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ON THE SETS OF DISCONTINUITY POINTS OF FUNCTIONS SATISFYING SOME APPROXIMATE QUASI-CONTINUITY CONDITIONS

Abstract

In this paper the sets of discontinuity points and the sets of approximate discontinuity points of function $f : \mathcal{R} \to \mathcal{R}$, satisfying some special approximate quasi-continuity conditions introduced in [2], are investigated.

Let $\mathcal{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ ($D_l(A, x)$) of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h} \text{ respectively}).$$

If $D_u(A, x) = D_l(A, x)$, then $D(A, x) = D_u(A, x)$ is called the outer density of the set A at x. In the case where the set A is measurable in the Lebesgue sense, the outer densities $D_u(A, x)$, $D_l(A, x)$ and respectively $D(A, x)$ are said to be in short the densities. A point x is called an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

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The family T_d of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

holds, is a topology called the density topology ([1, 4]). The sets $A \in T_d$ are Lebesgue measurable [1].

If T_e denotes the Euclidean topology in $\mathbb{R}$, then the continuity of a function f as an application from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called approximate continuity ([1, 4]).

For a function $f : \mathbb{R} \to \mathbb{R}$ denote by $C(f)$ the set of all continuity points of f, by $A(f)$ the set of all approximate continuity points of f, by $D(f)$ the set $\mathbb{R} \setminus C(f)$ and by $D_{ap}(f)$ the set $\mathbb{R} \setminus A(f)$.

Denote by $\mathcal{A}$ the family of all functions $f : \mathbb{R} \to \mathbb{R}$ which are approximately continuous at each point $x \in \mathbb{R}$.

In [2] the following properties are investigated:

1. A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_0) at a point x ($f \in s_0(x)$) if for each positive real r and for each set $U \ni x$ belonging to T_d, there is a point $t \in C(f) \cap U$ such that $|f(t) - f(x)| < r$.

2. A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_1) [(s_2)] at a point x ($f \in s_1(x)$ [$f \in s_2(x)$]) if for each positive real r and for each set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$ [$\emptyset \neq I \cap U \subset A(f)$] and $|f(t) - f(x)| < r$ for all points $t \in I \cap U$.

3. For $i = 0, 1, 2$ a function f has the property (s_i) if $f \in s_i(x)$ for every point $x \in \mathbb{R}$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_3) if for each nonempty set $U \in T_d$ there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$.

Evidently each function f having the property (s_1) has also the properties (s_2), (s_0) and (s_3) and for each function f having the property (s_3) the set $D(f) = \mathbb{R} \setminus C(f)$ is nowhere dense and of Lebesgue measure 0. But the closure $\mathrm{cl}(D(f))$ of some functions f having the property (s_1) may be of positive measure.

For example, if $A \subset [0, 1]$ is a Cantor set of positive measure and (I_n) is a sequence of all components of the set $[0, 1] \setminus A$ such that $I_n \neq I_m$ for $n \neq m$. Let $J_n \subset \mathrm{int}(I_n)$ be nondegenerate closed intervals such that $\frac{\mu(J_n)}{\mu(I_n)} < \frac{1}{n}$ for $n = 1, 2, \dots$ ($\mathrm{int}(I_n)$ denotes the interior of I_n). On each interval J_n we define a function $f_n : J_n \to [0, \frac{1}{n}]$ which is discontinuous only at one point $a_n \in \mathrm{int}(J_n)$

and such that $f_n(x) = 0$ if $x < a_n$ or x is the right endpoint of J_n, $f_n(a_n) = \frac{1}{n}$ and f_n is linear otherwise on J_n. Then the function

$$f(x) = f_n(x) \text{ for } x \in J_n, \quad n = 1, 2, \ldots$$

and $f(x) = 0$ otherwise on $\mathbb{R}$ has the property (s_1) but $\mu(\mathrm{cl}(D(f))) > 0$.

For a nonempty family $\mathcal{H}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ denote by $X(\mathcal{H})$ (respectively by $X_{ap}(\mathcal{H})$) the family of all sets $A \subset \mathbb{R}$ for which there are the functions $f \in \mathcal{H}$ such that $A = D(f)$ (resp. $A = D_{ap}(f)$).

Evidently, if $\mathcal{H}_1 \subset \mathcal{H}_2$, then $X(\mathcal{H}_1) \subset X(\mathcal{H}_2)$.

Let S_i, where $i = 0, 1, 2, 3$, be the family of all functions having the property (s_i).

Theorem 1. *The equalities $X(\mathcal{A} \cap S_1) = X(S_1) = X(S_3)$ are true and a set $A \in X(\mathcal{A} \cap S_1)$ if and only if it is an F_σ set of measure zero and satisfies the following condition*

(a) for each nonempty set $U \in T_d$ contained in the closure $\mathrm{cl}(A)$ *of the set A the set $U \cap A$ is nowhere dense in U.*

PROOF. The inclusions $X(\mathcal{A} \cap S_1) \subset X(S_1) \subset X(S_3)$ are obvious.

If $A \in X(S_3)$, then there is a function $f \in S_3$ such that $D(f) = A$. Since the set of all discontinuity points of an arbitrary function is an F_σ-set, the set A is the same. From the definition of the property (s_3) follows that $\mu(A) = 0$. If $\mu(\mathrm{cl}(A)) = 0$, then the set $D(\mathrm{cl}(A))$ of all density points of the closure $\mathrm{cl}(A)$ is empty and $A \cap U$ is nowhere dense in U for every $U \subset \mathrm{cl}(A)$ belonging to T_d. So, we suppose that $\mu(\mathrm{cl}(A)) > 0$ and fix a nonempty set $U \in T_d$ contained in $\mathrm{cl}(A)$. If an open interval I is such that $\emptyset \neq I \cap U$, then $I \cap U \in T_d$ and, by the property (s_3), there is an open interval $J \subset I$ such that

$$\emptyset \neq J \cap U \subset C(f).$$

So, the set $A \cap U$ is nowhere dense in U.

Now let A be an F_σ-set of measure zero satisfying the condition (a). We will construct a function $f \in \mathcal{A} \cap S_1$ such that $D(f) = A$. Since A is of the first category, there are closed sets A_n such that

$$A = \bigcup_n A_n, \text{ and } A_n \cap A_m = \emptyset \text{ for } n \neq m, \quad n, m = 1, 2, \ldots. \qquad ([3])$$

Without loss of generality we may suppose that the sets $A_n \neq \emptyset$ for $n = 1, 2, \ldots$.

Fix a positive integer k. If (a, b), $a, b \in \mathbb{R}$, is a component of the complement $\mathbb{R} \setminus A_k$, then we find two monotone sequences of points

$$a < \cdots < a_{n+1} < a_n < \cdots < a_1 = b_1 < \cdots < b_n < b_{n+1} < \cdots < b$$

such that

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b,$$

and

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{b - b_{n+1}} = \lim_{n\to\infty} \frac{a_n - a_{n+1}}{a_{n+1} - a} = 0.$$

In each interval (a_{n+1}, a_n) ((b_n, b_{n+1})) we find a nondegenerate closed interval $I_n \subset (a_{n+1}, a_n)$ ($J_n \subset (b_n, b_{n+1})$) such that

$$\frac{d(I_n)}{a_n - a_{n+1}} > 1 - \frac{1}{8^{k+n}} \quad (\frac{d(J_n)}{b_{n+1} - b_n} > 1 - \frac{1}{8^{k+n}}),$$

where $d(I_n)$ denotes the length of I_n.

If (a, b) is an unbounded component of the complement $\mathbb{R} \setminus A_k$; i.e., $a = -\infty$ or $b = \infty$, then we find only one sequence (I_n) or (J_n) satisfying the above conditions (as a_1 or b_1 we take arbitrary point in this component). For $x \in (a, b)$ let

$$f_{k,(a,b)}(x) = \begin{cases} \frac{1}{4^k} & if & x = a_n \text{ or } x = b_n, \\ & & n = 1, 2, \ldots \\ 0 & if & x \in I_n \cup J_n, \quad n = 1, 2, \ldots \\ linear & on\ the\ components\ of & [a_n + 1, a_n] \setminus \int(I_n), \\ & & \quad n = 1, 2, \ldots \\ linear & on\ the\ components\ of & [b_n, b_n + 1] \setminus \int(J_n), \\ & & \quad n = 1, 2, \ldots . \end{cases}$$

Define

$$f_k(x) = f_{k,(a,b)}(x) \text{ on the components } (a, b) \text{ of the set } \mathbb{R} \setminus A_k$$

and

$$f_k(x) = 0 \text{ on } A_k$$

and observe that the function f_k is continuous at each point $x \in \mathbb{R} \setminus A_k$, and discontinuous at each point $x \in A_k$. Since for every $x \in A_k$ the density

$$D((f_k)^{-1}(0), x) = 1 \text{ and } f_k(x) = 0,$$

the function f_k is approximately continuous. Let

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Since $|f_k| \leq \frac{1}{4^k}$ for $k = 1, 2, \ldots$ and $\sum_{k=1}^{\infty} \frac{1}{4^k} < \infty$, the series $\sum_{k=1}^{\infty} f_k$ uniformly converges to f. So, the function f is continuous at each point $x \in \mathbb{R} \setminus A$ and approximately continuous everywhere on $\mathbb{R}$. If $x \in A$, then there is a positive integer k_1 such that

$$x \in A_{k_1} \text{ and } x \in \mathbb{R} \setminus A_k \text{ for } k \neq k_1.$$

So the function f_{k_1} is discontinuous at x and for $k \neq k_1$ the functions f_k are continuous at x. Consequently

$$f = f_{k_1} + \sum_{k \neq k_1} f_k$$

is discontinuous at x and

$$A = \bigcup_{k=1}^{\infty} A_k = D(f).$$

We will prove that $f \in S_1$. For this fix a real $r > 0$, a point x and a set $U \ni x$ belonging to T_d. If $x \in \mathbb{R} \setminus A$, then f is continuous at x and there is a real $s > 0$ such that

$$|f(t) - f(x)| < r \text{ for } t \in (x - s, x + s).$$

Since

$$U \cap (x - s, x + s) \neq \emptyset \text{ and } U \cap (x - s, x + s) \in T_d$$

and A satisfies the condition (a), there is an open interval $I \subset (x - s, x + s)$ such that $\mathbb{R} \setminus A \supset U \cap I \neq \emptyset$ and in the considered case $f \in s_1(x)$.

So we suppose that $x \in A_k$ for some integer k. Since the function $h = f - f_k$ is continuous at x, there is a real $s > 0$ such that

$$|h(t) - h(x)| < \frac{r}{2} \text{ for } t \in (x - s, x + s).$$

But the density

$$D(\text{int}((f_k)^{-1}(0)), x) = D((f_k)^{-1}(0), x) = 1,$$

so

$$D((x-s,x+s)\cap U\cap \operatorname{int}((f_k)^{-1}(0)),x)=1.$$

If

$$((x-s,x+s)\cap U\cap \operatorname{int}((f_k)^{-1}(0)))\not\subset \operatorname{cl}(A),$$

then there is an open interval

$$I\subset (x-s,x+s)\cap \operatorname{int}((f_k)^{-1}(0))=W,$$

such that $\emptyset\neq I\cap U\subset C(f)$. Suppose that $T_d\ni W\cap U\subset \operatorname{cl}(A)$. Since the set $A\cap W\cap U$ is nowhere dense in $W\cap U$, there is an open interval $I\subset W\cap(\mathbb{R}\setminus A)$ such that $\emptyset\neq I\cap U\subset C(f)$. For $t\in I\cap U\subset W$ we have

$$|f(t)-f(x)|\leq |f_k(t)-f_k(x)|+|h(t)-h(x)|<0+\frac{r}{2}<r,$$

thus $f\in s_1(x)$ and the proof is complete. □

Next example shows that the condition (a) from Theorem 1 can't be replaced by the condition

(b) the set $A\cap D(\operatorname{cl}(A))$ is nowhere dense in $D(\operatorname{cl}(A))$.

Example 1. Let $C\subset[0,1]$ be a Cantor set of positive measure such that $\mu(I\cap C)>0$ for every open interval I with $I\cap C\neq\emptyset$. Let $B\subset C$ be a compact set of positive measure which is nowhere dense in C. Let (I_n) be a sequence of all components of the set $[0,1]\setminus C$ such that $I_n\neq I_m$ for $n\neq m$. For each $n=1,2,\ldots$ let $c_n\in \operatorname{int}(I_n)$ be a fixed point. Let $E\subset D(B)$ be a countable set dense in $D(B)$. Then the set $A=E\cup\{c_n; n=1,2,\ldots\}$ is countable (so it is an F_σ-set of measure zero) and satisfies the condition (b), but it does not satisfy the condition (a).

Theorem 2. *The equality $X(S_0)=X(S_2)$ is true. Moreover a set $A\in X(S_0)$ if and only if A is an F_σ-set of measure zero.*

PROOF. In [2] it is observed that $S_2\subset S_0$ and that each function $f\in S_0$ is almost everywhere continuous. So if $f\in S_0$, then the set $D(f)$ is an F_σ-set of measure zero.

On the other hand if A is an F_σ-set of measure zero, then the same as in the proof of Theorem 1 we construct an approximately continuous function f with $D(f)=A$. We will show that $f\in S_2$. For this fix a point $x\in\mathbb{R}$, a real $r>0$, and a set $U\ni x$ belonging to T_d. Since f is an approximately continuous function, the set

$$W=f^{-1}((f(x)-\frac{r}{2},f(x)+\frac{r}{2}))\in T_d$$

and consequently $U \cap W \in T_d$ is of positive measure. But f is almost everywhere continuous, so there is a point $u \in C(f) \cap U \cap W$. Let $s > 0$ be a real such that
$$|f(t) - f(u)| < \frac{r}{2} \text{ for } t \in I = (u - s, u + s).$$
Consequently, $I \cap U \neq \emptyset$ and for $t \in I \cap U$ we obtain
$$|f(t) - f(x)| \leq |f(t) - f(u)| + |f(u) - f(x)| < \frac{r}{2} + \frac{r}{2} = r.$$
This completes the proof. □

The same as in the proof of Theorem 1 we can prove that for each function $f \in S_3$ the set $D_{ap}(f)$ is a set satisfying the condition (a) from Theorem 1. Since $D_{ap}(f) \subset D(f)$ and the function $f \in S_3$ is almost everywhere continuous, for $f \in S_3$ the set $D_{ap}(f)$ is contained in an F_σ-set of measure zero.

Theorem 3. *The inclusion $X(S_1) \subset X_{ap}(S_1)$ is true.*

PROOF. Suppose that A is an F_σ-set of measure zero satisfying the condition (a). Without loss of generality we can suppose that the set A is the union of an infinite family of pairwise disjoint compact sets $A_n \neq \emptyset$.

Fix a positive integer k. Let
$$U_1 = \{x : \text{dist}(x, A_k) < 1\},$$
where
$$\text{dist}(x, A_k) = \inf\{|t - x|; t \in A_k\}.$$
Observe that the set U_1 is open and since A_k is compact, the family of the components of U_1 is finite. Let $\{I_{1,1}, \ldots, I_{1,i(1)}\}$ be the family of all components of U_1. For each positive integer $i \leq i(1)$ there are pairwise disjoint nondegenerate closed intervals
$$K_{1,i,1}, \ldots, K_{1,i,k(1,i)} \subset I_{1,i} \setminus A_k$$
such that
$$\frac{\mu(K_{1,i,1} \cup \ldots \cup K_{1,i,k(1,i)})}{\mu(I_{1,i})} > 1 - \frac{1}{2}.$$
Let
$$r_2 = \text{dist}(\bigcup_{i=1}^{i(1)} \bigcup_{j=1}^{k(1,i)} K_{1,i,j}, A_k) = \inf\{|t - x|; t \in \bigcup_{i=1}^{i(1)} \bigcup_{j=1}^{k(1,i)} K_{1,i,j},\ x \in A_k\},$$

let

$$U_2 = \{x : \operatorname{dist}(x, A_k) < \frac{r_2}{2}\}$$

and let $I_{2,1}, \ldots, I_{2,i(2)}$ be the components of the set U_2. In each component $I_{2,i}$, $i \leq i(2)$, we find pairwise disjoint nondegenerate closed intervals

$$K_{2,i,1}, \ldots, K_{2,i,k(2,i)} \subset I_{2,i} \setminus A_k$$

such that

$$\frac{\mu(K_{2,i,1} \cup \ldots \cup K_{2,i,k(2,i)})}{\mu(I_{2,i})} > 1 - \frac{1}{4}.$$

In general in the n-th step we define

$$r_n = \operatorname{dist}(\bigcup_{i=1}^{i(n-1)} \bigcup_{j=1}^{k(n-1,i)} K_{n-1,i,j}, A_k),$$

$$U_n = \{x : \operatorname{dist}(x, A_k) < \frac{r_n}{2}\},$$

and in each component $I_{n,i}$, $i \leq i(n)$, of the set U_n we find pairwise disjoint nondegenerate closed intervals

$$K_{n,i,1}, \ldots, K_{n,i,k(n,i)} \subset I_{n,i} \setminus A_k$$

such that

$$\frac{\mu(K_{n,i,1} \cup \ldots \cup K_{n,i,k(n,i)})}{\mu(I_{n,i})} > 1 - \frac{1}{2^n}.$$

Now for each triple (n, i, j), $n \geq 1$, $i \leq i(n)$, $j \leq k(n, i)$, we find closed intervals $J_{n,i,j} \subset I_{n,i}$ such that

$$K_{n,i,j} \subset \operatorname{int}(J_{n,i,j}) \text{ and } J_{n,i,j_1} \cap J_{n,i,j_2} = \emptyset \text{ for } j_1 \neq j_2$$

and define a continuous function $f_{n,i,j} : J_{n,i,j} \to [0, \frac{1}{2^k}]$ such that

$$f_{n,i,j}(K_{n,i,j}) = \{\frac{1}{2^k}\} \text{ and } f_{n,i,j}(x) = 0 \text{ if } x \text{ is an endpoint of } J_{n,i,j}.$$

Let $f_k(x) = f_{2n,i,j}(x)$ for $x \in J_{2n,i,j}$, $n \geq 1$, $i \leq i(2n)$, $j \leq k(2n, i)$ and $f_k(x) = 0$ otherwise on $\mathbb{R}$. Then $C(f_k) = \mathbb{R} \setminus A_k$. If $x \in A_k$, then $f_k(x) = 0$ and for each positive integer n there is a positive integer $a(x) \leq i(2n)$ such that $x \in I_{2n,a(x)}$. Since

$$\lim_{n\to\infty} \frac{\mu(\bigcup_{j=1}^{k(2n,a(x))} K_{2n,a(x),j})}{\mu(I_{2n,a(x)})} = 1,$$

the function f_k is not approximately continuous at x. Let $f = \sum_{k=1}^{\infty} f_k$. Since the convergence of the above series is uniform, we have

$$C(f) = \mathbb{R} \setminus A \text{ and } D_{ap}(f) = A.$$

We will prove that $f \in S_1$. For this fix a real $r > 0$, a point x and a set $U \ni x$ belonging to T_d. If $x \in \mathbb{R} \setminus A = C(f)$, then the proof of the relation $f \in s_1(x)$ is the same as one in the proof of Theorem 1.

So we suppose that $x \in A_k$ for some integer $k > 0$. Since the function $h = f - f_k$ is continuous at x, there is a real $s > 0$ such that

$$|h(t) - h(x)| < \frac{r}{2} \text{ for } t \in (x - s, x + s).$$

But

$$\lim_{n\to\infty} \frac{\mu(K_{2n-1,a(x),1} \cup \ldots \cup K_{2n-1,a(x),k(2n-1,a(x))})}{\mu(I_{2n-1,a(x)})} = 1,$$

so there is a positive integer $j \leq k(2n - 1, a(x))$ such that

$$T_d \ni \operatorname{int}(K_{2n-1,a(x),j}) \cap U \cap (x - s, x + s) \neq \emptyset.$$

If

$$(x - s, x + s) \cap \operatorname{int}(K_{2n-1,a(x),j}) \cap U \not\subset \operatorname{cl}(A),$$

then there is an open interval

$$I \subset ((x - s, x + s) \cap K_{2n-1,a(x),j}) \setminus \operatorname{cl}(A)$$

such that $C(f) \supset I \cap U \neq \emptyset$. Similarly if

$$T_d \ni (x - s, x + s) \cap \operatorname{int}(K_{2n-1,a(x),j}) \cap U \subset \operatorname{cl}(A),$$

then by the condition (a) there is an open interval

$$I \subset ((x - s, x + s) \cap K_{2n-1,a(x),j}) \setminus A$$

such that $C(f) \supset I \cap U \neq \emptyset$. For $t \in I \cap U$ we have

$$|f(t) - f(x)| \leq |f_k(t) - f_k(x)| + |h(t) - h(x)| < 0 + \frac{r}{2} < r.$$

This completes the proof. □

Problem 1. Does there exist a function $f \in S_1$ such that the set $D_{ap}(f)$ is not an F_σ-set?

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