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STRONG AND WEAK VITALI PROPERTIES

Abstract

Let X be a metric space and let μ be a Borel measure on X. We say that μ satisfies the strong Vitali property if for any Borel subset E of X with $\mu(E) < \infty$ and for any fine cover $\mathcal{V}$ of E, we may extract a countable disjoint subcollection $\pi = \{B_i\}$ from $\mathcal{V}$ such that $\mu(E \setminus \cup B_i) = 0$. If we require π to satisfy the condition that if $B(x_i, r_i),\ B(x_j, r_j) \in \pi$ with $i \neq j$, then $x_i \notin B(x_j, r_j)$, then μ is said to satisfy the weak Vitali property. Besicovitch showed that every finite Borel measure in $\mathbb{R}^n$ must satisfy the strong Vitali property. It is also true for certain other metric spaces. In a general metric space it is no longer pertinent to ask if all Borel measures possess a certain property. To this end we construct a metric space Ω and identify two subsets $A, B \subseteq \Omega$ such that for any Borel probability measure μ on Ω,

1. if $\mu(A) = 1$, then μ must satisfy the strong Vitali property;
2. if $\mu(B) = 1$, then μ must satisfy the weak Vitali property but not necessarily the strong Vitali property;

We introduce the notion of a centralized Vitali property and give an example of a measure for which this property fails.

Introduction

Let $E \subseteq \mathbb{R}^n$. A collection β of closed balls is called a centered Vitali cover of E if for every $x \in E$ and every $\epsilon > 0$, there is a ball $B(x, r) \in \beta$ with $0 < r \leq \epsilon$. A Borel measure μ is said to satisfy the Vitali covering theorem if for every Borel subset E of $\mathbb{R}^n$ with $\mu(E) < \infty$, and any centered Vitali cover $\mathcal{V}$ of E, we can extract a countable disjoint subcollection $\{B_i\}$ from $\mathcal{V}$ such that

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$\mu(E \setminus \cup B_i) = 0$. One of the fundamental properties of $\mathbb{R}^n$ is that every finite Borel measure on $\mathbb{R}^n$ satisfies the Vitali covering theorem. This was shown by Besicovitch [1]. Larman [11] extended this to "finite dimensional" compact metric spaces. The property is true for compact ultrametric spaces, finite-dimensional Banach spaces and sufficiently smooth Riemannian manifolds. Davies [5] constructed a metric space Ω and a measure μ on Ω that fails to satisfy the Vitali covering theorem. A more general covering theory was developed by Morse [13].

The proof of the fact that every Borel measure in $\mathbb{R}^n$ satisfies the Vitali covering theorem depends essentially on the following geometric property of $\mathbb{R}^n$. If we let $c = 16^n + 1$, then for any centered Vitali cover β of a bounded set $E \subseteq \mathbb{R}^n$ we can extract a countable subcover $\beta' = \{B_i\}$ such that for any k, the ball B_k intersects at most $c - 1$ of the previous $k - 1$ balls. In fact the same proof shows that in a general metric space a similar geometric condition is sufficient to ensure that every Borel measure satisfies the Vitali covering theorem. Are there such metric spaces? We show how Davies' space Ω may be suitably modified to construct such a metric space X. (We obtain an entire class of such examples). It turns out that we need to restrict our attention to a subset of X. If we change our subset then this aforementioned geometric condition could fail. But the following weaker condition might hold. There is a constant K such that for any centered Vitali cover β of a bounded set $E \subseteq X$ we can extract a countable subcover $\beta' = \{B_i\}$ such that for any k, the ball B_k contains the centers of at most $K - 1$ of the previous $k - 1$ balls. Note that now there is no longer a limit to the number of balls that might intersect each other. In fact this number could grow with k. Of course in a general metric space balls do not have unique centers and so some natural analogous definitions are required.

This suggests that in a general metric space, the geometry may very well differ from place to place and so it is no longer pertinent to ask if a certain Vitali type result holds for all measures. Instead it becomes a property of the support of the measure, which seems like an obvious statement anyway. This paper is merely a reinforcement of this simple observation.

1 Definitions

Let (X, ρ) be a metric space. A *constituent* in a metric space is an ordered pair (x, r) where $x \in X$ and $r > 0$. The constituent (x, r) may be thought of as the ball $B(x, r)$. In a general metric space it is possible for two balls $B(x, r)$ and $B(y, s)$ to be equal as point sets even though $x \neq y$ and/or $r \neq s$. The *centralizer* of a constituent (y, s) is defined to be the set $C_{(y,s)} = \{x \in X :$

there is $r > 0$ with $B(x, r) = B(y, s)\}$.

A collection of constituents is a *packing* iff $B(x, r) \cap B(y, s) = \emptyset$ for all $(x, r) \neq (y, s)$. For $E \subseteq X$, a *fine cover* of E is a collection of constituents centered in E such that for every $x \in E$ and every $\epsilon > 0$, there is a constituent (x, r) in the collection with $r < \epsilon$. A Borel measure μ on X is said to satisfy the *strong Vitali property* if and only if for every Borel subset E of X with $\mu(E) < \infty$, and any fine cover $\mathcal{V}$ of E, we can extract a countable packing $\{B_i\}$ from $\mathcal{V}$ such that $\mu(E \setminus \cup B_i) = 0$.

A collection π of constituents centered in E is a *weak-packing* if and only if for every (x, r), $(y, s) \in \pi$ with $(x, r) \neq (y, s)$, we must have $x \notin B(y, s)$. Edgar [7] defines packings as (b)-packings and weak-packings as (a)-packings. A Borel measure μ on X is said to satisfy the *weak Vitali property* if and only if for every Borel subset E of X with $\mu(E) < \infty$, and any fine cover $\mathcal{V}$ of E, we can extract a countable weak-packing $\{B_i\}$ from $\mathcal{V}$ such that $\mu(E \setminus \cup B_i) = 0$. Since every packing is also a weak-packing, it is clear that any measure that has the strong Vitali property also has the weak Vitali property.

Definition 1.1. A metric space X has the *Besicovitch packing property* if there exists an integer K such that for any $E \subseteq X$ and any fine cover β of E, we can extract a countable subcover $\beta' = \{B_i\}$ such that for any n, the ball $B(x_n, r_n)$ intersects at most $K - 1$ of the previous $n - 1$ balls.

The proof of the following Theorem is the same as in [7, Cor. 1.3.12, Thm. 1.3.13].

Theorem 1.2. *Let X be a metric space that has the Besicovitch packing property. Then every finite Borel measure on X has the strong Vitali property.*

Definition 1.3. A metric space X has the *Besicovitch weak-packing property* if there exists an integer K such that for any $E \subseteq X$ and any fine cover β of E, we can extract a countable subcover $\beta' = \{B_i\}$ such that

1. for any n, the ball B_n intersects the centralizers of at most $K - 1$ of the previous $n - 1$ balls;

2. if $(x, r), (y, s) \in \beta'$, then $B(x, r) \cap C_{(y,s)} = \emptyset \iff B(y, s) \cap C_{(x,r)} = \emptyset$.

The next theorem then follows.

Theorem 1.4 ([3])**.** *Let X be a metric space that has the Besicovitch weak-packing property. Then every finite Borel measure on X has the weak Vitali property.*

2 Construction of the Metric Space

Our metric space will be a modification of Davies' construction [5]. We will essentially follow Edgar's terminology [6] where he describes Davies' construction in detail.

Let K_1, K_2 be two fixed natural numbers. For a given integer N, let $G(N)$ be a set of vertices labelled as follows. There are N vertices $(i, 0, 0), 1 \leq i \leq N$ which we call *central* vertices and $(K_1 + K_2)N^2$ additional vertices that are labelled $(i, j, k), 1 \leq i, j \leq N, 1 \leq k \leq K_1 + K_2$. Vertices $(i, j, k), 1 \leq k \leq K_1$ are called *peripheral* vertices; vertices $(i, j, k), K_1 + 1 \leq k \leq K_1 + K_2$ are called *outer* vertices.

For a fixed i, j, each outer vertex (i, j, k) is joined to every peripheral vertex (i, j, k'), but not to each other. The peripheral vertices (i, j, k') are joined to their central neighbor $(i, 0, 0)$, but not to each other. A central vertex $(i, 0, 0)$ is joined to all the peripheral vertices $(i, j, k),\ 1 \leq j \leq N,\ 1 \leq k \leq K_1$ and all other central vertices $(i', 0, 0)$.

Given two vertices u, v we will write $u \sim v$ if $u = v$ or u is joined to v by an edge. We will write $u \not\sim v$ if not $u \sim v$. Let (N_n) be an increasing sequence of natural numbers. Our metric space is defined as $\Omega = \prod_{n=1}^{\infty} G(N_n)$. Let $u = (u_1, u_2, \ldots) \in \Omega$ where $u_i \in G(N_i)$. We define the metric ρ as follows. First $\rho(u, u) = 0$ for every $u \in \Omega$. If $u, v \in \Omega$ and $u \neq v$, let n be the least integer such that $u_n \neq v_n$. If $u_n \sim v_n$ in $G(N_n)$, then we let $\rho(u, v) = (1/2)^n$ and if $u_n \not\sim v_n$ in $G(N_n)$, then we let $\rho(u, v) = (1/2)^{n-1}$.

Given a finite sequence $w_1 \in G(N_1), w_2 \in G(N_2), \ldots, w_n \in G(N_n)$ define a cylinder $\Omega(w_1, w_2, \ldots, w_n) = \{u \in \Omega : u_1 = w_1, u_2 = w_2, \ldots, u_n = w_n\}$. The diameter of $\Omega(w_1, w_2, \ldots, w_n)$ is $(1/2)^n$. A cylinder will be called central, peripheral or outer according as the last coordinate is central, peripheral or outer. The open balls in Ω are as follows. Let $u \in \Omega$ and r be given with $0 < r < 1$. Let n be such that $(1/2)^n < r \leq (1/2)^{n-1}$. Then $B(u, r) = \{v : u_1 = v_1, u_2 = v_2, \ldots, u_{n-1} = v_{n-1}, u_n \sim v_n\}$. It follows that if u_n is central, then $B(u, r)$ is the union of N_n central and $K_1 N_n$ peripheral cylinders. If u_n is peripheral, then $B(u, r)$ is the union of one central, one peripheral and K_2 outer cylinders. If u_n is outer, then $B(u, r)$ is the union of K_1 peripheral cylinders and one outer cylinder.

If we let $B = B(u, r)$, then $C_{(u,r)} = \Omega(u_1, u_2, \ldots, u_n)$ for n with $2^{-n} < r \leq 2^{-n+1}$. Suppose two balls $B(u, r)$ and $B(v, s)$ are of the same size in the sense that $2^{-n} < r, s \leq 2^{-n+1}$. First suppose that u_n, v_n are peripheral or outer. If u_n and v_n correspond to different central neighbors, then $B(u, r)$ and $B(v, s)$ are disjoint. Suppose they correspond to the same central neighbor $(i, 0, 0)$. If

$u_n = (i, j, k)$ and $v_n = (i, j', k')$ where $j \neq j'$, then $B(u, r), B(v, s)$ may or may not be disjoint but their centralizers will be. If $u_n = (i, j, k)$ and $v_n = (i, j, k')$ where $k \neq k'$, then we have the following. If u_n and v_n are both peripheral or both outer, then $B(u, r)$ and $B(v, s)$ are not disjoint but their centralizers are; if u_n is peripheral and v_n is outer, then $B(u, r)$ contains the centralizer of $B(v, s)$ and vice versa. Similar arguments can be used to study the other cases.

For $u = (u_1, u_2, \ldots) \in \Omega$ we say that u is a *central point* if u_k is a central vertex in $G(N_k)$ for each k. We say that u is an *eventually central point* if u_k is a central vertex for all but finitely many values of k. We define peripheral, outer, eventually peripheral and eventually outer points in the same way.

3 Measures and Vitali Properties

Let A denote the set of all eventually outer points and let B denote the set of all eventually peripheral or eventually outer points.

Lemma 3.1. *A satisfies the Besicovitch packing property with $K = K_2$.*

Proof. Let β be a fine cover of A. If $u = (u_1, u_2, \ldots) \in A$, then there exists a smallest integer $n(u)$ such that u_k is an outer vertex for all $k \geq n(u)$. For each u we may discard the balls $B(u, r)$ for which $r > 2^{-n(u)}$. We decompose this new collection into countably many families, each family consisting of balls of the "same size". In other words, let β_n denote the collection of all balls $B(u, r)$ with $(u, r) \in \beta$ such that $2^{-n} < r \leq 2^{-n+1}$ $(r > 2^{-n(u)})$ and such that for all $k = 1, \ldots, n-1$ and for all balls $B \in \beta_k$ we have $u \notin B$. Then for all n and for all $B(u, r) \in \beta_n$ with $u = (u_1, u_2, \ldots)$, u_n is an outer vertex in $G(N_n)$. Further, each family β_n contains at most $C(K_1, K_2, N_n) < \infty$ distinct balls. The constant $C(K_1, K_2, N_n)$ may be estimated by noting that in $G(N_n)$ one has to take an outer vertex and in $G(N_k)$ for $k = 1, \ldots, n-1$, one can take anything.

Now if $B_1, \ldots, B_k \in \cup_{k=1}^{\infty}\{B | B \in \beta_n\}$ (enumerated in a suitable way), then there are at most $K_2 - 1$ balls B_j $(j \neq k)$ with $B_j \cap B_k \neq \emptyset$. In fact, if $B_j \cap B_k \neq \emptyset$ for some $j = 1, \ldots, k-1$, then there is l such that $B_k, B_j \in \beta_l$, giving the claim. □

A similar proof gives us the following.

Lemma 3.2. *B satisfies the Besicovitch weak-packing property with $K =$* $\max\,(K_1, K_2)$.

Proposition 3.3. *Let $A, B \subseteq \Omega$ as above. Let μ be any probability measure on Ω.*

(i) *If $\mu(A) = 1$, then μ must satisfy the strong Vitali property.*

(ii) *If $\mu(B) = 1$, then μ must satisfy the weak Vitali property but not necessarily the strong Vitali property.*

PROOF. (i) follows immediately from Lemma 3.1 and Theorem 1.2.

The first part of (ii) follows from Lemma 3.2 and Theorem 1.4. For the second part we take μ to be the "uniform measure" defined by Edgar [6]. We briefly describe the construction of μ. We define an additive set function for each cylinder. This may then be extended to a Borel probability measure in the usual way. First we take N_n to be a sequence of integers ≥ 2 such that

$$\prod_{n=1}^{\infty} \frac{N_n - 1}{N_n + 1} \geq \frac{1}{3}.$$

Next we let $\gamma_0 = 1,\ \gamma_n = \dfrac{\gamma_{n-1}}{N_n(N_n+1)}$. In each finite graph $G(N_n)$ we delete the vertices $(i, j, k),\ 2 \leq k \leq K_1 + K_2$ for each i, j. This is equivalent to assigning measure 0 to all the corresponding cylinders. This leaves us with N_n central cylinders and N_n^2 peripheral cylinders. Let $\mu(\Omega) = \gamma_0 = 1$ and $\mu(\Omega(u_1, u_2, \ldots u_n)) = \gamma_n$. It turns out (see [6]) that $\mu(B) = 1$ and μ fails to satisfy the strong Vitali property. □

For $u \in \Omega$ let $u = (u_1, u_2, u_3, \ldots)$ and let $\mu_n = \max\{\mu(\Omega(u_1, u_2, \ldots u_n)) : u \in \Omega\}$. Let $\gamma_0 = 1$ and $\gamma_n = \dfrac{\gamma_{n-1}}{N_n(N_n+1)}$.

Proposition 3.4. *Let C denote the set of eventually central points. Let μ be a finite measure on Ω such that $\mu(C) = \alpha > 0$. If the series $\displaystyle\sum_{n=1}^{\infty} \frac{\mu_n}{(N_n+1)\gamma_n}$ converges, then μ fails the weak Vitali property.*

PROOF. Let $m \in \mathbb{N}$ and let $u \in C$. There exists $n(u) \in \mathbb{N}$ such that if $u = (u_1, u_2, u_3, \ldots)$, then u_k is central for all $k \geq n(u)$. Define a gauge on C by letting $\Delta(u) = \min\{2^{-n(u)}, m\}$. Let $\mathcal{V}$ be a Δ-fine cover of C.

Let us recall the geometry of the central balls. Suppose two balls $B(u, r)$ and $B(v, s)$ are of the same size in the sense that $2^{-n} < r, s \leq 2^{-n+1}$ and u_n and v_n are central in $G(N_n)$. If $u_n \sim v_n$, then $B(u, r) \cap C_{(v,s)} \neq \emptyset$ and if $u_n \not\sim v_n$, then $B(u, r) \cap B(v, s) = \emptyset$. Let π be a countable weak packing chosen from $\mathcal{V}$. Inside a given cylinder $\Omega(u_1, u_2, \ldots u_{n-1})$, among the balls $B(u, r)$ with $2^{-n} < r \leq 2^{-n+1}$, the weak packing contains at most one central ball.

A central ball is the union of N_n central and N_n peripheral cylinders, so that $\mu(B(u,r)) \leq 2N_n\mu_n$. Thus

$$\sum_{(u,r)\in\pi} \mu(B(u,r)) \leq \sum_{n=m}^{\infty} N_1(N_1+1)N_2(N_2+1)\dots N_{n-1}(N_{n-1}+1)\ 2N_n\mu_n$$
$$\leq \sum_{n=m}^{\infty} \frac{2\mu_n}{(N_n+1)\gamma_n.}$$

Since the sum converges, we may choose m large enough so that

$$\sum_{n=m}^{\infty} \frac{2\mu_n}{(N_n+1)\gamma_n} < \alpha. \qquad \square$$

It is not clear though if such a μ exists. For instance if μ is the uniform measure on Ω (as in the proof Proposition 3.3 (ii)), then $\mu_n = \gamma_n$ and so the series in Proposition 3.4 converges; but $\mu(C) = 0$. On the other hand we may construct a measure μ with $\mu(C) = 1$ as follows. We define the measure on the cylinders and then extend it to the entire space. Let $\gamma_0 = 1$, $\mu(\Omega(u_1, u_2, \dots u_n)) = \gamma_n$ where

$$\gamma_n = \begin{cases} \frac{1}{N_n+1}\gamma_{n-1} & \text{if } u_n \text{ is central} \\ \frac{1}{N_n^2(N_n+1)}\gamma_{n-1} & \text{if } u_n \text{ is peripheral} \end{cases}$$

So in each graph $G(N_n)$ we keep the central and peripheral vertices and delete all others. This is equivalent to assigning measure 0 to all the corresponding cylinders. This set function is clearly additive on the semi-ring of cylinders and may be extended to a measure μ.

The proof that $\mu(C) = 1$ is exactly the same as the proof of Proposition 4.2 in [6]. Let $C_m = \{u = (u_1, u_2, \dots) \in \Omega : u_k \text{ is central for all } k \geq m\}$. In $G(N_k)$ there are N_k central vertices; so

$$\mu(C_m) = \prod_{k=m}^{\infty} \frac{N_k}{N_k+1}.$$

By assumption, the infinite product $\prod \frac{N_k-1}{N_k+1}$ converges so that $\prod \frac{N_k}{N_k+1}$ also converges. Therefore the tail products must approach 1 which means that $\mu(C_m) \to 1$ as $m \to \infty$. But C_m increases to C and so $\mu(C) = 1$. But in this case the series in Proposition 3.4 diverges.

4 Centralized Vitali Property

Definition 4.1. Let X be a metric space. A Borel measure μ on X is said to satisfy the *centralized Vitali property* if and only if for every Borel subset E of X with $\mu(E) < \infty$, and any fine cover $\mathcal{V}$ of E, we can extract a countable weak-packing π from $\mathcal{V}$ such that

$$\mu(E \setminus \bigcup_{(x,r)\in\pi} C_{(x,r)}) = 0.$$

This property is clearly false in metric spaces like $\mathbb{R}^n$ where the center consists of a single point. At the other extreme, in an ultrametric space, every weak packing is a packing and the centralizer of a constituent is the constituent itself. The notions of strong, weak and centralized Vitali properties therefore coincide in this case. Davies' space and its modifications suggest that this notion is nontrivial. We therefore give an example of a measure that does not satisfy the centralized Vitali property.

Let C denote the set of central points in Davies' space Ω. We construct a measure μ with $\mu(C) = 1$ as follows: In $G(N_n)$ there are exactly N_n central vertices. Let $\gamma_0 = 1,\ \gamma_n = \dfrac{\gamma_{n-1}}{N_n}$. Let $\mu(\Omega) = \gamma_0 = 1$. If u_k is central for every k then $\mu(\Omega(u_1, u_2, \ldots u_n)) = \gamma_n$; else $\mu(\Omega(u_1, u_2, \ldots u_n)) = 0$. This set function is clearly additive on the semi-ring of cylinders and may be extended to a measure μ. Clearly $\mu(C) = 1$. We will show that μ fails the centralized Vitali property. So fix $m \in \mathbb{N}$, and let $\epsilon = 2^{-m+1}$. Let $\mathcal{V}$ be an ϵ-fine cover of C and let π be a countable weak packing chosen from $\mathcal{V}$. Inside a given cylinder $\Omega(u_1, u_2, \ldots u_{n-1})$, among the balls $B(u, r)$ with $2^{-n} < r \le 2^{-n+1}$, the weak packing contains at most one central ball. Thus

$$\begin{aligned}\sum_{(x,r)\in\pi} \mu(C_{(x,r)}) &\le \sum_{n=m}^{\infty} N_1 N_2 \ldots N_{n-1}\ \gamma_n \\ &\le \sum_{n=m}^{\infty} \frac{1}{N_n},\end{aligned}$$

which is the tail of a convergent series and can therefore be made smaller than one for m large enough.

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