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ON FUNCTIONS OF TWO VARIABLES WHOSE VERTICAL SECTIONS HAVE CLOSED GRAPHS

Abstract

In this article we investigate some properties of functions $f : X \times Y \to \mathcal{R}$ (the cliquishness, the Baire property, the measurability) whose vertical sections f_x have closed graphs.

We say that a function $f : X \to Y$, where X and Y are topological spaces, is a function with a closed graph, if the graph of the function f; i.e., the set

$$G(f) = \{(x, y) \in X \times Y; x \in X \text{ and } y = f(x)\},$$

is a closed subset of the product space $X \times Y$.

Let $\mathcal{R}$ be the space of all reals with the Euclidean topology T_e. In this article we will show some properties of functions of two variables $f : X \times Y \to \mathcal{R}$ whose sections $f_x(y) = f(x, y)$, $x \in X$ and $y \in Y$, have closed graphs $G(f_x)$ in the product space $Y \times \mathcal{R}$.

It is well known that there are discontinuous functions $f : \mathcal{R}^2 \to [0, 1]$ with continuous sections f_x and $f^y(x) = f(x, y)$, $x, y \in \mathcal{R}$. Evidently the graphs of such functions are not closed in $\mathcal{R}^3$. While continuous real functions defined on topological spaces have closed graphs, there are functions $f : \mathcal{R}^2 \to [0, 1]$ having continuous sections f^y and f_x with closed graphs such that the graphs $G(f)$ are not closed in $\mathcal{R}^2 \times [0, 1]$. However the following theorem holds.

Theorem 1. *Let (X, T_X) and (Y, T_Y) be topological spaces. If the sections f^y, $y \in Y$, are equi-continuous at each point $x \in X$ and if the graphs $G(f_x)$ of the sections f_x, $x \in X$, are closed in $Y \times \mathcal{R}$, then the graph $G(f)$ of the function f is closed in $X \times Y \times \mathcal{R}$.*

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PROOF. Assume, to the contrary, that the graph $G(f)$ is not closed. Then there is a point

$$(*) \qquad (x,y,z) \in cl(G(f)) \setminus G(f) \subset X \times Y \times \mathcal{R},$$

where cl denotes the closure operation. Since the graph $G(f_x)$ of the section f_x is closed and $z \neq f(x,y)$, there are an open neighborhood $V \in T_Y$ of the point y and an open interval $(z-3r, z+3r)$, $r>0$, such that

$$(**) \qquad (f_x)^{-1}([z-3r, z+3r]) \cap V = \emptyset.$$

From equi-continuity of the sections f^y, $y \in Y$, at the point x it follows that there is a neighborhood $U \in T_X$ of the point x such that

$$|f(u,v) - f(x,v)| < r \text{ for } u \in U \text{ and } v \in Y.$$

We will prove that

$$(***) \qquad (U \times V \times (z-r, z+r)) \cap G(f) = \emptyset.$$

Of course, if there is a point

$$(u_1, v_1) \in U \times V \text{ with } |f(u_1, v_1) - z| < r,$$

then

$$|f(x,v_1) - z| \leq |f(x,v_1) - f(u_1,v_1)| + |f(u_1,v_1) - z| < r + r = 2r,$$

is a contradiction with $(**)$.

So the relation $(***)$ is true, a contradiction with $(*)$, and the proof is complete. □

Theorem 2. *Suppose that (X, T_X) and (Y, T_Y) are topological spaces with countable bases and moreover (Y, T_Y) is a perfectly normal topological space. If the sections f_x, $x \in X$, have closed graphs and the sections f^y, $y \in Y$, have the Baire property, then f has the Baire property as a function of two variables.*

In the proof of this theorem we will apply the following lemma:

Lemma 1. *If (Z, T_Z) is a topological space with a countable basis and if a function $f : Z \to \mathcal{R}$ satisfies the condition*

(a) for each positive real η and for each set $A \subset Z$ with the Baire property and of the second category there is a set $B \subset A$ with the Baire property and of the second category on which $osc_B f \leq \eta$,

then f has the Baire property.

PROOF OF LEMMA 1. For a given real $\eta > 0$ we can find a countable family of pairwise disjoint sets $A_n \subset Z$ having the Baire property and of the second category such that $osc_{A_n} f \leq \eta$ for $n \geq 1$ and

$$Z \setminus \bigcup_{n=1}^{\infty} A_n \text{ is of the first category.}$$

For each integer $n \geq 1$ we choose a point $a_n \in A_n$ and define

$$g(x) = \begin{cases} f(a_n) & \text{for} & x \in A_n,\ n = 1, 2, \ldots \\ f(x) & \text{otherwise on} & Z. \end{cases}$$

Then the function g has the Baire property and $|g - f| \leq \eta$, so f has the Baire property as the uniform limit of a sequence of functions with the Baire property. This completes the proof of Lemma 1. $\square$

PROOF OF THEOREM 2. Since the space $(X \times Y, T_X \times T_Y)$ has a countable basis, it suffices to prove that the function f satisfies the condition (a) from the above Lemma 1. Fix a real $\eta > 0$ and a set $A \subset X \times Y$ with the Baire property and of the second category. There are nonempty open sets $U \in T_X$ and $V \in T_Y$ such that the set $(U \times V) \setminus A$ is of the first category.

Since (Y, T_Y) is a perfectly normal topological space and the sections f_x have closed graphs, by Dobos's theorem from [5] the sets of all discontinuity points $D(f_x)$ of the sections f_x are nowhere dense in Y. So, for each point $u \in U$ there are an open set $W(u) \subset V$ from a countable basis $\mathcal{B}(Y)$ of the space Y and an open interval $I(u) = (a(u), b(u))$ with rational endpoints such that

$$b(u) - a(u) < \frac{\eta}{2} \text{ and } f_u(W(u)) \subset I(u).$$

Since the set U is of the second category and the set of all pairs $(W(u), I(u))$ is countable, there are a nonempty open set $W \in T_Y$ and an open interval $I = (a, b)$ such that the set

$$C = \{u \in U; W(u) = W \text{ and } I(u) = I\}$$

is of the second category. Let $S \in T_X$ be a nonempty set such that each subset $E \subset S \setminus C$ with the Baire property is of the first category. Let $[c, d] = J \supset cl(I) = [a, b]$ be a closed interval such that

$$c < a, \quad d > b \text{ and } d - c < \eta.$$

We will prove that the set

$$H = (S \times W) \setminus f^{-1}(J)$$

is of the first category. Of course, if H is of the second category, then for each point

$$u \in Pr_X(H) = \{u \in S; \exists_{v \in W}(u, v) \in H\}$$

there is an open set $K(u) \in \mathcal{B}(Y)$ such that

$$f_u(K(u)) \subset \mathcal{R} \setminus J \text{ and } K(u) \subset W.$$

Consequently, there is a nonempty set $K \in T_Y$ such that the set

$$M = \{u \in Pr_X(H); K(u) = K\}$$

is of the second category. Fix a point $y \in K$ and consider the section f^y. Since for $x \in C$ we have $f(x, y) \in I$ and for $x \in M$ we have $f(x, y) \in \mathcal{R} \setminus J$, we obtain a contradiction with the Baire property of the section f^y. So the set H is of the first category. The set

$$B = A \cap [(S \times W) \setminus H] = A \cap (S \times W) \cap f^{-1}(J)$$

has the Baire property and it is of the second category. Moreover,

$$B \subset A \text{ and } osc_B f \leq \eta,$$

so by Lemma 1 our theorem is proved. □

Remember that a function $h : X \to \mathcal{R}$ is cliquish at a point $x \in X$ if for each real $\eta > 0$ and for each set $U \in T_X$ containing x there is a nonempty set $V \subset U$ belonging to T_X such that $osc_V f < \eta$ ([2]).

Theorem 3. *Let (X, T_X) and (Y, T_Y) be topological spaces satisfying the second countability axiom such that (Y, T_Y) is a Baire space. If all sections f_x, $x \in X$, of a function $f : X \times Y \to \mathcal{R}$ have closed graphs and all sections f^y, $y \in Y$, are cliquish at all points $x \in X$ then f is cliquish at each point $(x, y) \in X \times Y$.*

PROOF. Fix a nonempty set $A \in T_X \times T_Y$ and a positive real η. Let $U \in T_X$ and $V \in T_Y$ be nonempty sets such that $U \times V \subset A$. Let $\mathcal{B}(X)$ be a countable basis in the space (X, T_X). Since the sections f^y, $y \in Y$, are cliquish, for each point $y \in Y$ there are a set $U(y) \in \mathcal{B}(X)$ and an open interval $I(y) = (a(y), b(y))$ with rational endpoints such that

$$b(y) - a(y) < \eta \text{ and } f^y(U(y)) \subset I(y) \text{ and } U(y) \subset U.$$

(Y, T_Y) is a Baire space, so the set V is of the second category. Consequently, there are a nonempty set $W \in T_X$ and an open interval $I = (a, b)$ such that the set

$$C = \{y \in V; U(y) = W \text{ and } I(y) = I\}$$

is of the second category. There is a nonempty set $S \in T_Y$ in which the set $S \cap C$ is dense in S. For $x \in W$ the restrictions f_x/S have closed graphs (in $S \times \mathcal{R}$) and are bounded on $S \cap C$, so they are continuous on S. From this

$$f(W \times S) \subset cl(I) = [a, b] \text{ and } osc_{W \times S} f \leq b - a < \eta$$

and the proof is complete. $\square$

Theorem 4. *Let $(X, T_X) = (Y, T_Y) = (\mathcal{R}, T_e)$ and let $f : \mathcal{R}^2 \to \mathcal{R}$ be a function such that the sections f_x, $x \in \mathcal{R}$, have closed graphs and the sections f^y, $y \in \mathcal{R}$, are measurable (in the Lebesgue sense). Then f is measurable (in the Lebesgue sense) as the function of two variables.*

In the proof of this theorem we will apply the density topology T_d. For this denote by μ_e the outer Lebesgue measure in $\mathcal{R}$ and remember ([3]) that $x \in \mathcal{R}$ is said to be an outer density point of a set $A \subset \mathcal{R}$ if

$$\lim_{h \to 0^+} \frac{\mu_e([x - h, x + h] \cap A)}{2h} = 1.$$

If A is a Lebesgue measurable set then an outer density point x of A is said to be a density point of A. The family

$$T_d = \{A \subset \mathcal{R}; A \text{ is measurable and if } x \in A \text{ then } x \text{ is a density point of } A\}$$

is a topology called the density topology ([3, 8]).

Moreover in this proof we will apply the following Davies's Lemma from [4] (Lemma 2).

Lemma 2. *Let $(X, \mathcal{M}, v)$ be a finite measure space, and let $f : X \to \mathcal{R}$ be such that for every $\varepsilon > 0$ the class*

$$\mathcal{D}_\varepsilon = \{D \in \mathcal{M} : oscf \leq \varepsilon \text{ on } D\}$$

satisfies the following condition: for every set $A \in \mathcal{M}$ of positive v-measure, there exists a set $D \in \mathcal{D}_\varepsilon$ such that $D \subset A$ and $v(D) > 0$. Then f is $\bar{v}$-measurable (where $\bar{v}$ is the completion of v).

Remark 1. *Evidently the above Davies's lemma is also true for every σ-finite measure v.*

PROOF OF THEOREM 4. We will prove that for each Lebesgue measurable set $A \subset \mathcal{R}^2$ of positive Lebesgue measure and for each positive real η there is a Lebesgue measurable set $B \subset A$ of positive Lebesgue measure $\mu_2(B) > 0$ on which $osc_B f \leq \eta$. By Davies's Lemma 2 this condition implies the measurability of the function f. Let $A \subset \mathcal{R}^2$ be a Lebesgue measurable set with $\mu_2(A) > 0$ and let $\eta > 0$ be a real. As is known (compare Saks [7], pp. 130–131) there is a measurable set $E \subset A$ such that all sections

$$E_x = \{y; (x,y) \in E\} \in T_d \text{ and } \mu_2(A \setminus E) = 0.$$

For each point $(x,y) \in E$ the restriction $f_x/cl(E_x)$ of the section f_x to the closure $cl(E_x)$ of the section E_x has a closed graph, and by Dobos's theorem from [5], the set $D(f/cl(E_x))$ of all discontinuity points of the restriction $f/cl(E_x)$ is nowhere dense in $cl(E_x)$. So for each point $(x,y) \in E$ there are an open interval $I(x,y)$ with rational endpoints and a closed interval $J(x,y)$ with rational endpoints and of length $|J(x,y)| < \frac{\eta}{2}$ such that

$$I(x,y) \cap cl(E_x) \neq \emptyset \text{ and } f_x(I(x,y) \cap cl(E_x)) \subset J(x,y).$$

Let $(I_n)_n$ be an enumeration of all open intervals with rational endpoints and let $(J_n)_n$ be a sequence of all closed intervals with rational endpoints and of the length $|J_n| < \frac{\eta}{2}$. For $n, m = 1, 2, \ldots$ let

$$A_{n,m} = \{(x,y) \in E; I(x,y) = I_n \text{ and } J(x,y) = J_m\}.$$

Since the set of all pairs of intervals with rational endpoints is countable and $\mu_2(E) > 0$, there is a pair (I, J) of intervals such that the set

$$H = \{(x,y) \in E; I(x,y) = I \text{ and } J(x,y) = J\}$$

is not of measure zero. Consequently,

$$\mu_e(Pr_X(H) = \{x; \exists_y (x,y) \in H\}) > 0,$$

and there is a nonempty set $K \in T_d$ such that each measurable set $L \subset K \setminus Pr_X(H)$ is of measure zero. Evidently the set

$$G = E \cap (K \times I)$$

is Lebesgue measurable and by Fubini's theorem $\mu_2(G) > 0$. Let $M \subset G$ be a measurable set such that all sections

$$M_x \in T_d \text{ and } \mu_2(G \setminus M) = 0.$$

If $(x, y) \in M$ is a point such that $f(x, y) \in \mathcal{R} \setminus J$ then there is an open interval $I'(x, y) \subset I$ with rational endpoints such that

$$I'(x, y) \cap cl(M_x) \neq \emptyset \text{ and } f_x(I'(x, y) \cap cl(M_x)) \subset \mathcal{R} \setminus J.$$

Of course, this is evident if $f_x/cl(M_x)$ is continuous at y. In the case where $f_x/cl(M_x)$ is discontinuous at y it suffices to observe that the set $D(f_x/cl(M_x))$ of all discontinuity points of $f_x/cl(M_x)$ is nowhere dense in $cl(M_x)$ and for each sequence of points

$$y_n \in cl(M_x) \setminus D(f_x/cl(M_x)) \setminus \{y\}$$

converging to y and such that the corresponding sequence $(f_x(y_n))$ has a limit (finite or infinite) we have

$$\lim_{n \to \infty} f_x(y_n) \in \{f_x(y), \infty, -\infty\}.$$

If the set

$$M \setminus f^{-1}(J)$$

is not of measure zero, then there is an open interval $P \subset I$ such that the set

$$M_1 = \{(x, y) \in M; f(x, y) \in \mathcal{R} \setminus J \text{ and } I'(x, y) = P\}$$

is not of measure zero. So there is a nonempty set $S \in T_d$ such that $S \subset K$ and such that each measurable subset $L \subset S \setminus Pr_X(M_1)$ is of measure zero. Then for $y \in P$ such that $\mu_e(M^y) > 0$ we obtain

$$f^y(S \cap Pr_X(H)) \subset J \text{ and } f^y(S \cap Pr_X(M_1)) \subset \mathcal{R} \setminus J,$$

a contradiction with the measurability of the section f^y. This contradiction completes the proof. □

If (X, ρ_X) and (Y, ρ_Y) are separable metric space then each function $f : X \times Y \to \mathcal{R}$ having the continuous sections f^y, $y \in Y$, and the sections f_x, $x \in X$, with the closed graphs must be of the second Baire class (since the sections f^y are of the first Baire class ([6])). However there are nonborelien functions $f : \mathcal{R}^2 \to \mathcal{R}$ such that all their sections f_x and f^y, $x, y \in \mathcal{R}$, have closed graphs.

Example 1. *For* $x \in \mathcal{R} \setminus \{0\}$ *let*

$$g(x) = \frac{1}{|x|}.$$

Let

$$\mathcal{R} = A \cup B, \quad \text{where } \ A \cap B = \emptyset \ \text{ and } \ A \ \text{ is not borelien.}$$

For $(x, y) \in \mathcal{R}^2$ *let's put*

$$f(x,y) = \begin{cases} g(x-y) & if \quad x - y \neq 0 \\ 1 & if \quad x \in A \text{ and } x - y = 0 \\ 2 & if \quad x \in B \text{ and } x - y = 0. \end{cases}$$

Then f *is a nonborelien function but the sections* f_x *and* f^y*,* $x, y \in \mathcal{R}$*, have closed graphs.*

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