INROADS

Real Analysis Exchange
Vol. 27(2), 2001/2002, pp. 801–808

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SCRAMBLED SETS FOR TRANSITIVE MAPS

Abstract

We deal with two types of chaos: the well known chaos in the sense of Li and Yorke and ω-chaos which was introduced in [S. Li, *Trans. Amer. Math. Soc.* 339 (1993)]. In this paper we prove that every bitransitive map $f \in C(I, I)$ is conjugate to $g \in C(I, I)$, which satisfies the following conditions,

1. there is a c-dense ω-scrambled set for g,
2. there is an extremely LY-scrambled set for g with full Lebesgue measure,
3. every ω-scrambled set of g has zero Lebesgue measure.

1 Introduction

Let (X, d) be a compact metric space, by $C(X, X)$ we denote the set of all continuous maps $f : X \to X$, and by I the unit interval $[0, 1]$.

For any integer $n \geq 0$, let f^n denotes the n^{th} iteration of f. For any x in X, the sequence of iterations $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$, is the *trajectory* of x; and the set $\omega_f(x)$ of all limit points of the trajectory is the *ω-limit set* of x under f.

Let $A \subset X$. By $\sharp A$ we denote the cardinality of A, by $(A)'$ the set of all limit points of the set A, and by $\lambda(A)$ the Lebesgue measure of A. By an interval we mean a non-degenerate one.

The next definition has been introduced by Shihai Li in [8].

Key Words: LY-chaos, ω-chaos, scrambled sets, transitive maps
Mathematical Reviews subject classification: Primary 26A18, 37D45, 37E05; Secondary 54H20, 26A30
Received by the editors January 14, 2002
*The research was supported, in part, by the contract No. 201/01/P134 from the Grant Agency of Czech Republic, and the contract No. CEZ: J10/98:192400002 from the Czech Ministry of Education.

Definition 1.1. Let $f \in C(X,X)$ and $S \subset X$. We say that S is an ω-*scrambled set for* f if, for any $x, y \in S$ with $x \neq y$,

1. $\omega_f(x) \backslash \omega_f(y)$ is uncountable,
2. $\omega_f(x) \cap \omega_f(y)$ is non-empty,
3. $\omega_f(x)$ is not contained in the set of periodic points.

We say that f is ω-*chaotic*, if there exists an uncountable ω-scrambled set.

Remark. If $X = I$, then the third condition is superfluous (see, e.g., [8]).

Theorem 1.1. *([8], Theorem). For $f \in C(I,I)$ the following statements are equivalent,*

1. *f has a positive topological entropy,*
2. *f is ω-chaotic,*
3. *there is an uncountable ω-scrambled set S such that $\bigcap_{x \in S} \omega_f(x) \neq \emptyset$.*

Definition 1.2. Let $f \in C(X,X)$. A subset S of X containing no periodic point is called an *LY-scrambled set for* f if, for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0$,
2. $\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0$.

We say that f is *chaotic in the sense of Li and Yorke* (briefly, *LY-chaotic*), if there exists an uncountable LY-scrambled set.

In the case that $\limsup_{n \to \infty} d(f^n(x), f^n(y)) = \operatorname{diam}(X)$ we speak about *extremal* LY-scrambled set and *extremal* LY-chaotic map.

Remark. It is known that positive topological entropy implies LY-chaos, but not conversely, and hence ω-chaos implies LY-chaos, but not conversely.

The measure of LY-scrambled sets of $f \in C(I,I)$ was studied by many authors. In [10] there is given an example of a function whose scrambled set has full outer Lebesgue measure, maps in [7] and [11] have scrambled sets with positive Lebesgue measure, [5] and [9] give examples of function chaotic almost everywhere. Babilonová in [1] (resp. [2]) improved these results by showing that any bitransitive continouos map of the interval is conjugate to a map extremely LY-chaotic (resp. distributionally chaotic) almost everywhere.

It is natural to ask, what can be said about a "size" of ω-scrambled set. We prove, that every bitransitive $f \in C(I,I)$ is conjugate to $g \in C(I,I)$, which satisfies the following conditions:

1. there is a c-dense ω-scrambled set for g,
2. there is an extremely LY-scrambled set for g with full Lebesgue measure,
3. every ω-scrambled set of g has zero Lebesgue measure.

2 Main Result

Let us recall some known definitions. The map $f \in C(I, I)$ is called *(topologically) transitive* if for any intervals U, V in I, there exists a positive integer k, such that $f^k(U) \cap V \neq \emptyset$; f is called *bitransitive* if f^2 is transitive. Two maps f, $g \in C(I, I)$ are (*topologically*) *conjugate* (resp. *semiconjugate*) if there is a homeomorphism (resp. surjective map) $h \in C(I, I)$, such that $h \circ f = g \circ h$. We say that a subset A of I is *c-dense* in I if, for every interval $J \subset I$, the set $A \cap J$ has the continuum cardinality; i.e., $\sharp(A \cap J) = c$. Throughout this paper c-dense set means set, which is c-dense in I.

Lemma 2.1. *([5], Proposition 4.4 or [3], Proposition 44). Let $f \in C(I, I)$ be a bitransitive map, and $J, K \subset (0, 1)$ compact intervals. Then $f^n(J) \supset K$, for any sufficiently large n.*

Lemma 2.2. *A map $f \in C(I, I)$ is transitive if and only if,*

$$\textit{for any interval } J \subset I, \textit{ the set } I \setminus \bigcup_{n=0}^{\infty} f^n(J) \textit{ has at most three points.} \quad (1)$$

PROOF. Let $J \subset I$ be an interval. From Lemma 2.1 it follows, that $I \setminus \bigcup_{n=0}^{\infty} f^n(J) = \{0, 1\}$ for any bitransitive map f.

If f is transitive, but not bitransitive, then there exists $c \in (0, 1)$ such that $f([0, c]) = [c, 1]$, $f([c, 1]) = [0, c]$ and both $f^2|_{[0,c]}$ and $f^2|_{[c,1]}$ are bitransitive (see, e.g., [3], Proposition 42). Hence $I \setminus \bigcup_{n=0}^{\infty} f^n(J) = \{0, c, 1\}$.

The opposite implication is obvious. □

The following theorem is the main result of this paper.

Theorem 2.3. *Every bitransitive $f \in C(I, I)$ is conjugate to $g \in C(I, I)$, which satisfies the following conditions,*

1. *there is a c-dense ω-scrambled set for g (hence g is ω-chaotic),*
2. *there is an extremely LY-scrambled set for g with full Lebesgue measure (consequently, g is extremely LY-chaotic),*
3. *every ω-scrambled set of g has zero Lebesgue measure.*

Before proving this theorem we formulate the following auxiliary lemmas.

Lemma 2.4. *Let $f \in C(I,I)$ be transitive, then f has a c-dense ω-scrambled set.*

Proof. It is known, that each transitive map $f \in C(I,I)$ has positive topological entropy (see e.g. [3]). By Theorem 1.2 f has an ω-scrambled set S of cardinality c. Order S into a transfinite sequence $S = \{x_\alpha\}_{\alpha<\Omega}$, where Ω is the first ordinal number of the power of continuum. Let $\mathcal{I} = \{I_\alpha\}_{\alpha<\Omega}$ be a transfinite sequence of all subintervals of I. Let $S^\star = \{x^\star_\alpha\}_{\alpha<\Omega}$ be such that $x^\star_\alpha \in I_\alpha$ and there exists n_α such that $f^{n_\alpha}(x^\star_\alpha) = x_\alpha$; this is possible by (1). It is easy to see, that $\omega_f(x_\alpha) = \omega_f(x^\star_\alpha)$ and hence $S^\star$ is an ω-scrambled set for f. The c-density of $S^\star$ is obvious. $\square$

The following two lemmas and most of all the constructions from their proofs, are almost the same as in Lemma 2 and Theorem 1 from [1]. But to prove Lemma 2.5 we need slightly modify the original proof, and hence it is necessary to rewrite it. And although the proof of Lemma 2.6 is similar to the proof of Theorem 1 from [1], we give it here for completeness.

Lemma 2.5. *Let $f \in C(I,I)$ be bitransitive, $J \subset (0,1)$ a compact interval, M an infinite set of positive integers, and let $p_n \in (0,1)$ be such that the accumulation points of $\{p_n\}_{n=1}^\infty$ are in $\{0,1\}$, let $\{r_n\}_{n=1}^\infty$ be a sequence of all rational numbers in I. Then there are a non-empty nowhere dense perfect set $P \subset J$, and an increasing sequence $\{k(n)\}_{n=1}^\infty$ in M with the following properties,*

$$f^{k(n)}(P) \subset \left[p_n - \frac{1}{n}, p_n + \frac{1}{n}\right],\ \textit{for any } n, \tag{2}$$

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1,\ \textit{for any } x, y \in P,\ x \neq y, \tag{3}$$

and

$$\omega_f(x) = I,\ \textit{for any } x \in P. \tag{4}$$

Proof. Put $q_n = 1 - p_n$. We let the set P be in the form $P = \bigcap_{n=1}^\infty P_n$ where, for any n, P_n is the union of pairwise disjoint compact intervals U_s, $s \in \{0,1\}^n$, and $P_{n+1} \subset P_n$. The intervals U_s are defined inductively by n.

Stage 1: Let U_0, U_1 be disjoint closed subintervals of J. Put $P_1 = U_0 \cup U_1$, and let $t(1,0) < t(1,1) < t(1,2)$ be any numbers in M.

Stage $n+1$: Sets $P_1, ..., P_n$, and positive integers

$t(1,0) < t(1,1) < t(1,2) < t(2,0) < t(2,1) < t(2,2) < t(2,3) < ... < t(n,0) < t(n,1) < ... < t(n,n+1)$ in M are available from stage n such that, for any $s = s_1 ... s_v \in \{0,1\}^v$, $1 \le v \le n$, $1 \le j \le v+1$,

$$|U_s| \le \frac{1}{v}, \tag{5}$$

$$f^{t(v,j)}(U_s) \subset \left[p_v - \frac{1}{v}, p_v + \frac{1}{v}\right] \text{ if } j = 1 \text{ or } s_j = 0, \tag{6}$$

$$f^{t(v,j)}(U_s) \subset \left[q_v - \frac{1}{v}, q_v + \frac{1}{v}\right] \text{ if } s_j = 1, \tag{7}$$

$$f^{t(v,0)}(U_s) \subset \left[r_v - \frac{1}{v}, r_v + \frac{1}{v}\right]. \tag{8}$$

By Lemma 2.1 there is an integer $t(n+1,0) > t(n,n)$ in M such that, for any $s \in \{0,1\}^n$, $r_{n+1} \in f^{t(n+1,0)}(U_s)$. Hence, for any $s \in \{0,1\}^n$, there is a compact interval $V_s \subset U_s$ such that

$$f^{t(n+1,0)}(V_s) \subset \left[r_{n+1} - \frac{1}{n+1}, r_{n+1} + \frac{1}{n+1}\right]. \tag{9}$$

Again by Lemma 2.1 there is an integer $t(n+1,1) > t(n+1,0)$ in M such that, for any $s \in \{0,1\}^n$, $p_{n+1} \in f^{t(n+1,0)}(U_s)$. Hence, for any $s \in \{0,1\}^n$, there is a compact interval $V_s^1 \subset V_s$ such that

$$f^{t(n+1,1)}(V_s^1) \subset \left[p_{n+1} - \frac{1}{n+1}, p_{n+1} + \frac{1}{n+1}\right]. \tag{10}$$

Next, there is a $t(n+1,2) > t(n+1,1)$ in M such that, for any $s \in \{0,1\}^n$, $\{p_{n+1}, q_{n+1}\} \subset f^{t(n+1,2)}(V_s^1)$. Hence, for any $s = s_1 s_2 ... s_n \in \{0,1\}^n$ there is a compact interval $V_s^2 \subset V_s^1$ such that $x \in f^{t(n+1,2)}(V_s^2)$, where $x = p_{n+1}$ if $s_1 = 0$, and $x = q_{n+1}$ otherwise, and such that $|f^{t(n+1,2)}(V_s^2)| \le 1/(n+1)$. Applying this process n times we obtain integers $t(n+1,2) < t(n+1,3) < ... < t(n+1,n+1)$ in M, and compact intervals $V_s^2 \supset V_s^3 \supset ... \supset V_s^{n+1}$ such that, for any $s = s_1 s_2 ... s_n \in \{0,1\}^n$, and any $2 \le j \le n+1$,

$$f^{t(n+1,j)}(V_s^j) \subset \left[p_{n+1} - \frac{1}{n+1}, p_{n+1} + \frac{1}{n+1}\right] \text{ if } s_j = 0, \tag{11}$$

and

$$f^{t(n+1,j)}(V_s^j) \subset \left[q_{n+1} - \frac{1}{n+1}, q_{n+1} + \frac{1}{n+1}\right] \text{ if } s_j = 1. \tag{12}$$

Finally, let $t(n+1,n+2) > t(n+1,n+1)$ be from M and such that, for any $s \in \{0,1\}^n$, $f^{t(n+1,n+2)}(V_s^{n+1}) \supset \{p_{n+1}, q_{n+1}\}$. Then there are disjoint compact intervals $U_{s0}, U_{s1} \subset V_s^{n+1}$ such that

$$|U_{s0}|, |U_{s1}| \leq \frac{1}{n+1}, \tag{13}$$

$$f^{t(n+1,n+2)}(U_{s0}) \subset \left[p_{n+1} - \frac{1}{n+1}, p_{n+1} + \frac{1}{n+1}\right], \tag{14}$$

and

$$f^{t(n+1,n+2)}(U_{s1}) \subset \left[q_{n+1} - \frac{1}{n+1}, q_{n+1} + \frac{1}{n+1}\right]. \tag{15}$$

Thus we have sets U_s defined for any $s \in \{0,1\}^{n+1}$. They satisfy (5) by (13), (6) by (10), (11), (14); (7) by (12), and (15); and (8) by (9). This completes the induction.

Put $P = \bigcap_{k=1}^{\infty} \bigcup_{s \in \{0,1\}^k} U_s$ and for any n , $k(n) = t(n,1)$. Then P is a nowhere dense perfect set; this follows by (5). By (6), P satisfies (2). Let us prove (3).

Let x and y be distinct points in P. Then for any positive integer K there are $s, \tilde{s} \in \{0,1\}^K$ such that $x \in U_s$, $y \in U_{\tilde{s}}$. Take K sufficiently large so that the sets U_s and $U_{\tilde{s}}$ are disjoint. Thus, $s \neq \tilde{s}$ and hence, $s_j \neq \tilde{s}_j$ for some j, where $s = s_1...s_K$, $\tilde{s} = \tilde{s}_1...\tilde{s}_K$. Without loss of generality assume $s_j = 0$ and $\tilde{s}_j = 1$. Let $n > K$. Then by (6) and (7), $|f^{t(n,j)}(x) - f^{t(n,j)}(y)| \geq |p_n - q_n| - 2/n \to 1$ for $n \to \infty$, which implies (3).

Finally, it remains to prove (4). Since $(\{r_n\}_{n=1}^{\infty})' = I$ and (8) is fulfilled, it is easy to see, that (4) is satisfied. $\square$

Lemma 2.6. *Let $f \in C(I,I)$ be bitransitive. Then there exists a set S, which is c-dense, F_σ, extremely LY-chaotic and $\omega_f(x) = I$, for every $x \in S$.*

PROOF. Define by induction a sequence $S_1 \subset S_2 \subset ...$ of perfect extremely LY-scrambled sets, and a sequence $M_1 \supset M_2 \supset ...$ of infinite sets of positive integers such that

$$f^{d(n)}(S_m) \subset \left[0, \frac{1}{n}\right] \text{ for every } n, \tag{16}$$

where $\{d(n)\}_{n=1}^{\infty}$ is an enumeration of M_m. Apply Lemma 2.5 to $p_n = 1/(n+1)$, $J = [1/3, 2/3]$, and the set of positive integers M, to get P and $\{k(n)\}_{n=1}^{\infty}$. Denote $S_1 = P$ and $M_1 = \{k(n)\}_{n=1}^{\infty}$.

Now, assume there are sets S_m and M_m satisfying (16). Let J be the middle third of the interval complementary to S_m of maximal length. Apply Lemma 2.5 to $M = M_m$, $p_n = 1/(n+1)$ for n odd, $p_n = 1 - 1/(n+1)$ for

n even, and $q_n = 1 - p_n$, to get P and $\{k(n)\}_{n=1}^{\infty}$. Then $S_{m+1} = S_m \cup P$ is a perfect extremely LY-scrambled set since, for $x \in S_m$ and $y \in P$, by (2) and (16), $\lim_{n\to\infty} |f^{k(n)}(x) - f^{k(n)}(y)| = 1$. To complete the induction put $M_{m+1} = \{k(2n+1)\}_{n=1}^{\infty}$.

Let $S = \bigcup_{m=1}^{\infty} S_m$. Then S is an extremely LY-scrambled set of type F_σ, which is c-dense in I. By (4) it is obvious that $\omega_f(x) = I$, for every $x \in S$. $\square$

Now, we can prove Theorem 2.3.

PROOF OF THEOREM 2.3. Let $f \in C(I, I)$ be bitransitive. Let S be the set from Lemma 2.6, and $\varphi \in C(I, I)$ be a homeomorphism such that $\lambda(\varphi(S)) = 1$ (such a homeomorphism exists since, by [6], any c-dense F_σ set is homeomorphic to a set of full Lebesgue measure). Put $g = \varphi \circ f \circ \varphi^{-1}$. So f and g are conjugate.

1. Since f and g are conjugate, they are both bitransitive and it suffices to apply Lemma 2.4 to obtain our assertion.

2. Since S is extremely LY-scrambled for f, and f and g are conjugate, $\varphi(S)$ must be extremely LY-scrambled for g.

3. Let S_ω be an ω-scrambled set for g. It is easily seen that $\sharp(S_\omega \cap \varphi(S)) \leq 1$. Since $\lambda(\varphi(S)) = 1$, every ω-scrambled set S_ω has zero Lebesgue measure.

$\square$

Remark. The first condition from Theorem 2.3 is true for any transitive map, since Lemma 2.4 is true for transitive maps.

Theorem 2.7. *Let $f \in C(I, I)$ be a map with positive topological entropy. Then, for some $k \geq 1$, f^k is semiconjugate to a map $g \in C(I, I)$, which satisfies the following conditions,*

1. *there is a c-dense ω-scrambled set for g,*

2. *there is an extremely LY-scrambled set for g with full Lebesgue measure,*

3. *every ω-scrambled set of g has zero Lebesgue measure.*

PROOF. There exists a positive integer k such that f^k is semiconjugate to a bitransitive map $g \in C(I, I)$ (see [4]). Using Theorem 2.3. we obtain the assertion. $\square$

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