood W of x such that $W \subset U$ and $f(\mathrm{bd}(W)) \subset V$.

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3. D - f is a Darboux function if $f(J)$ is connected for each connected set $J \subset \mathbb{R}$.

4. $Conn$ - f is a connectivity function if the graph of f restricted to J is a connected subset of $\mathbb{R}^2$ for each connected set $J \subset \mathbb{R}$.

5. AC - $f : \mathbb{R} \to \mathbb{R}$ is almost continuous if each open neighborhood in $\mathbb{R}^2$ of the graph of f contains the graph of a continuous function $g : \mathbb{R} \to \mathbb{R}$.

6. Ext - f is extendable if there is a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(x, 0) = f(x)$ for all $x \in \mathbb{R}$ and the graph of the restriction $F \restriction J$ is connected for each connected set $J \subset \mathbb{R}^2$.

7. PR - f has a perfect road if for each $x \in \mathbb{R}$ there exists a perfect set P having x as a bilateral limit point such that $f \restriction P$ is continuous at x. We refer to P as a perfect road at x.

8. $WCIVP$ - f has the weak Cantor intermediate value property if for each $x < y$ with $f(x) \neq f(y)$ there is a Cantor set $C \subset (x, y)$ such that $f(C)$ lies between $f(x)$ and $f(y)$.

9. $CIVP$ - f has the Cantor intermediate value property if for each $x < y$ with $f(x) \neq f(y)$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set $C \subset (x, y)$ such that $f(C) \subset K$.

10. $SCIVP$ - f has the strong Cantor intermediate value property if in the preceding definition C can be chosen so that $f \restriction C$ is also continuous.

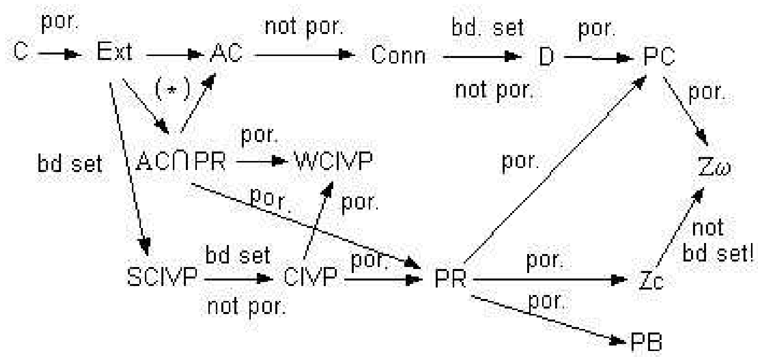
11. PB - f has property B if for each pair I, J of open intervals $I \cap f^{-1}(J)$ whenever uncountable contains a nonempty perfect set.

12. Zc - f satisfies Zahorski's condition if for each a, each set $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ whenever nonempty is bilaterally $\mathfrak{c}$-dense in itself.

13. $Z\omega$ - f satisfies condition Zω if for each a, each set $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ whenever nonempty is bilaterally dense in itself.

These definitions could have been given instead for classes of functions $f : [0, 1] \to \mathbb{R}$. Each function space has on it the metric d of uniform convergence defined by $d(f, g) = \min \{1, \sup \{|f(x) - g(x)| : x \in \mathbb{R}\}\}$, and unless otherwise specified, the closure of a function space is taken in the class $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ into $\mathbb{R}$.

The following chart, in which $\to$ indicates proper inclusion, was lifted from [10] and [6]. In [13], we determine which spaces are porous or boundary sets for the chain of spaces of functions $f:\mathbb{R}\to\mathbb{R}$ in the top row of the chart. We indicate in the rest of the chart which function spaces are determined in this paper to be porous or boundary sets.



Whether or not Ext is porous in SCIVP and what the situation is for the commutative diagram $(*)$ are left as open problems.

If A is a subspace of B and B is porous in C, then A is porous in C. This is also true if "porous" is replaced by "a boundary set." For example, according to the chart, $Conn$ is porous in PC because D is porous in PC.

We will invoke the following construction often. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous at some point $x_0\in\mathbb{R}$, and suppose $h:[0,1]\to[0,1]$ is any function. Given $r>0$ there exists a $\delta>0$ such that $f\ [x_0-\delta,x_0+\delta]\subset(f(x_0)-\frac{r}{4},\ f(x_0)+\frac{r}{4})$. We describe what it means to say that "g is the function obtained from f by gluing a copy of h at x_0." Let f_0 be a scaled-down copy of h to the rectangle $\left[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}\right]\times\left[f(x_0)-\frac{r}{4},f(x_0)+\frac{r}{4}\right]$ instead of $[0,1]\times[0,1]$. Namely, $f_0(x)=\frac{r}{2}h\left(\frac{1}{\delta}\left(x-x_0+\frac{\delta}{2}\right)\right)+f(x_0)-\frac{r}{4}$. We obtain a function $g:\mathbb{R}\to\mathbb{R}$ by gluing this copy, f_0, of h into the rectangle $\left[x_0-\frac{\delta}{2},x_0+\frac{\delta}{2}\right]\times\left[f(x_0)-\frac{r}{4},f(x_0)+\frac{r}{4}\right]$ and connecting it linearly to the rest of the graph of f outside the larger rectangle $[x_0-\delta,x_0+\delta]\times\left[f(x_0)-\frac{r}{4},f(x_0)+\frac{r}{4}\right]$. That is,

$$g(x)=\begin{cases} f(x) & \text{if } x\notin(x_0-\delta,x_0+\delta)\\ l_1(x) & \text{if } x\in\left[x_0-\delta,x_0-\dfrac{\delta}{2}\right]\\ f_0(x) & \text{if } x\in\left(x_0-\dfrac{\delta}{2},x_0+\dfrac{\delta}{2}\right)\\ l_2(x) & \text{if } x\in\left[x_0+\dfrac{\delta}{2},x_0+\delta\right]\ , \end{cases}$$

where l_1 and l_2 are linear functions such that $l_1(x_0-\delta)=f(x_0-\delta), l_1(x_0-\frac{\delta}{2})=f_0(x_0-\frac{\delta}{2})$, $l_2(x_0+\frac{\delta}{2})=f_0(x_0+\frac{\delta}{2})$, and $l_2(x_0+\delta)=f(x_0+\delta)$. Notice that $d(f,g)<\frac{r}{2}$. In the proofs, g will inherit certain properties h has.

Theorem 1. *$AC\cap PR$ is porous in $WCIVP$.*

Proof. Suppose $f\in\overline{AC\cap PR}$. (This closure taken in $WCIVP$ is the same as the closure taken in $\mathbb{R}^{\mathbb{R}}$.)

Case 1: f is continuous at some $x_0\in\mathbb{R}$.

Let $h:[0,1]\to[0,1]$ be the function defined by

$$h(x)=\begin{cases}1 & \text{if } x=0,1\\ \left|x-\frac{1}{2}\right| & \text{if } 0<x<1.\end{cases}$$

Notice $h\in WCIVP\setminus PR$. For each $0<r\le 1$, we can glue in a copy of h at x_0 to obtain from f a function $g:\mathbb{R}\to\mathbb{R}$ in $WCIVP\setminus PR$ with $d(f,g)<\frac{r}{2}$. In the space $WCIVP$, $B(g,\frac{r}{8})\subset B(f,r)\setminus PR$. Since $\gamma(f,r,AC\cap PR)\ge\frac{r}{8}$, $p(f,AC\cap PR)\ge\frac{1}{8}>0$ implies $AC\cap PR$ is porous at f.

Case 2: f is continuous at no point.

Then since $f\in\overline{AC}\subset\overline{D}$, $\overline{f}\ \cap(\{x\}\times\mathbb{R})$ is a nondegenerate interval for every $x\in\mathbb{R}$. It follows from [12], [5] that the graph of f is somewhere dense in $\mathbb{R}^2$. Therefore there is an open rectangle $U=(a-\delta,b+\delta)\times(c,d)\subset\overline{f}$ where $a<b$ and $\delta>0$. Let $0<r\le 1$ and we suppose $r\le d-c$. Pick a point $(a,p)\in\overline{U}$ such that $c+\frac{r}{4}<p<d-\frac{r}{4}$. L_1 is the line segment joining the points (a,p) and $\left(b,p-\frac{r}{4}\right)$, and L_2 is the line segment joining the points $\left(a,p+\frac{r}{8}\right)$ and $\left(b,p+\frac{r}{4}\right)$. V denotes the trapezoidal region in U that lies

between L_1 and L_2. A function $g : \mathbb{R} \to \mathbb{R}$ can be obtained from f by shifting all points of $f \cap V$ vertically up to L_2. Then $d(f,g) \le \frac{r}{2}$ and $g \notin D \supset AC \cap PR$. To see $g \in WCIVP$, suppose $x < y$ and $g(x) \neq g(y)$. Then since f is dense in $U \setminus \overline{V}$, there is a Cantor set $C \subset (x,y)$ such that $f(C) = g(C)$ lies between $g(x)$ and $g(y)$. In $WCIVP$, $B(g, \frac{r}{16}) \subset B(f,r) \setminus D \subset B(f,r) \setminus (AC \cap PR)$. Since $\gamma(f, r, AC \cap PR) \ge \frac{r}{16}$, $p(f, AC \cap PR) \ge \frac{1}{16}$ implies $AC \cap PR$ is porous at f. $\square$

Theorem 2. *$AC \cap PR$ is porous in PR.*

Proof. Suppose $f \in \overline{AC \cap PR}$. (This closure taken in PR is the same as the closure taken in $\mathbb{R}^{\mathbb{R}}$.)

Case 1: f has a point of continuity x_0.

According to [6], the characteristic function $h : [0,1] \to [0,1]$ for C^0 belongs to $PR \setminus D \subset PR \setminus AC$, where C^0 is the set of nonendpoints of the Cantor ternary set C. Given $0 < r \le 1$, let $g : \mathbb{R} \to \mathbb{R}$ be the function obtained from f by gluing a copy of h at x_0. Then $g \in PR \setminus AC$ and $d(f,g) < \frac{r}{2}$. Suppose $\phi \in PR$ and $d(\phi, g) < \frac{r}{4}$. Since ϕ misses the horizontal line segment $\left[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right] \times \{f(x_0)\}$, $\phi \notin D$ and so $\phi \notin AC$. Therefore $B(g, \frac{r}{4}) \subset B(f,r) \setminus AC$.

Case 2: f has no point of continuity.

There is an open rectangle $U = (a,b) \times (c,d) \subset \overline{f}$. Let $0 < r \le \min\{1, d-c\}$. V is the open rectangle in U with the same width and center as U but with height $\frac{r}{2}$. We may suppose $f(a), f(b) \in \overline{U} \setminus \overline{V}$. We obtain a function $g : \mathbb{R} \to \mathbb{R}$ by moving all points of $f \cap V$ vertically to points of the top horizontal side H of V and letting $g = f$ elsewhere. Then $g \in PR \setminus AC$ because f is dense in $U \setminus V$, $f \in PR$, and $g \notin D$. By construction, $d(f,g) \le \frac{r}{2}$ and in the space PR, $B(g, \frac{r}{4}) \subset B(f,r) \setminus D \subset B(f,r) \setminus AC$.

Both cases show $AC \cap PR$ is porous at f. $\square$

Theorem 3. *If the graph of $f : \mathbb{R} \to \mathbb{R}$ is dense in $\mathbb{R}^2$ and $f \in PR$, then $f \in \overline{AC} \cap \overline{CIVP}$.*

Proof. We first show $f \in \overline{D}$. Let $\mathcal{U}$ denote the class of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every interval $J \subset \mathbb{R}$ and every set A of cardinality less than $\mathfrak{c}$, $f(J \setminus A)$ is dense in $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$. According to [8], $\mathcal{U} = \overline{D}$.

Let J be an interval and suppose $\inf_{x\in J} f(x) < y < \sup_{x\in J} f(x)$. Since f is dense in $\mathbb{R}^2$, for each set $O_n = J \times (y - \frac{1}{n}, y + \frac{1}{n})$ there exists $x_n \in J$ such that $(x_n, f(x_n)) \in O_n$. Because $f \in PR$, there exists a perfect set P_n in J containing x_n such that $f \restriction P_n \subset O_n$. For every set A of cardinality less than $\mathfrak{c}$, $y \in \mathrm{cl}\left(f\left(\cup_{n=1}^{\infty} P_n \setminus A\right)\right) \subset \mathrm{cl}\left(f\left(J \setminus A\right)\right)$. Therefore $f \in \mathcal{U} = \overline{D}$. Then $f \in \overline{CIVP}$ because $\overline{CIVP} = PR \cap \overline{D}$ [2]. According to [7], since $f \in \overline{D}$ is dense in $\mathbb{R}^2$, $Conn$ is dense in each open ball in $\overline{D}$ of radius ≤ 1 centered at f. Therefore $f \in \overline{Conn}$. By [13], since $f \in \overline{Conn}$ is dense in $\mathbb{R}^2$, AC is dense in each open ball in $\overline{Conn}$ of radius ≤ 1 with center f. So $f \in \overline{AC}$. □

Theorem 4. *Ext is a boundary set in SCIVP.*

Proof. Suppose $f \in \overline{Ext}$. (This closure is taken in $SCIVP$.)

Case 1: f is continuous at some point x_0.

According to [11] ,the following type example $h : [0,1] \to [0,1]$ belongs to $Conn \setminus AC$, and the graph of each function within a vertical distance .05 from h is not in AC. That is, $B(h, .05) \cap AC = \emptyset$. Let C be the Cantor ternary set in $[0,1]$, $\{(a_n, b_n) : n = 1,2,3,\dots\}$ the set of components of $[0,1] \setminus C$, and

$$h(x) = \begin{cases} \dfrac{x - a_n}{b_n - a_n} & \text{if } x \in [a_n, b_n] \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

It turns out $h \in SCIVP$. For each $0 < r \leq 1$, the function $g : \mathbb{R} \to \mathbb{R}$ obtained from f by gluing a copy of h at x_0 belongs to $SCIVP \setminus AC \subset SCIVP \setminus Ext$, $d(f,g) < \frac{r}{2}$, and $B(g, .025r) \subset B(f, r) \setminus Ext$. It follows that Ext is porous at f.

Case 2: f is discontinuous at every point.

Since the graph of f is somewhere dense in $\mathbb{R}^2$, $\overline{f}$ contains some upright open rectangleU of height t. We may suppose $0 < r \leq t$, and pick $a < b$ so that $b - a < r$ and the horizontal lines $H_a = \mathbb{R} \times \{a\}$ and $H_b = \mathbb{R} \times \{b\}$ meet U. Move all points of f lying on $H_a \cap U$ vertically to points on $H_b \cap U$ and leave the remaining points of f alone in order to obtain a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \notin D$ (because $g \cap H_a \cap U = \emptyset$) and $d(f,g) < r$. Also $g \in SCIVP$, really. For, suppose $x < y$, $g(x) \neq g(y)$, and K is a Cantor set between $g(x)$ and $g(y)$. Let K_0 be a Cantor set in $K \setminus \{a\}$. Since f is dense in U and $f \in SCIVP$, there is a Cantor set $C \subset (x,y)$ such that $f(C) \subset K_0$, $f \restriction C$ is continuous, and $g \restriction C = f \restriction C$, which implies $g \in SCIVP$. Therefore $g \in SCIVP \setminus Ext$.

Both cases show that Ext is a boundary set in $SCIVP$. □

The next theorem follows immediately from a result in [4] that both $SCIVP$ and $CIVP \setminus SCIVP$ are dense in $\overline{CIVP}$.

Theorem 5. *$SCIVP$ is not porous in $CIVP$, but $SCIVP$ is a boundary set in $CIVP$.*

Theorem 6. *$CIVP$ is porous in $WCIVP$.*

Proof. Let $f \in \overline{CIVP}$.

Case 1: Some x_0 is a point of continuity of f.

The same function h and argument given for Case 1 of Theorem 1 can be used here to show that $CIVP$ is porous at f because $h \in WCIVP \setminus PR \subset WCIVP \setminus CIVP$.

Case 2: f has only points of discontinuity.

Since $\overline{CIVP} = WCIVP \cap \overline{D}$ [2], $f \in \overline{D}$ and so by [12] there is an open rectangle $U = (a-\delta, b+\delta) \times (c,d) \subset \overline{f}$ where $a < b$ and $\delta > 0$. If $0 < r \leq 1$, $r \leq d-c$, and $V = [a,b] \times \left(\frac{c+d}{2} - \frac{r}{4}, \frac{c+d}{2} + \frac{r}{4}\right] \subset U$, then $g : \mathbb{R} \to \mathbb{R}$ is obtained from the graph of f by shifting all the points of $f \cap V$ vertically up to the horizontal segment $H = [a,b] \times \left\{\frac{c+d}{2} + \frac{3r}{8}\right\}$ and letting $g = f$ elsewhere. H lies in U but above V. Then $g \in WCIVP \setminus CIVP$, $d(f,g) \leq \frac{5r}{8}$,and $B\left(g, \frac{r}{5}\right) \subset B\left(f, r\right) \setminus CIVP$. It follows that $CIVP$ is porous at f. □

Theorem 7. *If $f \in \overline{WCIVP}$ and the graph of f is dense in $\mathbb{R}^2$,then $f \in \overline{CIVP}$.*

Proof. We show $f \in WCIVP$ and $f \in \overline{D}$ and then use the fact that $\overline{CIVP} = WCIVP \cap \overline{D}$ according to [2]. Since $f \in \overline{WCIVP}$, f is the uniform limit of a sequence of functions $f_n \in WCIVP$. Suppose $x < y$ and we may suppose $f(x) < f(y)$. Because f is dense in $\mathbb{R}^2$, if $w \in (f(x), f(y))$ and V is an open neighborhood of w in $(f(x), f(y))$, there exist numbers $x_1 < y_1$ in (x,y) such that $w \in (f(x_1), f(y_1)) \subset V$. For $\varepsilon = \frac{1}{2} \min \{f(x_1) - f(x), f(y) - f(y_1)\}$, there exists a positive integer n such that for all $z \in \mathbb{R}, |f(z) - f_n(z)| < \varepsilon$. There is a Cantor set $C_1 \subset (x_1, y_1)$ such that $f_n(C_1) \subset (f_n(x_1), f_n(y_1))$. Then $C_1 \subset (x,y)$ and $f(C_1) \subset (f(x), f(y))$. Therefore $f \in WCIVP$. There is a Cantor set $C_2 \subset (x_1, y_1)$ such that $f(C_2) \subset (f(x_1), f(y_1))$. Therefore since $\mathfrak{c}$-many points of $[x,y]$ map into V, $f \in \mathcal{U} = \overline{D}$. □

Remark 1. If the graph of $f : \mathbb{R} \to \mathbb{R}$ is dense in $\mathbb{R}^2$, then $f \in WCIVP$ if and only if $f \in PR$.

Theorem 8. *$CIVP$ is porous in PR.*

PROOF. Suppose $f \in \overline{CIVP}$.

Case 1: There is a point of continuity x_0 of f.

The characteristic function for the set of nonendpoints of the Cantor ternary set, which is the same function $h \in PR \setminus D$ given for Case l of Theorem 2, can be used again here because $h \notin CIVP$. If $0 < r \leq 1$ and $g : \mathbb{R} \to \mathbb{R}$ is the function obtained from f by gluing a copy of h at x_0, then $g \in PR \setminus CIVP$, $d\left(f, g\right) < \frac{r}{2}$,and $B\left(g, \frac{r}{5}\right) \subset B\left(f, r\right) \setminus CIVP$.

Case 2: f is discontinuous everywhere.

Let $U = (a, b) \times (c, d) \subset \overline{f}$. We can use the same construction given in Theorem 2 to obtain a function $g \in PR \setminus CIVP$ such that $d\left(f, g\right) \leq \frac{r}{2}$. Namely, if $0 < r \leq 1$, $r \leq d - c, V = (a, b) \times \left(\frac{c+d}{2} - \frac{r}{4}, \frac{c+d}{2} + \frac{r}{4}\right) \subset U$, and $f\left(a\right), f\left(b\right) \in \overline{U} \setminus \overline{V}$, then g is obtained from the graph of f by shifting all points of $f \cap V$ vertically up to the top horizontal side H of V and letting $g = f$ elsewhere. This time, in the space PR, $B\left(g, \frac{r}{8}\right) \subset B\left(f, r\right) \setminus CIVP$ because if $\phi \in B\left(g, \frac{r}{8}\right)$, then there exist points $x < y$ in (a, b) such that $\phi\left(x\right) < \frac{c+d}{2} - \frac{r}{8} < \frac{c+d}{2} + \frac{r}{8} < \phi\left(y\right)$. If K is a Cantor set in $\left(\frac{c+d}{2} - \frac{r}{8}, \frac{c+d}{2} + \frac{r}{8}\right)$, then there is no Cantor set $C \subset (x, y)$ such that $\phi\left(C\right) \subset K$ since $(a, b) \cap \phi^{-1}\left(\frac{c+d}{2} - \frac{r}{8}, \frac{c+d}{2} + \frac{r}{8}\right) = \emptyset$.

Both cases show $CIVP$ is porous in PR. $\square$

Theorem 9. *(a) PC is porous in $Z\omega$ and (b) PR is porous in Zc.*

PROOF. For part (a), let $f \in \overline{PC} = PC$ [1], and for part (b), let $f \in \overline{PR} = PR$ [1].

Case 1: f has some point of continuity x_0.

The function $h : [0, 1] \to [0, 1]$ defined by

$$h(x) = \begin{cases} \chi_{C^o}(x) & \text{if } x \in [0,1] \setminus \left\{\frac{1}{4}\right\} \\ \frac{1}{2} & \text{if } x = \frac{1}{4}, \end{cases}$$

where C^0 is the set of nonendpoints of the Cantor ternary set, belongs to $Zc \setminus PC \subset (Z\omega \setminus PC) \cap (Zc \setminus PR)$ [6]. For each $0 < r \leq 1$, the function

$g:\mathbb{R}\to\mathbb{R}$ obtained from f by gluing a copy of h at x_0 belongs to $Z\omega\setminus PC$ for part (a) and belongs to $Zc\setminus PR$ for part (b) and $d(f,g)<\frac{r}{2}$. By construction, (a) PC will be porous at f in $Z\omega$ and (b) PR will be porous at f in Zc.

Case 2: f has only discontinuity points.

Since f might not belong to $\overline{D}$, we cannot conclude in general that the graph of f is somewhere dense in $\mathbb{R}^2$, but we can in the following first subcase.

Subcase (i): Suppose for every $x\in\mathbb{R}$, $\overline{f}\cap(\{x\}\times\mathbb{R})$ contains a nondegenerate interval.

For each integer k and positive integer n, let $Q_{nk}=\{x\in\mathbb{R}:$ some component of $\overline{f}\cap(\{x\}\times\mathbb{R})$ meets both $\mathbb{R}\times\left\{\frac{k}{n}\right\}$ and $\mathbb{R}\times\left\{\frac{k+1}{n}\right\}\}$. Each Q_{nk} is closed and $\mathbb{R}=\cup_{n,k}Q_{nk}$. By the Baire Category Theorem, some Q_{nk} is somewhere dense in $\mathbb{R}$ and therefore contains an interval (a,b). This shows $U=(a,b)\times\left(\frac{k}{n},\frac{k+1}{n}\right)\subset\overline{f}$. Let $0<r\leq\frac{1}{n}$. Choose a point $(c,f(c))\in U$ such that the distance from $f(c)$ to each of $\frac{k}{n}$ and $\frac{k+1}{n}$ is $>\frac{r}{4}$, and let $V=(a,b)\times\left(f(c)-\frac{r}{4},f(c)+\frac{r}{4}\right)$. A function $g:\mathbb{R}\to\mathbb{R}$ for which $d(f,g)\leq\frac{r}{2}$ can be obtained from f by moving all points of $f\cap V$ except for $(c,f(c))$ vertically to $(a,b)\times\left\{f(c)+\frac{r}{4}\right\}$. Then for part (a), $g\in Z\omega\setminus PC$ (because f is dense in U), $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PC$, and so PC is porous at f in $Z\omega$. For part (b), $g\in Zc\setminus PC$, $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PR$, and so PR is porous at f in Zc.

Subcase (ii): Suppose there exists an $x_0\in\mathbb{R}$ such that the closed set $\overline{f}\cap(\{x_0\}\times\mathbb{R})$ is totally disconnected.

Let $0<r\leq 1$, and choose a point $(x_0,a)\notin\overline{f}\cap(\{x_0\}\times\mathbb{R})$ such that $\frac{r}{4}<a-f(x_0)<\frac{r}{2}$. For part (a), since $f\in PC$, the graph of f is bilaterally dense in itself, and so there exist disjoint countable dense subsets A_1 and A_2 of the graph of f. Then A_1 and A_2 are each bilaterally dense in itself and bilaterally dense in the graph of f. For part (b), since $f\in PR$, there exist disjoint bilaterally c-dense subsets A_1 and A_2 of the graph of f. For both parts (a) and (b), define $g:\mathbb{R}\to\mathbb{R}$ for which $d(f,g)\leq\frac{3r}{4}$ by

$$g(x)=\begin{cases} a+\dfrac{r}{4} & \text{if } (x,f(x))\in A_1\cap\left(\mathbb{R}\times\left(a-\dfrac{r}{2},a+\dfrac{r}{2}\right)\right) \text{ and } x\neq x_0\\ a-\dfrac{r}{4} & \text{if } (x,f(x))\in (f\setminus A_1)\cap\left(\mathbb{R}\times\left(a-\dfrac{r}{4},a+\dfrac{r}{4}\right)\right)\\ a & \text{if } x=x_0\\ f(x) & \text{otherwise.}\end{cases}$$

Then in part (a), $g\in Z\omega\setminus PC$ because each A_i is dense in the graph of f and (x_0,a) is an isolated point of the graph of g. Since $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PC$, PC is porous at f in $Z\omega$. In part (b), $g\in Zc\setminus PR$ and since $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PR$, PR is porous at f in Zc. $\square$

Theorem 10. *PR is porous in PC.*

PROOF. Suppose $f\in\overline{PR}=PR$ [1].

Case 1: f is continuous at some x_0.

If $h:[0,1]\to[0,1]$ is the characteristic function of the irrationals in $[0,1]$, then $h\in PC\setminus PR$ [6]. For every $0<r\leq 1$, glue a copy of h at x_0 in order to obtain from f a function $g\in PC\setminus PR$ such that $d(f,g)<\frac{r}{2}$ and to see that PR is porous at f.

Case 2: f is continuous nowhere.

Subcase (i): For every $x\in\mathbb{R}$, $\overline{f}\cap(\{x\}\times\mathbb{R})$ contains a nondegenerate interval.

Since the graph of f is somewhere dense in $\mathbb{R}^2$, there is an open rectangle $U=(a,b)\times(c,d)\subset\overline{f}$. Let $0<r\leq 1$, $r\leq d-c$, and pick a point $(p,f(p))\in U$ such that the rectangle $V=(a,b)\times\left(f(p)-\frac{r}{4},f(p)+\frac{r}{4}\right)\subset(a,b)\times\left[f(p)-\frac{r}{4},f(p)+\frac{r}{4}\right]\subset U$. Let A be a countable dense subset of $f\cap V$. A function $g:\mathbb{R}\to\mathbb{R}$ can be obtained from f by moving all points of $f\cap V$ except for $\{(p,f(p))\}\cup A$ vertically to $(a,b)\times\left\{f(p)+\frac{r}{4}\right\}$. Then $g\in PC\setminus PR$ (since A is dense in $f\cap V$ and f is dense in U), $d(f,g)\leq\frac{r}{2}$, and $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PR$. So PR is porous at f.

Subcase (ii): There is an $x_0\in\mathbb{R}$ such that $\overline{f}\cap(\{x_0\}\times\mathbb{R})$ is totally disconnected.

For each $0<r\leq 1$, there is a point $(x_0,a)\notin\overline{f}$ such that $\frac{r}{4}<a-f(x_0)<\frac{r}{2}$. Let B_1 and B_2 be disjoint countable subsets of f. Then B_1 and B_2 are each bilaterally dense in itself and bilaterally dense in the graph of f. Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases} a+\frac{r}{4} & \text{if } (x,f(x))\in(f\setminus B_1)\cap\left(\mathbb{R}\times\left(a-\frac{r}{4},a+\frac{r}{4}\right)\right)\\ a & \text{if } x=x_0 \text{ or}\\ & (x,f(x))\in B_2\cap\left[\mathbb{R}\times\left(\left(a-\frac{r}{2},a-\frac{r}{4}\right)\cup\left(a+\frac{r}{4},a+\frac{r}{2}\right)\right)\right]\\ f(x) & \text{elsewhere.}\end{cases}$$

Then $g\in PC\setminus PR$, $d(f,g)\le\frac{r}{2}$, and $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PR$. Therefore PR is porous at f. □

Theorem 11. *PR is porous in PB.*

Proof. Let $f\in\overline{PR}=PR$ and let $x_0\in\mathbb{R}$. Since f has a perfect road at x_0, there exists a perfect set P_0 having x_0 as a bilateral limit point such that $f\restriction P_0$ is continuous at x_0. Let $0<r\le 1$ and $S=\mathbb{R}\times\left(f(x_0)-\frac{r}{4},f(x_0)+\frac{r}{4}\right)$. Define $g(x)=f(x_0)+\frac{r}{4}$ whenever

(1) $(x,f(x))\in S$ and $x\ne x_0$ or

(2) $(x,f(x))\in\mathbb{R}\times\left\{f(x_0)-\frac{r}{4}\right\}$ and there exists a perfect road P at x such that either of $f\restriction(P\cap(-\infty,x))$ or $f\restriction(P\cap(x,\infty))$ is contained in $\overline{S}$, and $(x,f(x))$ is not a limit point of $f\setminus\overline{S}$.

Both $P\cap(-\infty,x]$ and $P\cap[x,\infty)$ are perfect sets containing x. Define $g(x)=f(x)$ for all other x. Suppose I and J are open intervals with $I\cap g^{-1}(J)$ uncountable. We may suppose J contains $f(x_0)-\frac{r}{4}$ or $f(x_0)+\frac{r}{4}$. Let $w\in I\cap g^{-1}(J)$. Therefore $g(w)\in J$. If $g(w)=f(w)$, then there exists a perfect set P_1 containing w such that $g\restriction P_1$ is continuous at w, and so $I\cap g^{-1}(J)$ contains a perfect subset of P_1. If $g(w)\ne f(w)$, then $g(w)=f(x_0)+\frac{r}{4}$ and either by (1), $(w,f(w))\in S$ (in which case f has a perfect road P_2 at w) or by (2), $f(w)=f(x_0)-\frac{r}{4}$ and there exists a perfect set P_3 containing w such that $f\restriction(P_3\setminus\{w\})\subset\overline{S}$. So $I\cap g^{-1}(J)$ contains a perfect subset of either P_2 or P_3. Therefore $g\in PB\setminus PR$ and $d(f,g)\le\frac{r}{2}$. In PB, the open ball $B\left(g,\frac{r}{8}\right)\subset B(f,r)\setminus PR$. Since $p(f,PR)\ge\frac{1}{8}$, PR is porous at f. □

Theorem 12. $\overline{PB}=PB$.

PROOF. Suppose $f \in \overline{PB}$ and I and J are open intervals such that $I \cap f^{-1}(J)$ is uncountable. Then f is the uniform limit of a sequence of functions $f_n \in PB$. There exist a positive number $\varepsilon \leq 1$, a positive integer N, and an interval $(c,d) \subset (c-\varepsilon, d+\varepsilon) \subset J$ such that $I \cap f^{-1}(c,d)$ is uncountable and $|f(x) - f_n(x)| < \dfrac{\varepsilon}{2}$ for all $n \geq N$ and for all $x \in \mathbb{R}$. Therefore for all $n \geq N$, $I \cap f^{-1}(c,d) \subset I \cap f_n^{-1}\left(c - \dfrac{\varepsilon}{2}, d + \dfrac{\varepsilon}{2}\right)$, which since uncountable contains a perfect set P_n. Because $f(P_n) \subset f\left(I \cap f_n^{-1}\left(c - \dfrac{\varepsilon}{2}, d + \dfrac{\varepsilon}{2}\right)\right) \subset J$, $P_n \subset I \cap f^{-1}(J)$. Therefore $f \in PB$. $\square$

The following result is different from all the others.

Theorem 13. *Zc is not a boundary set in $Z\omega$.*

PROOF. We must show that Zc contains an open ball of $Z\omega$. Let $\mathbb{Q}$ denote the set of rational numbers and $\{F_q : q \in \mathbb{Q}\}$ denote a collection of pairwise disjoint $\mathfrak{c}$-dense subsets of $\mathbb{R}$. Like in [9], define $f = \sum_{q\in\mathbb{Q}} q\chi_{F_q}$. Then $f \in Zc$. Let $0 < \varepsilon \leq 1$ and suppose $g \in Z\omega$ with $d(f,g) < \varepsilon$. Let $a \in \mathbb{R}$ and choose a rational number $q > a + \varepsilon$. If $x \in F_q$, then $f(x) = q$,and since $|q - g(x)| = |f(x) - g(x)| \leq d(f,g) < \varepsilon \leq 1$, then $g(x) > a$. Therefore $F_q \subset \{x : g(x) > a\}$, which implies $\{x : g(x) > a\}$ is $\mathfrak{c}$-dense in $\mathbb{R}$ and hence bilaterally $\mathfrak{c}$-dense in itself. Similarly, $\{x : g(x) < a\}$ is bilaterally $\mathfrak{c}$-dense in itself. Then $g \in Zc$. This shows $B(f,\varepsilon) \subset Zc$. So Zc is not a boundary set in $Z\omega$ (and consequently not porous in $Z\omega$). $\square$

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