INROADS

Real Analysis Exchange
Vol. 27(2), 2001/2002, pp. 735–738

Marianna Csörnyei, Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom. e-mail: mari@math.ucl.ac.uk

LIPSCHITZ IMAGE OF LIPSCHITZ FUNCTIONS

Abstract

We show that there exists a bi-Lipschitz homeomorphism $f : \mathbb{R} \to \mathbb{R}$ and a dense open set U of the space of Lipschitz functions defined on $[0, 1]$, for which $\{f \circ g : g \in U\}$ is of first category.

In order to study Fréchet differentiability of Lipschitz maps, J. Lindenstrauss and D. Preiss introduced recently a new concept of null sets in Banach spaces. This concept enabled, among many others, the proof of the existence of Fréchet derivatives of Lipschitz maps between certain infinite dimensional Banach spaces. No such results have been known previously.

The definition involves both category and measure: Let $T = [0, 1]^{\mathbb{N}}$ be endowed with the product Lebesgue measure μ, and for every Banach space X let $\Gamma(X)$ denote the space of all continuous mappings $\gamma : T \to X$ having continuous partial derivatives $D_j\gamma$, equipped with the topology generated by the semi-norms $\|\gamma\|_0 = \sup_{t\in T} \|\gamma(t)\|$ and $\|\gamma\|_k = \sup_{t\in T} \|D_k\gamma(t)\|$. The space $\Gamma(X)$ with this topology is a Polish space.

Definition 1. A Borel set $N \subset X$ is called Γ null, if $\mu\{t \in T : \gamma(t) \in N\} = 0$ for residually many $\gamma \in \Gamma(X)$. A possibly non-Borel set $A \subset X$ is Γ null, if it is contained in a Borel Γ null set.

It is easy to check that in $\mathbb{R}^n$, Γ null sets and Lebesgue null sets coincide. However, in the infinite dimensional case Γ null sets form a completely different σ-ideal then the previously known σ-ideals of null sets. For instance, an infinite dimensional super-reflexive space X can always be decomposed into the union of a Gauss null set and a Γ null set.

Key Words: measure and category, null set, Lipschitz function
Mathematical Reviews subject classification: 26A24
Received by the editors July 30, 2001

The authors of [1] asked whether the σ-ideal of Γ null sets is stable under Lipschitz equivalences: let f be a Lipschitz equivalence between separable Banach spaces with RNP (say, between separable reflexive Banach spaces), and let $A \subset X$ be a Γ null set. Does this imply that $f(A) \subset Y$ is Γ null?

One difficulty in studying questions like this comes from the fact that $\gamma \in \Gamma(X)$ does not necessarily imply $f \circ \gamma \in \Gamma(X)$. However, for spaces with RNP, Γ null sets can be defined in the following equivalent way (see in [1]): For $\gamma : T \to X$ define $\mathrm{Lip}_i(\gamma) = \sup \|\gamma(t + re_i) - \gamma(t)\|/|r|$, where e_i denotes the ith unit vector and the supremum is taken over all $t, t+re_i \in T$. Consider the space of continuous mappings $\gamma : T \to X$ for which $\mathrm{Lip}_i(\gamma) < \infty$ for every i, with the topology generated by $\sup_{t\in T} \|\gamma(t)\|$ and $\mathrm{Lip}_k(\gamma)$, $k \in \mathbb{N}$. Let $\tilde{\Gamma}(X)$ denote the closure of the set of those mappings which depend only on finitely many coordinates. Then a Borel set $N \subset X$ is Γ null if and only if $\mu\{t : \gamma(t) \in N\} = 0$ for residually many $\gamma \in \tilde{\Gamma}(X)$.

Let $f : X \to Y$ be a Lipschitz equivalence, and let $U \subset \tilde{\Gamma}(X)$ be a residual subset. J. Lindenstrauss and D. Preiss asked whether $f \circ U \subset \tilde{\Gamma}(Y)$ is residual. Of course, this would imply a positive answer to the problem about the stability of Γ null sets.

Unfortunately in this note we give a negative answer to this question. We show that the answer is negative already "in the first coordinate": Consider the (non-separable) space of real valued Lipschitz functions defined on $[0, 1]$, with the topology generated by the supremum norm of the Lipschitz functions and the essential supremum of their derivatives. We prove the following theorem:

Theorem 2. *There exists a bi-Lipschitz homeomorphism $f : \mathbb{R} \to \mathbb{R}$ and a dense open set U of the space of Lipschitz functions defined on $[0, 1]$, for which $\{f \circ g : g \in U\}$ is of first category.*

PROOF. Put $I^0 \overset{\text{def}}{=} \mathbb{R}$, $J^0 \overset{\text{def}}{=} [1, 2]$, and let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \setminus \{0\}$. Let $I_1^n \cup I_2^n \cup \ldots I_n^n = \mathbb{R} \setminus \{q_1, q_2, \ldots, q_{n-1}\}$, where $I_i^n = (a_i^n, b_i^n)$ are disjoint open intervals, $b_i^n = a_{i+1}^n$.

Thus, by induction, we can choose closed intervals $J_1^n, \ldots, J_n^n$ and open intervals $K_1^n, \ldots, K_n^n$ such that

(1) $\bigcup_{i=1}^n (I_i^n \times J_i^n) \subset \bigcup_{i=1}^{n-1} (I_i^{n-1} \times J_i^{n-1})$

(2) $|J_i^n| \le 1/n$

(3) $q_i \in K_i^n$

(4) $\mathrm{cl}(K_i^n \cdot J_j^n) = \mathrm{cl}(\{xy : x \in K_i^n, y \in J_j^n\}) \subset \mathbb{R}$ are pairwise disjoint intervals for every $i, j \in \{1, \ldots, n\}$ (in particular, $0 \notin K_i^n \, \forall i, n$).

$(1),(2) \implies \lim_{n\to\infty} \bigcup_{i=1}^{n} \big(I_i^n \times J_i^n\big)$ is the graph of a function $\phi : \mathbb{R}\setminus\mathbb{Q} \to \mathbb{R}$, and $f(x) \stackrel{\text{def}}{=} \int_0^x \phi(y)\,dy$ is a bi-Lipschitz homeomorphism.

Let $K^n \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} K_i^n$, then $(4) \implies d_n \stackrel{\text{def}}{=} \min_{i\neq j} \operatorname{dist}(K^n\cdot J_i^n, K^n\cdot J_j^n) > 0$. Let U_n be the set of those Lipschitz functions g for which $g'(x) \in K^n$ a.e. x, and for $1 \le i < n$ let U_{in} be the set of those Lipschitz functions $g \in U_n$ for which $g^{-1}(I_i^n)$ and $g^{-1}(I_{i+1}^n)$ are nonempty. Then from (3) it follows that $\bigcup_{i,n} U_{in}$ contains a dense open set. We show that $\{f\circ g :\ g \in U_{in}\}$ is nowhere dense for every $1 \le i < n$.

For any Lipschitz function $g \in U_n$, $1 \le j \le n$ and for a.e. $x \in g^{-1}(I_j^n)$ we have $(f\circ g)'(x) \in J_j^n \cdot K^n$. Therefore by the definition of d_n, for any two $g, h \in U_n$ for which $g^{-1}(I_j^n) \setminus h^{-1}(I_j^n)$ or $h^{-1}(I_j^n) \setminus g^{-1}(I_j^n)$ has positive measure, $\operatorname{dist}(f\circ g, f\circ h) \ge d_n$.

Fix some $1 \le i < n$, and for any function $g \in U_{in}$ let $A_j^g \stackrel{\text{def}}{=} g^{-1}(I_j^n)$. Then A_i^g and A_{i+1}^g are nonempty, moreover, $F^g \stackrel{\text{def}}{=} \operatorname{cl}(A_i^g) \cap \operatorname{cl}(A_{i+1}^g) \neq \emptyset$. We have $\operatorname{dist}(f\circ g, f\circ h) \ge d_n$ for every $g, h \in U_{in}$ for which $F^g \neq F^h$. So it is enough to show that for any fixed closed set $F \subset [0,1]$ and $B \stackrel{\text{def}}{=} \{g \in U_{in} :\ F^g = F\}$, the set $\{f\circ g :\ g \in B\}$ is nowhere dense.

This is immediate, since $g(t_0) = b_i^n$ for any fixed $t_0 \in F$, and therefore $\{f\circ g :\ g \in B\} \subset \{h :\ h(t_0) = f(b_i^n)\}$, which is a nowhere dense set. □

References

[1] J. Lindenstrauss and D. Preiss, *Fréchet differentiability of Lipschitz maps between Banach spaces*, to appear in Ann. Math.