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SETS WHOSE HAUSDORFF MEASURE EQUALS METHOD I OUTER MEASURE

Abstract

A set $E \subseteq \mathbb{R}^n$ is s-straight for $s > 0$ if E has finite Method II outer
s-measure equal to its Method I outer s-measure. If E is Method II
s-measurable this means E has finite Hausdorff s-measure equal to its
Hausdorff s-content. Here we make a first study of such sets, following their 1995 introduction by Foran. Primary facts are proved about
subsets, intersections, unions, and some mappings of s-straight s-sets.
Basic examples of 1-straight and countable unions of 1-straight 1-sets
are constructed from line segments. It is noted that self-similar s-sets
are s-straight. Verifying a conjecture of Foran, the circle is proved to
be the countable union of perfect 1-straight 1-sets along with a set of
Hausdorff 1-measure zero. Such perfect sets are then further examined.
Also examined are subsets of 1-straight sets E maximal in the sense that
their Hausdorff 1-measure equals the diameter of E.

1 Introduction

In [8], Foran introduced the notion of an s-straight set in $\mathbb{R}^n$. (See Definition
1.2 below.) A countable union of s-straight sets is then naturally called σs-
straight. These ideas were developed further in [2], and here we present a first
study of such sets. In section 2, two criteria for a finite union of s-straight
s-sets to be s-straight are established, and primary facts are proved about
subsets, intersections, unions, and some mappings of s-straight s-sets. In the
first part of section 3, it is noted that self-similar s-sets are s-straight. In the
second part of section 3, basic examples of 1-straight and $\sigma 1$-straight 1-sets

are constructed from line segments, and further results are proved for such sets. In the first part of section 4, through a detailed construction, the unit circle is proved $\sigma 1$-straight as the countable union of perfect 1-straight 1-sets along with a set of Hausdorff 1-measure zero, verifying a conjecture posed in [8]. In the second part of section 4, such perfect 1-straight 1-sets are further examined.

Note that in [3], graphs of convex functions $f : [a, b] \to \mathbb{R}$ are shown to be $\sigma 1$-straight. That result is then further extended in [4] to graphs of continuously differentiable, absolutely continuous, and increasing continuous functions. Finally, in [5], we prove a general result which implies that every s-set is σs-straight.

Let d be the standard distance function on $\mathbb{R}^n$ where $n \geq 1$. The diameter of an arbitrary nonempty set $U \subseteq \mathbb{R}^n$ is defined by $|U| = \sup\{d(x, y) : x, y \in U\}$, with $|\emptyset| = 0$. Given $0 < \delta \leq \infty$, let C_δ^n represent the collection of subsets of $\mathbb{R}^n$ with diameter less than δ.

Definition 1.1. For $s > 0$ and $E \subseteq \mathbb{R}^n$, let

$$\text{s-}m_\delta^*(E) = \inf\left\{\sum |E_i|^s : E \subseteq \bigcup E_i \text{ where } E_i \in C_\delta^n \text{ for } i = 1, 2, ...\right\}.$$

Define $\text{s-}m_I^*(E) = \text{s-}m_\infty^*(E)$ and $\text{s-}m_{II}^*(E) = sup_{\delta>0} \text{ s-}m_\delta^*(E)$. The outer measure $\text{s-}m_I^*(E)$ is constructed by what is called Method I. The outer measure $\text{s-}m_{II}^*(E)$ is constructed by what is called Method II, and when restricted to the σ-field of $\text{s-}m_{II}^*$-measurable sets is called Hausdorff s-measure, or $\mathcal{H}^s$-measure. A set $E \subseteq \mathbb{R}^n$ is called an s-set if it is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$.

Since for any $0 < \alpha < \beta \leq \infty$ it follows that $\text{s-}m_\beta^*(E) \leq \text{s-}m_\alpha^*(E)$, we always have $\text{s-}m_I^*(E) \leq \text{s-}m_{II}^*(E)$. Also, $\text{s-}m_{II}^*$ is a metric outer measure, Borel sets are $\mathcal{H}^s$-measurable, and $\mathcal{H}^s$-measure is Borel regular. (See [7, 2.10.2 (1)], and [10, p. 9 and pp. 26-40] for details.) This paper studies sets of finite measure for which the last inequality is in fact an equality.

Definition 1.2. [2] Define $E \subseteq \mathbb{R}^n$ to be s-straight if $\text{s-}m_I^*(E) = \text{s-}m_{II}^*(E) < \infty$. A set which is the countable union of s-straight sets is called σs-straight. When E is $\mathcal{H}^s$-measurable, we write this definition more cleanly as

$$\mathcal{H}_\infty^s(E) = \mathcal{H}^s(E) < \infty.$$

2   Basic Results

In general, we will consider s-sets in $\mathbb{R}^n$ for $s > 0$, and often only 1-sets. However, an s-straight set need not be an s-set, as it may have zero $\mathcal{H}^s$-measure. This follows from Theorem 2.1 of Foran, which provides a useful

equivalent definition of an s-straight set that does not require the calculation of s-m_I^*. Henceforth we will often use this result without reference. We include the proof for completeness.

Theorem 2.1. *[8, p. 733]. Let $E \subseteq \mathbb{R}^n$ satisfy s-$m_{II}^*(E) < \infty$. Then, E is s-straight if and only if s-$m_{II}^*(A) \leq |A|^s$ for each s-m_{II}^*-measurable $A \subseteq E$. This last condition can be written $\mathcal{H}^s(A) \leq |A|^s$. In particular, sets of zero $\mathcal{H}^s$-measure are s-straight.*

PROOF. (Based on [8, p. 733].) On the one hand, suppose for each s-m_{II}^*-measurable $A \subseteq E$ it follows that s-$m_{II}^*(A) \leq |A|^s$. Then

$$\text{s-}m_{II}^*(E) \geq \text{s-}m_I^*(E) = \inf \left\{ \sum |E_i|^s : E = \bigcup E_i \right\}$$
$$\geq \inf \left\{ \sum \text{s-}m_{II}^*(E_i) : E = \bigcup E_i \right\} \geq \text{s-}m_{II}^*(E),$$

where the infima are over countable covers $\{E_i\}$ of E. So it follows that s-$m_I^*(E) = $ s-$m_{II}^*(E) < \infty$. Hence E is s-straight. Conversely, suppose s-$m_I^*(E) = $ s-$m_{II}^*(E) < \infty$. If there were an s-m_{II}^*-measurable subset $A \subseteq E$ such that s-$m_{II}^*(A) > |A|^s$, then since s-$m_I^*(E \backslash A) \leq $ s-$m_{II}^*(E \backslash A) \leq $ s-$m_{II}^*(E) < \infty$, we have

$$\text{s-}m_{II}^*(E) = \text{s-}m_{II}^*(A) + \text{s-}m_{II}^*(E \backslash A) > |A|^s + \text{s-}m_I^*(E \backslash A)$$
$$\geq \text{s-}m_I^*(A) + \text{s-}m_I^*(E \backslash A) \geq \text{s-}m_I^*(E),$$

contradicting the assumption that s-$m_I^*(E) = $ s-$m_{II}^*(E)$. $\square$

We can now begin to catalog what s-sets are s-straight. Note that if $A \subseteq \mathbb{R}^n$ is unbounded, then $|A| \geq M$ for any $M > 0$, so the condition $\mathcal{H}^s(A) \leq |A|^s$ is trivially satisfied. Subsequent proofs therefore need only consider bounded subsets.

Theorem 2.2. *If $E \subseteq \mathbb{R}^n$ and $\mathcal{H}^n(E) < \infty$, then E is n-straight.*

PROOF. Let $A \subseteq E$ be bounded and $\mathcal{H}^n$-measurable. By Theorem 2.1, we must show that $\mathcal{H}^n(A) \leq |A|^n$. Denote n-dimensional Lebesgue measure by λ^n. By the isodiametric inequality [6, p. 13], we have $\lambda^n(A) \leq c_n |A|^n$, where $c_n = \pi^{n/2}/2^n(n/2)!$ which depends only upon n. It is well-known [6, p. 13] that $\lambda^n(A) = c_n \mathcal{H}^n(A)$. Thus $\mathcal{H}^n(A) = \frac{1}{c_n}\lambda^n(A) \leq |A|^n$, as desired. $\square$

So, only when $s < n$ might there exist s-sets which are not s-straight, as in Example 2.3, in which $s = 1$ and $n = 2$.

Example 2.3. *A semicircle $E \subseteq \mathbb{R}^2$ of diameter 1 is not 1-straight, since* $\mathcal{H}^1_\infty(E) = 1 < \frac{\pi}{2} = \mathcal{H}^1(E)$.

In [8], Corollary 2.4 appears without proof. For completeness, we provide two proofs here.

Corollary 2.4. *[8, p. 734]. Every $\mathcal{H}^s$-measurable subset A of an s-straight s-set $E \subseteq \mathbb{R}^n$ is s-straight. In particular, intersections of s-straight s-sets are s-straight.*

PROOF. Let $A \subseteq E$ be $\mathcal{H}^s$-measurable. We prove this result in two different ways. First by Theorem 2.1, since E is an s-straight s-set, for any $\mathcal{H}^s$-measurable set $B \subseteq A \subseteq E$ it follows that $\mathcal{H}^s(B) \leq |B|^s$. So, A is also s-straight. A second, direct proof which does not require Theorem 2.1 is as follows. By definition, s-$m_I^*(E) =$ s-$m_{II}^*(E) < \infty$. Since for all $U \subseteq \mathbb{R}^n$ it follows that s-$m_I^*(U) \leq$ s-$m_{II}^*(U)$, in particular s-$m_I^*(A) \leq$ s-$m_{II}^*(A) < \infty$. For the reverse inequality, since A is $\mathcal{H}^s$-measurable, and s-$m_I^*(E \backslash A) \leq$ s-$m_{II}^*(E \backslash A) < \infty$, we have s-$m_{II}^*(A) =$ s-$m_{II}^*(E) -$ s-$m_{II}^*(E \backslash A) =$ s-$m_I^*(E) -$ s-$m_{II}^*(E \backslash A) \leq$ s-$m_I^*(A) - ($s-$m_{II}^*(E \backslash A) -$ s-$m_I^*(E \backslash A)) \leq$ s-$m_I^*(A)$. So, s-$m_I^*(A) =$ s-$m_{II}^*(A) < \infty$, and A is s-straight.   □

Remark 2.5. *We make no further use of the s-m_I^* or s-m_{II}^* notation in this paper.*

Corollary 2.6. *[8, p. 734]. Let $E \subseteq \mathbb{R}^n$ be an $\mathcal{H}^s$-measurable s-straight s-set. For $a \in \mathbb{R}^n$ and $k > 0$, let $E + a = \{x + a : x \in E\}$ and $kE = \{kx : x \in E\}$. Then each translation $E + a$ and each dilation kE is an s-straight s-set. Similarly, rotations and reflections of E are s-straight s-sets.*

PROOF. By [9, p. 57], for any bounded $\mathcal{H}^s$-measurable subset $A \subseteq E$, it follows that $\mathcal{H}^s(A + a) = \mathcal{H}^s(A)$, $|A + a| = |A|$, $\mathcal{H}^s(kA) = k^s \cdot \mathcal{H}^s(A)$, and $|kA| = k \cdot |A|$. Since E is s-straight, $\mathcal{H}^s(A) \leq |A|^s$. Then $\mathcal{H}^s(A+a) = \mathcal{H}^s(A) \leq |A|^s = |A + a|^s$, and $\mathcal{H}^s(kA) = k^s \cdot \mathcal{H}^s(A) \leq k^s \cdot |A|^s = (k \cdot |A|)^s = |kA|^s$. So, $E + a$ and kE are s-straight. The arguments for rotations and reflections of E are similar.   □

As we will see in section 3, a union of s-straight s-sets need not be s-straight. However, it is true that an increasing union of s-straight s-sets is s-straight.

Theorem 2.7. *Let each $E_i \subseteq \mathbb{R}^n$, for $i = 1, 2, \ldots$, be an s-straight s-set.*
(i) If $E_1 \subseteq E_2 \subseteq \cdots$, and $E = \bigcup_{i=1}^\infty E_i$ with $\mathcal{H}^s(E) < \infty$, then E is s-straight.
(ii) The set $\underline{\lim} E_i = \bigcup_{k=1}^\infty \left[\bigcap_{i>k} E_i \right]$ is s-straight.

PROOF. (i) Let $A \subseteq E$ be $\mathcal{H}^s$-measurable and bounded, and let $A_i = A \cap E_i$. Since the E_i are increasing, for each $k = 1, 2, \dots$ it follows that $\bigcup_{i=1}^{k} A_i \subseteq E_k$. By Corollary 2.4 each $\bigcup_{i=1}^{k} A_i$ is therefore s-straight, so $\mathcal{H}^s(\bigcup_{i=1}^{k} A_i) \leq |\bigcup_{i=1}^{k} A_i|^s$. Then, since the sequence $\left\{\bigcup_{i=1}^{k} A_i\right\}_{k=1}^{\infty}$ is increasing, by the continuity of $\mathcal{H}^s$-measure we have

$$
\begin{aligned}
\mathcal{H}^s(A) =& \mathcal{H}^s\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathcal{H}^s\left(\lim_{k\to\infty} \bigcup_{i=1}^{k} A_i\right) = \lim_{k\to\infty} \mathcal{H}^s\left(\bigcup_{i=1}^{k} A_i\right) \\
&\leq \lim_{k\to\infty} \left|\bigcup_{i=1}^{k} A_i\right|^s \leq \left|\lim_{k\to\infty} \bigcup_{i=1}^{k} A_i\right|^s = \left|\bigcup_{i=1}^{\infty} A_i\right|^s = |A|^s.
\end{aligned}
$$

Since A is arbitrary, E is s-straight. (ii) The second statement follows from (i) since $\bigcap_{i>1} E_i \subseteq \bigcap_{i>2} E_i \subseteq \cdots$, and by Corollary 2.4 intersections of s-straight s-sets are s-straight. $\square$

Theorem 2.8 employs a standard argument.

Theorem 2.8. *Let $E \subseteq \mathbb{R}^n$ be an s-set. Every $\mathcal{H}^s$-measurable subset of positive $\mathcal{H}^s$-measure of E contains an s-straight set of positive $\mathcal{H}^s$-measure if and only if E is σs-straight.*

PROOF. Suppose E is an s-set. Then $\mathcal{H}^s(E) > 0$. Assume there exists an s-straight subset E_0 of E with $\mathcal{H}^s(E_0) > 0$. Let α be an ordinal number, and suppose for every ordinal number $\beta < \alpha$, an s-straight subset $E_\beta \subseteq E \setminus \bigcup_{\gamma < \beta} E_\gamma$ has been chosen with $\mathcal{H}^s(E_\beta) > 0$. If $\mathcal{H}^s(E \setminus \bigcup_{\beta < \alpha} E_\beta) = 0$, let $E_\alpha = E \setminus \bigcup_{\beta < \alpha} E_\beta$. Otherwise choose an s-straight subset $E_\alpha \subseteq E \setminus \bigcup_{\beta < \alpha} E_\beta$ such that $\mathcal{H}^s(E_\alpha) > 0$. Because $\mathcal{H}^s(E) < \infty$, at most a countable number of nonempty sets E_α can exist. Relabeling them as sets E_k, we have $\mathcal{H}^s(E \setminus \bigcup_{k=0}^{\infty} E_k) = 0$. Hence, $E = [\bigcup_{k=0}^{\infty} E_k] \cup [E \setminus \bigcup_{k=0}^{\infty} E_k]$ is by definition σs-straight. Conversely, suppose $E = \bigcup_{i=1}^{\infty} E_i$ where each E_i is s-straight. Let $A \subseteq E$ be a bounded $\mathcal{H}^s$-measurable set such that $\mathcal{H}^s(A) > 0$. Since each E_i is s-straight, by Corollary 2.4 the set $A \cap E_i$ is s-straight for each i. Finally, because $\mathcal{H}^s(A) = \mathcal{H}^s(\bigcup_{i=1}^{\infty}(A \cap E_i)) > 0$, at least one such $A \cap E_i \subseteq A$ must have positive $\mathcal{H}^s$-measure. $\square$

In general, if finitely many s-straight s-sets in $\mathbb{R}^n$ are separated by a large enough distance, then their union will be s-straight. This is roughly because separation increases diameter but not $\mathcal{H}^s$-measure, suggesting Definition 2.9.

Definition 2.9. Let $E_1, \ldots, E_m \subseteq \mathbb{R}^n$. Let $d(E_j, E_k) = \inf\{d(x,y) : x \in E_j,$ $y \in E_k\}$. We say that $E_1, \ldots, E_m$ are s-separated if for $j, k \in \{1, 2, \ldots, m\}$ we have

$$\frac{2}{m(m-1)} \cdot \sum_{j<k} [d(E_j, E_k)]^s \geq \sum_{i=1}^m |E_i|^s$$

Definition 2.10, first appearing in [4], provides another, finer condition which will also guarantee that a finite union of s-straight s-sets is s-straight.

Definition 2.10. [4] Let $E_1, \ldots, E_m \subseteq \mathbb{R}^n$. We say that $E_1, \ldots, E_m$ are s-aligned if for each bounded subset $A \subseteq \bigcup_{i=1}^m E_i$ we have $|A|^s \geq \sum_{i=1}^m |A \cap E_i|^s$.

Theorem 2.11. *Let $E_1, \ldots, E_m \subseteq \mathbb{R}^n$ be s-straight s-sets. Consider the statements:*
(i) $E_1, \ldots, E_m$ are s-separated.
(ii) $E_1, \ldots, E_m$ are s-aligned.
(iii) $E = \bigcup_{i=1}^m E_i$ is s-straight.
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). In general, (iii) $\not\Rightarrow$ (i),(ii), and (ii) $\not\Rightarrow$ (i).

PROOF. Let $A \subseteq \bigcup_{i=1}^m E_i$ be bounded. Write $A = \bigcup_{i=1}^m A_i$, where $A_i = A \cap E_i$ for each $i \in \{1, 2, \ldots, m\}$. Each $A_i \subseteq E_i$ is s-straight by Corollary 2.4, so in particular, $\mathcal{H}^s(A_i) \leq |A_i|^s$.
$(i) \Rightarrow (ii)$: Now, for each pair (j, k) where $j, k \in \{1, 2, \ldots, m\}$ and $j < k$, we have $d(E_j, E_k) = \inf\{d(x,y) : x \in E_j, y \in E_k\} \leq \inf\{d(x,y) : x \in A_j, y \in A_k\} = d(A_j, A_k) \leq \sup\{d(x,y) : x, y \in A_j \cup A_k\} = |A_j \cup A_k| \leq |A|$. So, since the number of pairs (j, k) where $j, k \in \{1, 2, \ldots, m\}$ and $j < k$ is $\frac{m(m-1)}{2}$, we conclude that

$$\sum_{i=1}^m |A \cap E_i|^s = \sum_{i=1}^m |A_i|^s \leq \sum_{i=1}^m |E_i|^s \leq \frac{2}{m(m-1)} \cdot \sum_{j<k} [d(E_j, E_k)]^s$$
$$\leq \frac{2}{m(m-1)} \cdot \sum_{j<k} [d(A_j, A_k)]^s \leq |A|^s,$$

where the last inequality follows from the elementary fact that if a real number $|A|^s = x$ satisfies $x \geq y_i$ for each $i \in \{1, 2, \ldots, p\}$, then $x \geq \frac{1}{p} \sum_{i=1}^p y_i$. Since A was arbitrary, $E_1, \ldots, E_m$ are therefore s-aligned.
$(ii) \Rightarrow (iii)$: The union E will be s-straight if for each bounded $\mathcal{H}^s$-measurable subset $A \subseteq E$ it follows that $\mathcal{H}^s(A) \leq |A|^s$. In fact,

$$\mathcal{H}^s(A) = \mathcal{H}^s\left(\bigcup_{i=1}^m A_i\right) \leq \sum_{i=1}^m \mathcal{H}^s(A_i) \leq \sum_{i=1}^m |A_i|^s \leq |A|^s,$$

where the last inequality follows from the definition of s-aligned. Since A was arbitrary, $E = \bigcup_{i=1}^{m} E_i$ is therefore s-straight.

Finally, Example 3.16 will show that the implication $(ii) \Rightarrow (i)$ fails in general. Example 4.8 will show that there exist 1-straight 1-sets E_1, E_2 whose union is 1-straight, but which are neither 1-separated nor 1-aligned, so that the implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ also fail in general. $\qquad\square$

3 Examples of s-Straight s-Sets

3.1 Self-similar s-Sets are s-Straight

In [8, p. 736], Foran notes that self-similar s-sets are s-straight. We include the argument here for completeness.

Definition 3.1. See [6, pp. 119-122] and [9, p. 65]. A function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is called a contraction if there exists $0 \leq c < 1$ such that for all $x, y \in \mathbb{R}^n$ it follows that $d(\psi(x), \psi(y)) \leq c \cdot d(x, y)$. The infimum of all such values c is called the contraction ratio for ψ. A function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is called a similitude if there exists $0 < c < 1$ such that for all $x, y \in \mathbb{R}^n$ it follows that $d(\psi(x), \psi(y)) = c \cdot d(x, y)$. Of course, a similitude is a particular case of a contraction. A set $A \subseteq \mathbb{R}^n$ is invariant for a finite collection of contractions $\psi_1, \ldots, \psi_m$ with contraction ratios $0 \leq r_j < 1$ for $j = 1, \ldots, m$, if $A = \bigcup_{j=1}^{m} \psi_j(A)$. In fact, given a finite collection of contractions, there exists a unique compact invariant set E. If these contractions are similitudes, and for some $t > 0$ both $\mathcal{H}^t(E) > 0$ and $\mathcal{H}^t(\psi_i(E) \cap \psi_k(E)) = 0$ for $i \neq k$, then we say that E is self-similar. The similarity dimension of this self-similar invariant set E is defined to be the unique $s > 0$ such that $\sum_{j=1}^{m} r_j^s = 1$.

Foran's note in [8] is an immediate corollary of the following theorem of Bandt and Graf [1].

Theorem 3.2. *[1, p. 1000]. If $\psi_1, \ldots, \psi_m$ is a collection of similitudes, s is the similarity dimension of the associated compact invariant set E, and E is self-similar (though not necessarily an s-set), then for any $\mathcal{H}^s$-measurable subset $B \subseteq E$, it follows that $\mathcal{H}^s(B) = \mathcal{H}^s_\infty(B)$.*

Corollary 3.3. *[8, p. 736]. Self-similar s-sets, where s is the associated similarity dimension, are s-straight.*

3.2 Constructing 1-Straight 1-Sets

Since the notion of an s-straight set is recent and no examples exist in the literature beyond those in [8] and [4], we prove some further results and con-

struct here some basic examples of 1-straight 1-sets in $\mathbb{R}^2$. For this purpose we will need a few standard definitions.

Definition 3.4. A (closed) line segment in $\mathbb{R}^n$ is the image under an isometry of a closed (non-degenerate) interval in $\mathbb{R}$. The length $\mathcal{L}(E)$ of a line segment E with endpoints x and y is defined by $\mathcal{L}(E) = |E| = d(x,y)$. Following [12, p. 197], an arc in $\mathbb{R}^n$ is defined to be the image of a homeomorphism $f : [0,1] \to \mathbb{R}^n$. In particular, an arc does not cross itself. A set $E \subseteq \mathbb{R}^n$ is said to be arc-connected if each pair of distinct points $x, y \in E$ is connected by an arc $\Lambda = f([0,1]) \subseteq E$ such that $f(0) = x$ and $f(1) = y$. By definition a line segment is both an arc and arc-connected. The length of an arc Λ is defined to be $\mathcal{L}(\Lambda) = \sup \sum_{i=1}^{m} d(f(t_{i-1}), f(t_i))$, where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0,1]$.

A well-known fact will be helpful.

Theorem 3.5. *[6, p. 29] If $\Lambda \subseteq \mathbb{R}^n$ for $n \geq 1$ is an arc, then $\mathcal{H}^1(\Lambda) = \mathcal{L}(\Lambda)$.*

The following Theorem is basic.

Theorem 3.6. *If $E \subseteq \mathbb{R}^n$ for $n \geq 1$ is a (non-degenerate) line segment, then $0 < |E| = \mathcal{L}(E) = \mathcal{H}^1(E) < \infty$, and E is a 1-straight 1-set.*

Proof. Since a set of one or two endpoints has measure zero and does not affect the calculation of diameter or length, E can be assumed to be closed. Let $n = 1$. If $E = [a,b] \subseteq \mathbb{R}$, then by the definitions of diameter, length, and the Lebesgue measure $\lambda(E)$ of an interval, $0 < b - a = |E| = \mathcal{L}(E) = \lambda(E) < \infty$. Also, for any such $E \subseteq \mathbb{R}$ it is well-known (see [10, p. 40] or [6, p. 13]) that $\mathcal{H}^1(E) = \lambda(E)$. So a line segment (interval) E in $\mathbb{R}$ is a 1-set. Let $A \subseteq E = [a,b]$ be $\mathcal{H}^1$-measurable. Let $\overline{A}$ be the smallest line segment (interval) in E containing A. Then $\mathcal{H}^1(A) \leq \mathcal{H}^1(\overline{A}) = |\overline{A}| = |A|$. Since A is arbitrary, $E = [a,b]$ is 1-straight. Suppose $n > 1$. A line segment $E \subseteq \mathbb{R}^n$, as the image of an isometry of a closed interval of $\mathbb{R}$, similarly satisfies $0 < |E| = \mathcal{L}(E) = \mathcal{H}^1(E) < \infty$, using Theorem 3.5 for the second equality. Let $A \subseteq E$ be $\mathcal{H}^1$-measurable. Let $\overline{A}$ be the smallest line segment in E containing A. Then $\mathcal{H}^1(A) \leq \mathcal{H}^1(\overline{A}) = |\overline{A}| = |A|$ using Theorem 3.5 again. Since A is arbitrary, E is 1-straight. Thus any line segment $E \subseteq \mathbb{R}^n$ for $n \geq 1$ is a 1-straight 1-set.   □

An easy and intuitive example that shows the union of two s-straight s-sets need not be s-straight is an angle.

Example 3.7. *There exist 1-straight 1-sets $E_1, E_2 \subseteq \mathbb{R}^2$ such that $E = E_1 \cup E_2$ is not 1-straight.*

PROOF. Suppose $E_1, E_2 \subseteq \mathbb{R}^2$ are non-overlapping line segments which share an endpoint and form a 90° angle. Suppose $|E_1| = |E_2| = 1$. Let $E = E_1 \cup E_2$. Then using Theorem 3.6, we have $\mathcal{H}^1(E) = \mathcal{H}^1(E_1) + \mathcal{H}^1(E_2) = |E_1| + |E_2| = 2 > \sqrt{2} = |E|$. So E is not 1-straight. $\qquad\square$

Also, one would intuitively expect the following theorem characterizing line segments to hold.

Theorem 3.8. *A set $E \subseteq \mathbb{R}^n$ is a bounded, arc-connected, 1-straight 1-set if and only if E is a (non-degenerate) line segment.*

PROOF. If E is a line segment, then it is bounded and arc-connected. By Theorem 3.6, it follows that E is a 1-straight 1-set. For the converse, suppose E is a bounded, arc-connected, 1-straight 1-set. Let $x, y \in E$, where $x \neq y$. Suppose the arc Λ connecting x and y is not a line segment. Since Λ is closed, there exist points $z_1, z_2 \in \Lambda$ such that $d(z_1, z_2) = |\Lambda|$. Now, $z_1 \neq z_2$ because $|\Lambda| \geq d(x, y) > 0$. Without loss of generality assume these points occur on Λ in the order x, z_1, z_2, y. First, suppose both $x = z_1$ and $y = z_2$. Because Λ is not degenerate, there exists a point $z_3 \in \Lambda$ with $z_3 \neq x$, $z_3 \neq y$. Since Λ is not a line segment, by the definition of length as a supremum and using Theorem 3.5 we have

$$\mathcal{H}^1(\Lambda) = \mathcal{L}(\Lambda) \geq d(x, z_3) + d(z_3, y) > d(x, y) = |\Lambda|.$$

So $\Lambda \subseteq E$ is not 1-straight, contradicting Corollary 2.4 that an $\mathcal{H}^1$-measurable subset of a 1-straight set is 1-straight. Next, suppose either $x \neq z_1$ (so $d(x, z_1) > 0$) or $y \neq z_2$ (so $d(z_2, y) > 0$). Then

$$\begin{aligned}
\mathcal{H}^1(\Lambda) = \mathcal{L}(\Lambda) &\geq d(x, z_1) + d(z_1, z_2) + d(z_2, y) \\
&= d(x, z_1) + |\Lambda| + d(z_2, y) > |\Lambda|,
\end{aligned}$$

again contradicting the fact that $\Lambda \subseteq E$ is 1-straight. Thus the arc Λ connecting a pair of distinct points in E is a line segment contained in E. Now, suppose E is not contained in a line. Then, for $x, y \in E$, $x \neq y$, there exists another point $z \in E$, $z \neq x, z \neq y$, which does not lie on the line containing x and y. Thus there exist two additional distinct line segments contained in E, one connecting x and z, and one connecting y and z. These three segments form a triangle contained in E with vertices x, y, z. But by the triangle inequality this triangle is not 1-straight, which again by Corollary 2.4 contradicts the fact that E is 1-straight. So E must be contained in a line. Since E is bounded and arc-connected, hence a connected subset of a line [12, p. 197], it is in fact a line segment, [11, p. 38]. Of course, as a 1-set E is non-degenerate. $\qquad\square$

Suppose a set E satisfies $\mathcal{H}^1(E) \leq |E|$, that is, E satisfies $0 \leq |E| - \mathcal{H}^1(E)$. It sometimes also happens that $|E| - \mathcal{H}^1(E) \leq |A| - \mathcal{H}^1(A)$ holds for some collection of $\mathcal{H}^1$-measurable subsets $A \subseteq E$. If so, these two facts together imply, for such particular subsets A, that $\mathcal{H}^1(A) \leq |A|$. Although insufficient to prove that E is 1-straight, this situation is helpful, and Theorem 3.9 specifies some conditions under which it holds. In Theorem 3.9, the expression $A \subseteq E$ occurs in the form $E_0 \cup E_1 \subseteq E_0 \cup E_2$.

Theorem 3.9. *If $E_0, E_1, E_2 \subseteq \mathbb{R}^n$, where E_0 is a 1-set, E_2 is a line segment, and $E_1 \subseteq E_2$ is a nonempty $\mathcal{H}^1$-measurable subset, then*

$$|E_0 \cup E_2| - \mathcal{H}^1(E_0 \cup E_2) \leq |E_0 \cup E_1| - \mathcal{H}^1(E_0 \cup E_1),$$

or equivalently,

$$|E_0 \cup E_2| + \mathcal{H}^1(E_0 \cup E_1) \leq |E_0 \cup E_1| + \mathcal{H}^1(E_0 \cup E_2).$$

PROOF. We prove the equivalent statement. Since $E_0 \cup E_2$ is $\mathcal{H}^1$-measurable and $E_1 \subseteq E_2$, we have

$$\begin{aligned}
\mathcal{H}^1(E_0 \cup E_2) &= \mathcal{H}^1((E_0 \cup E_2)\backslash(E_0 \cup E_1)) + \mathcal{H}^1((E_0 \cup E_2) \cap (E_0 \cup E_1)) \\
&= \mathcal{H}^1(E_2\backslash(E_0 \cup E_1)) + \mathcal{H}^1(E_0 \cup E_1).
\end{aligned} \qquad (*)$$

Since $E_0 \cup E_1 \subseteq E_0 \cup E_2$ it follows that both $\mathcal{H}^1(E_0 \cup E_1) \leq \mathcal{H}^1(E_0 \cup E_2)$ and $|E_0 \cup E_1| \leq |E_0 \cup E_2|$. If $|E_0 \cup E_1| = |E_0 \cup E_2|$, the conclusion of the Theorem follows. Let K be the closure of $E_0 \cup E_1$. Suppose $|E_0 \cup E_1| < |E_0 \cup E_2|$. So there exists $x_2 \in E_2\backslash K$ and x_0 in the closure of $E_0 \cup E_2$ such that $d(x_0, x_2) = |E_0 \cup E_2|$. There exists $x_1 \in K$ such that $d(x_1, x_2) = d(K, x_2)$. Let $x_1' \in E_2$ satisfy $d(x_1', x_2) = d(x_1, x_2)$, and let $E_2(x_1', x_2) \subseteq E_2\backslash K \subseteq E_2\backslash(E_0 \cup E_1)$ represent the open line segment between x_1' and x_2. Note that $\mathcal{H}^1(E_2(x_1', x_2)) = d(x_1', x_2)$. If on the one hand $x_0 \in K$, then $d(x_0, x_1) \leq |K| = |E_0 \cup E_1|$. Let x_1^* represent either x_1 or x_1' as appropriate. Then by the triangle inequality we have $|E_0 \cup E_2| = d(x_0, x_2) \leq d(x_0, x_1^*) + d(x_1^*, x_2) \leq |E_0 \cup E_1| + \mathcal{H}^1(E_2(x_1', x_2)) \leq |E_0 \cup E_1| + \mathcal{H}^1(E_2\backslash(E_0 \cup E_1))$. So

$$|E_0 \cup E_2| \leq |E_0 \cup E_1| + \mathcal{H}^1(E_2\backslash(E_0 \cup E_1)). \qquad (**)$$

If on the other hand $x_0 \in E_2\backslash K$, then there exists $x_3 \in K$ such that $d(x_0, x_3) = d(x_0, K)$. Let $x_3' \in E_2$ satisfy $d(x_0, x_3') = d(x_0, x_3)$. Then with notation as above, $E_2(x_0, x_3') \subseteq E_2\backslash(E_0 \cup E_1)$ and $\mathcal{H}^1(E_2(x_0, x_3')) = d(x_0, x_3')$. Also $d(x_3, x_1) \leq |E_0 \cup E_1|$. Note that $\mathcal{H}^1(E_2(x_0, x_3') \cap E_2(x_1', x_2)) = 0$. Let x_1^*, x_3^* represent x_1, x_3 or x_1', x_3' as appropriate. Then $|E_0 \cup E_2| = d(x_0, x_2) \leq$

$d(x_0, x_3^*) + d(x_3^*, x_1^*) + d(x_1^*, x_2) \leq \mathcal{H}^1(E_2(x_0, x_3')) + |E_0 \cup E_1| + \mathcal{H}^1(E_2(x_1', x_2)) \leq |E_0 \cup E_1| + \mathcal{H}^1(E_2 \backslash (E_0 \cup E_1))$. Again $(**)$ follows. Finally, adding $(*)$ (written in the reverse direction) to $(**)$ yields $|E_0 \cup E_2| + \mathcal{H}^1(E_0 \cup E_1) \leq |E_0 \cup E_1| + \mathcal{H}^1(E_0 \cup E_2)$, as desired. $\square$

Example 3.10 shows that Theorem 3.9 does not generalize to arbitrary 1-straight 1-sets E_2. In this and subsequent examples, we use the notation $[a, b]$ to represent closed intervals in $\mathbb{R}$ or on the x-axis in $\mathbb{R}^2$.

Example 3.10. *There exist a 1-set E_0, a 1-straight 1-set E_2, and a nonempty $\mathcal{H}^1$-measurable subset $E_1 \subseteq E_2$ such that*

$$|E_0 \cup E_2| + \mathcal{H}^1(E_0 \cup E_1) > |E_0 \cup E_1| + \mathcal{H}^1(E_0 \cup E_2).$$

PROOF. Define $E_0, E_1, E_2 \subseteq \mathbb{R}$ as follows. Let $E_0 = [0, 1]$, $E_1 = [1, 2]$, and $E_2 = E_1 \cup \{3\}$. By construction E_2 is not a line segment. But, E_2 is 1-straight since by Theorem 2.2 all 1-sets in $\mathbb{R}$ are 1-straight. Theorem 3.9 fails since $|E_0 \cup E_2| + \mathcal{H}^1(E_0 \cup E_1) = 3 + 2 > 2 + 2 = |E_0 \cup E_1| + \mathcal{H}^1(E_0 \cup E_2)$. $\square$

Now we prove a geometric condition, interesting in its own right (recalling the Besicovitch circle-pair [6, p. 40]), for the union of two non-overlapping line segments in $\mathbb{R}^n$ to be 1-straight. Later in Theorem 3.14 we generalize to the union of $m \geq 2$ non-overlapping line segments in $\mathbb{R}^n$.

Theorem 3.11. *Let $E_1, E_2 \subseteq \mathbb{R}^n$ be non-overlapping line segments. Then $E = E_1 \cup E_2$ is a 1-straight 1-set if and only if an endpoint of E_2 lies outside or on the boundary of the open region R, defined to be the common part of two open balls each of radius $|E_1| + |E_2|$ and centered at the endpoints of E_1.*

PROOF. Non-overlapping line segments intersect in at most one point. So $\mathcal{H}^1(E_1 \cap E_2) = 0$. Thus, by Theorem 3.6, we have $\mathcal{H}^1(E) = \mathcal{H}^1(E_1 \cup E_2) = \mathcal{H}^1(E_1) + \mathcal{H}^1(E_2) = |E_1| + |E_2|$. Also, $|E|$ is the distance between two endpoints out of the four in E_1 and E_2. So if E is 1-straight, then $\mathcal{H}^1(E) = |E_1| + |E_2| \leq |E|$. Therefore some endpoint of E_2 is at least $|E_1| + |E_2|$ units from some endpoint of E_1, namely outside or on the boundary of R. To prove the converse, assume an endpoint of E_2 lies outside or on the boundary of R. So $|E| \geq |E_1| + |E_2|$. Suppose the set $E = E_1 \cup E_2$ is not 1-straight. Then by the definition of 1-straight, there exists an $\mathcal{H}^1$-measurable $A \subseteq E$ such that $\mathcal{H}^1(A) > |A|$. Write $A = A_1 \cup A_2$, with $A_1 \subseteq E_1$ and $A_2 \subseteq E_2$. Since A_1 and A_2 are, 1-straight, if $A = A_1$ or $A = A_2$, then $\mathcal{H}^1(A) \leq |A|$ is immediate. So assume that each of A_1, A_2 is nonempty.
Step (1). By Theorem 3.9,

$$|A_2 \cup E_1| + \mathcal{H}^1(A_2 \cup A_1) \leq |A_2 \cup A_1| + \mathcal{H}^1(A_2 \cup E_1)$$

$$|A_2 \cup E_1| + \mathcal{H}^1(A) \leq |A| + \mathcal{H}^1(A_2 \cup E_1).$$

Since by assumption $\mathcal{H}^1(A) > |A|$, we conclude $|A_2 \cup E_1| < \mathcal{H}^1(A_2 \cup E_1)$.
Step (2). By another application of Theorem 3.9,

$$|E_1 \cup E_2| + \mathcal{H}^1(E_1 \cup A_2) \leq |E_1 \cup A_2| + \mathcal{H}^1(E_1 \cup E_2)$$
$$|E| + \mathcal{H}^1(E_1 \cup A_2) \leq |E_1 \cup A_2| + |E_1| + |E_2|.$$

Since $|A_2 \cup E_1| < \mathcal{H}^1(A_2 \cup E_1)$ holds by Step (1), we conclude that

$$|E| < |E_1| + |E_2|,$$

which is a contradiction. So $\mathcal{H}^1(A) \leq |A|$ as desired. Since A is an arbitrary $\mathcal{H}^1$-measurable subset of E, then E is a 1-straight 1-set. □

One direction of Theorem 3.11 does not generalize to arbitrary 1-straight 1-sets E_2, as Example 3.12 shows.

Example 3.12. *There exists a line segment $E_1 \subseteq \mathbb{R}^2$, and a 1-straight 1-set $E_2 \subseteq \mathbb{R}^2$ such that a point of E_2 lies outside or on the boundary of the open region R, defined to be the common part of two open balls each of radius $|E_1| + |E_2|$ and centered at the endpoints of E_1, and the set $E = E_1 \cup E_2$ is not 1-straight.*

Proof. Define $E_1, E_2 \subseteq \mathbb{R}^2$ as follows. Let $E_1 = [-\sqrt{2}, 0]$. Let $E_2 = \{(x,y) : y = \sqrt{3}x \text{ and } x \in [0, \frac{1}{2}]\} \cup [\frac{1}{3}(2\sqrt{2}+1), \sqrt{2}]$; that is, E_2 consists of the interval $[\frac{1}{3}(2\sqrt{2}+1), \sqrt{2}]$ together with the line segment having endpoints $(0,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. So, $|E_1| = |E_2| = \sqrt{2}$, and the point $(\sqrt{2}, 0)$ of E_2 lies on the boundary of the open region R defined to be the common part of two open balls of radius $2\sqrt{2}$ and centered at the endpoints of E_1. By construction E_2 is not a line segment. Let $A = \{(x,y) : y = \sqrt{3}x \text{ and } x \in [0, \frac{1}{2}]\}$. Observe that $d(A, [\frac{1}{3}(2\sqrt{2}+1), \sqrt{2}]) = \frac{1}{3}\sqrt{15 - 2\sqrt{2}} > \frac{1}{3}\sqrt{12} = \frac{2\sqrt{3}}{3} > \frac{2(1.72)}{3} = \frac{3.44}{3} > \frac{3.42}{3} = \frac{2+1.42}{3} > \frac{2+\sqrt{2}}{3} = 1 + \frac{1}{3}(\sqrt{2} - 1) = |A| + |[\frac{1}{3}(2\sqrt{2}+1), \sqrt{2}]|$. Then, by Theorem 2.11 it follows that E_2 is 1-straight. The set $E = E_1 \cup E_2$ is not 1-straight since every subset of E containing the angle at $(0,0)$ is not 1-straight by reasoning as in Example 3.7. □

Corollary 3.13 is natural and useful for examples.

Corollary 3.13. *Let $E = E_1 \cup E_2$, where E_1, E_2 are the closed line segments forming the shorter pair of opposite sides of a rectangle in $\mathbb{R}^2$ with side lengths $0 < a \leq b$. Then, E is 1-straight if and only if $\sqrt{3}a \leq b$. In particular, the union of a pair of opposite sides of a square is not 1-straight.*

PROOF. Note that $|E| = \sqrt{a^2 + b^2}$ and $|E_1| = |E_2| = a$. So, by Theorem 3.11 it follows that E is 1-straight if and only if $\sqrt{a^2 + b^2} \geq 2a$, that is, $b \geq \sqrt{3}a$. □

Since the diameter of a set of non-overlapping line segments is determined by endpoints, Theorem 3.11 can be restated "Let $E_1, E_2 \subseteq \mathbb{R}^n$ be non-overlapping line segments. The set $E = E_1 \cup E_2$ is a 1-straight 1-set if and only if $|E_1 \cup E_2| \geq |E_1| + |E_2|$." This suggests the following generalization.

Theorem 3.14. *Let $m \geq 2$, and $E_1, \ldots, E_m \subseteq \mathbb{R}^n$ be non-overlapping closed line segments. The set $E = \bigcup_{i=1}^m E_i$ is a 1-straight 1-set if and only if for every subset $J \subseteq \{1, 2, \ldots, m\}$ we have*

$$\left| \bigcup_{j \in J} E_j \right| \geq \sum_{j \in J} |E_j| = \mathcal{H}^1 \left(\bigcup_{j \in J} E_j \right).$$

If $E = \bigcup_{i=1}^m E_i$ is not 1-straight, then in particular $\left| \bigcup_{i=1}^m E_i \right| < \mathcal{H}^1 \left(\bigcup_{i=1}^m E_i \right)$.

PROOF. Note that by Theorem 3.6 since each E_i is a closed line segment, we have $\mathcal{H}^1(E_i) = |E_i|$. Suppose first that $E = \bigcup_{i=1}^m E_i$ is 1-straight. Then by Corollary 2.4 for every $J \subseteq \{1, 2, \ldots, m\}$ the subset $\bigcup_{j \in J} E_j$ is also 1-straight. So, since the E_i are non-overlapping we have

$$\left| \bigcup_{j \in J} E_j \right| \geq \mathcal{H}^1 \left(\bigcup_{j \in J} E_j \right) = \sum_{j \in J} \mathcal{H}^1 \left(E_j \right) = \sum_{j \in J} |E_j|.$$

To prove the converse, suppose $E = \bigcup_{i=1}^m E_i$ is not 1-straight. Then by the definition of 1-straight, there exists an $\mathcal{H}^1$-measurable $A \subseteq E$ such that $\mathcal{H}^1(A) > |A|$. Write $A \cap E_i = A_i$ so that $A = \bigcup_{i=1}^m A_i$. We prove in a series of m steps for the particular set $J = \{1, 2, \ldots, m\}$ that $|\bigcup_{i=1}^m E_i| < \sum_{i=1}^m |E_i|$. Step (1): By Theorem 3.9, since $\bigcup_{i \neq 1} A_i$ is a 1-set, E_1 is a line segment, $A_1 \subseteq E_1$, the E_i are non-overlapping, and $A = \bigcup_{i=1}^m A_i = \bigcup_{i \neq 1} A_i \cup A_1$, we have

$$\left| \bigcup_{i \neq 1} A_i \cup E_1 \right| + \mathcal{H}^1 \left(\bigcup_{i \neq 1} A_i \cup A_1 \right) \leq \left| \bigcup_{i \neq 1} A_i \cup A_1 \right| + \mathcal{H}^1 \left(\bigcup_{i \neq 1} A_i \cup E_1 \right)$$

$$\left| \bigcup_{i \neq 1} A_i \cup E_1 \right| + \mathcal{H}^1 \left(A \right) \leq \left| A \right| + \mathcal{H}^1 \left(\bigcup_{i \neq 1} A_i \right) + \left| E_1 \right|.$$

Since by assumption, $\mathcal{H}^1(A) > |A|$, we conclude

$$\left| \bigcup_{i \neq 1} A_i \cup E_1 \right| < \mathcal{H}^1 \left(\bigcup_{i \neq 1} A_i \right) + \left| E_1 \right|.$$

Step (2). By Theorem 3.9 again, we similarly have

$$\left| \Big[\bigcup_{i \neq 1,2} A_i \cup E_1 \Big] \cup E_2 \right| + \mathcal{H}^1 \Big(\Big[\bigcup_{i \neq 1,2} A_i \cup E_1 \Big] \cup A_2 \Big)$$

$$\leq \left| \Big[\bigcup_{i \neq 1,2} A_i \cup E_1 \Big] \cup A_2 \right| + \mathcal{H}^1 \Big(\Big[\bigcup_{i \neq 1,2} A_i \cup E_1 \Big] \cup E_2 \Big)$$

so that

$$\left| \Big[\bigcup_{i \neq 1,2} A_i \cup E_1 \Big] \cup E_2 \right| + \mathcal{H}^1 \Big(\bigcup_{i \neq 1} A_i \Big) + \Big| E_1 \Big|$$

$$\leq \left| \bigcup_{i \neq 1} A_i \cup E_1 \right| + \mathcal{H}^1 \Big(\bigcup_{i \neq 1,2} A_i \Big) + \sum_{i=1}^{2} \Big| E_1 \Big|.$$

Using the result of Step (1), we conclude that

$$\left| \bigcup_{i \neq 1,2} A_i \cup \bigcup_{i=1}^{2} E_i \right| < \mathcal{H}^1 \Big(\bigcup_{i \neq 1,2} A_i \Big) + \sum_{i=1}^{2} \Big| E_i \Big|.$$

Continue this process. At Step $(m-1)$ we conclude that

$$\left| A_m \cup \bigcup_{i=1}^{m-1} E_i \right| < \mathcal{H}^1 \Big(A_m \Big) + \sum_{i=1}^{m-1} \Big| E_i \Big|.$$

Step (m). By Theorem 3.9 a last time, we have

$$\left| \bigcup_{i=1}^{m-1} E_i \cup E_m \right| + \mathcal{H}^1 \Big(\bigcup_{i=1}^{m-1} E_i \cup A_m \Big)$$

$$\leq \left| \bigcup_{i=1}^{m-1} E_i \cup A_m \right| + \mathcal{H}^1 \Big(\bigcup_{i=1}^{m-1} E_i \cup E_m \Big)$$

so that $\left| \bigcup_{i=1}^{m} E_i \right| + \sum_{i=1}^{m-1} \Big| E_i \Big| + \mathcal{H}^1 \Big(A_m \Big) \leq \left| \bigcup_{i=1}^{m-1} E_i \cup A_m \right| + \sum_{i=1}^{m} \Big| E_i \Big|.$
Using the result of Step $(m-1)$, we conclude $\left| \bigcup_{i=1}^{m} E_i \right| < \sum_{i=1}^{m} |E_i|$, as desired. $\square$

The following equivalence theorem for a finite number of non-overlapping line segments then holds.

Theorem 3.15. *Let $m \geq 2$, and $E_1, \ldots, E_m \subseteq \mathbb{R}^n$ be non-overlapping closed line segments. The following statements are equivalent:*

(i) For every $J \subseteq \{1, 2, \ldots, m\}$, we have $\left| \bigcup_{j \in J} E_j \right| \geq \sum_{j \in J} |E_j| = \mathcal{H}^1 \left(\bigcup_{j \in J} E_j \right)$.

(ii) $E_1, \ldots, E_m$ are 1-aligned.

(iii) $E = \bigcup_{i=1}^m E_i$ is 1-straight.

PROOF. The equivalence $(i) \Longleftrightarrow (iii)$ follows from Theorem 3.14. The implication $(ii) \Rightarrow (iii)$ follows from Theorem 2.11. For the implication $(iii) \Rightarrow (ii)$, suppose $E_1, \ldots, E_m$ are not 1-aligned. Then by Definition 2.10 of 1-aligned, there exists $A \subseteq \bigcup_{i=1}^m E$ such that $|A| < \sum_{i=1}^m |A \cap E_i|$. Write $A_i = A \cap E_i$ for each i. Let $\overline{A_i} \subseteq E_i$ be the smallest line segment in E_i containing A_i. So, $\mathcal{H}^1\left(\overline{A_i}\right) = |\overline{A_i}| = |A_i|$. Then since E is 1-straight and the E_i are non-overlapping,

$$|A| = \left| \bigcup_{i=1}^m A_i \right| < \sum_{i=1}^m |A_i| = \sum_{i=1}^m |\overline{A_i}| = \sum_{i=1}^m \mathcal{H}^1\left(\overline{A_i}\right) = \mathcal{H}^1\left(\bigcup_{i=1}^m \overline{A_i}\right)$$

$$\leq \left| \bigcup_{i=1}^m \overline{A_i} \right| = \left| \bigcup_{i=1}^m A_i \right| = |A|,$$

which is a contradiction. So $E_1, \ldots, E_m$ are 1-aligned, as desired. $\square$

Example 4.8 will show that the implication $(iii) \Rightarrow (ii)$ of Theorem 3.15 may fail if the m sets are not a collection of closed non-overlapping line segments. Now we exhibit three examples of interest. First, Example 3.16 shows why the converse implication $(ii) \Rightarrow (i)$ in Theorem 2.11 fails.

Example 3.16. *Let E_1, E_2 be closed line segments forming the shorter pair of opposite sides of a rectangle in $\mathbb{R}^2$ with side lengths $0 < a \leq b$ such that $\sqrt{3}a \leq b < 2a$. Then E_1, E_2 are 1-aligned, but not 1-separated.*

PROOF. By Corollary 3.13, the union $E = E_1 \cup E_2$ is 1-straight because $\sqrt{3}a \leq b$. By the implication $(iii) \Rightarrow (ii)$ of Theorem 3.15 it then follows that E_1, E_2 are 1-aligned. But $d(E_1, E_2) = b < 2a = |E_1| + |E_2|$, so by Definition 2.9 we see that E_1, E_2 are not 1-separated. $\square$

Each of the next two examples consists of a countable number of line segments. The idea of Example 3.17 is roughly to push the line segments close enough so that the union is bounded but not 1-straight.

Example 3.17. *There exists a bounded 1-set $B \subseteq \mathbb{R}^2$, consisting of a countable sequence of parallel line segments, which is not the finite union of 1-straight sets. Moreover, B necessarily contains specific subsets of arbitrarily small $\mathcal{H}^1$-measure which are not 1-straight.*

PROOF. Let $B = \bigcup B_{n,j}$ where $\{B_{n,j}\}$ is the countable collection of parallel line segments described below, numbered such that for $n = 0$, only $j = 1$, but for all $n \geq 1$, we have $j = 1, \ldots, 2^n + 2$. Each line segment is perpendicular to the unit interval $[0, 1]$ on the x-axis, and has one endpoint contained in $[0, 1]$. These will be the only endpoints referred to below. Let $a_n = \frac{1}{2}\frac{1}{3^n}$, so that $\sum_{n=1}^{\infty} 2^n a_n = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{2}(3-1) = 1$. Let each segment $B_{n,j}$ for each choice of j have length $b_n = \frac{1}{3^n}$. The endpoint of the first segment $B_{0,1}$ of length b_0 is at $(0,0)$, and the endpoints of segments $B_{1,1}$, $B_{1,2}$ both of length b_1, are at $(a_1, 0)$ and $(2a_1, 0)$. The endpoints of segments $B_{2,1}$, $B_{2,2}$, both of length b_2, are at $(2a_1 + a_2, 0)$ and $(2a_1 + 2^2 a_2, 0)$. In general, the endpoints of segments $B_{n,1}$, $B_{n,2}$, both of length b_n, are at $(2a_1 + 2^2 a_2 + \cdots + (2^n - 3)a_n, 0)$ and $(2a_1 + 2^2 a_2 + \cdots + 2^n a_n, 0)$. So, $|B| = \sqrt{2}$. Finally, for each $n \geq 1$ the endpoints of the 2^n segments $B_{n,3}, B_{n,4}, \ldots, B_{n,2^n+2}$, all of length b_n, are at arbitrary positions on $[0, 1]$ between $B_{n,1}$ and $B_{n,2}$, such that none of the segments $B_{n,j}$ coincide. Then $\mathcal{H}^1(B) = b_0 + 2\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} 2^n b_n = 1 + 2\left(\frac{3}{2} - 1\right) + (3 - 1) = 4$. Hence, B is not 1-straight because $\mathcal{H}^1(B) = 4 > \sqrt{2} = |B|$. For $n \geq 1$, let the union of the nth collection of parallel line segments $\bigcup_{j=1}^{2^n+2} B_{n,j} = B_n$. The outermost line segments $B_{n,1}$, $B_{n,2} \subseteq B_n$ each of length b_n are the vertical sides of a rectangle of horizontal width $3a_n$. Since $b_n = \frac{1}{3^n} = 2a_n$, it follows that $b_n < 3a_n$, but $\sqrt{3}b_n = \sqrt{3}\frac{1}{3^n} > \frac{3}{2}\frac{1}{3^n} = 3a_n$. So by Corollary 3.13, the set $B_{n,1} \cup B_{n,2} \subseteq B_n$ is not 1-straight. (That $B_{n,1} \cup B_{n,2}$ is not 1-straight also follows from $\mathcal{H}^1(B_{n,1} \cup B_{n,2}) = 2b_n = 4a_n > \sqrt{13}a_n = \sqrt{(3a_n)^2 + b_n^2} = |B_{n,1} \cup B_{n,2}|$.) So B_n is not 1-straight. Note that $|B_n| = \sqrt{13}a_n = \frac{\sqrt{13}}{2}\frac{1}{3^n}$ and $\mathcal{H}^1(B_n) = \frac{1}{3^n}(2^n + 2)$. For a fixed n there exists a least integer m_n large enough so that for this m_n (and in fact, for all $m \geq m_n$), $\frac{1}{m_n} \cdot \mathcal{H}^1(B_n) \leq |B_n|$. Then, $m_n \geq \frac{\mathcal{H}^1(B_n)}{|B_n|} = \frac{2}{\sqrt{13}}(2^n + 2)$. So for each n, if B_n is to be contained in a finite union of 1-straight sets, there must exist at least one 1-straight subset A such that $0 < \mathcal{H}^1(A) \leq |A| < \frac{1}{m_n} \cdot \mathcal{H}^1(B_n)$. But, $n \to \infty$ forces $m_n \to \infty$. So, a minimum number of 1-straight subsets sufficient to cover $\bigcup B_n = B$ does not exist. Hence B cannot be written as a finite union of 1-straight subsets. Finally, for any $n \geq 1$, any two line segments other than $B_{n,1}$, $B_{n,2}$ from B_n, each of length b_n, are the vertical sides of a rectangle of horizontal width x where $0 < x < 3a_n$. Let B' be the union of two such line segments. Then, $\mathcal{H}^1(B') = 2b_n = \mathcal{H}^1(B_{n,1} \cup B_{n,2}) > |B_{n,1} \cup B_{n,2}| = \sqrt{(3a_n)^2 + b_n^2} > \sqrt{x^2 + b_n^2} = |B'|$. So B' is not 1-straight. Therefore B contains these specific subsets of arbitrarily small $\mathcal{H}^1$-measure $2b_n = \frac{2}{3^n}$, which are not 1-straight. $\square$

Example 3.18 makes use of Theorem 2.7, Theorem 3.6, and Theorem 3.14. The idea of Example 3.18 is roughly to spread the line segments far enough

apart, following the geometry of Theorem 3.11, so that the union is 1-straight but remains bounded.

Example 3.18. *There exists a bounded 1-set $B \subseteq \mathbb{R}^2$, consisting of a countable sequence of parallel line segments, which is 1-straight.*

PROOF. Let $B = \bigcup_{k=0}^{\infty} B_k$, where $\{B_k\}$ is the countable collection of parallel line segments described below. Each line segment is perpendicular to the x-axis and has its midpoint contained in $[0,1]$. Suppose $\{a_k\}_{k=0}^{\infty}$ is a sequence of strictly decreasing positive real numbers such that $\sum_{k=0}^{\infty} a_k = \frac{1}{2}$. Let segment B_k have length $2a_k$. The midpoint of the first segment B_0 of length $2a_0$ is at $(x_0, 0) = (0, 0)$. The midpoints of all other segments B_k for $k = 1, 2, \ldots,$ are at points $(x_k, 0)$. The $x_k \in (0, 1)$ are increasing, such that for odd k we have $|x_k - x_{k-1}| = |x_{k+1} - x_k| = \sqrt{(2a_{k-1} + 2a_k)^2 - a_k^2} = c((k-1)/2)$, where for convenience we define

$$c(n) = \sqrt{(2a_{2n} + 2a_{2n+1})^2 - a_{2n+1}^2} = \sqrt{(2a_{2n} + 2a_{2n+1})(2a_{2n} + 3a_{2n+1})}$$

$$> 2a_{2n} + a_{2n+1}.$$

Thus, $|x_1 - x_0| = |x_2 - x_1| = c(0) = x_1$, and $x_2 = 2c(0)$. In general, for odd $k \geq 3$ it follows that $x_k = c((k-1)/2) + 2\sum_{n=0}^{(k-3)/2} c(n)$ and $x_{k+1} = 2\sum_{n=0}^{(k-1)/2} c(n)$. Let $\sum'$ represent summation over odd $k \geq 1$. Then, $\sum' c((k-1)/2) < \sum' \sqrt{(2a_{k-1} + 2a_k)^2} = 2\sum'(a_{k-1} + a_k) = 2\sum_{k=0}^{\infty} a_k = 1$. Also, $\mathcal{H}^1(B) = \sum_{k=0}^{\infty} 2a_k = 1$. We prove that $B = \bigcup_{k=0}^{\infty} B_k$ is 1-straight by induction on k. Let $k = 0$. By Theorem 3.6, as a line segment B_0 is 1-straight. Let $k = 1$. Since $c(0) > 2a_0 + a_1$, then $|B_0 \cup B_1| = \sqrt{(a_0 + a_1)^2 + c(0)^2} > (a_0 + a_1) + (2a_0 + a_1) > 2a_0 + 2a_1 = |B_0| + |B_1|$. So, by Theorem 3.14 it follows that $B_1 \cup B_0$ is 1-straight. As the induction hypothesis, assume that $\bigcup_{i=0}^{k-1} B_i$ is 1-straight. Suppose $\bigcup_{i=0}^{k} B_i$ is not 1-straight. Then by the second part of Theorem 3.14 we have

$$\left| \bigcup_{i=0}^{k} B_i \right| < \mathcal{H}^1 \left(\bigcup_{i=0}^{k} B_i \right). \tag{$*$}$$

But, since the a_k are strictly decreasing, it follows for odd $k \geq 3$ (the case for even k is similar) that

$$\left| \bigcup_{i=0}^{k} B_i \right| = \sqrt{(a_0 + a_k)^2 + x_k^2} > (a_0 + a_k) + x_k$$

$$= (a_0 + a_k) + c((k-1)/2) + 2\sum_{n=0}^{(k-3)/2} c(n)$$

$$\geq (a_0 + a_k) + (2a_{k-1} + a_k) + 2 \sum_{n=0}^{(k-3)/2} (2a_{2n} + a_{2n+1})$$

$$> 2a_0 + \cdots + 2a_k = \mathcal{H}^1 \Big(\bigcup_{i=0}^{k} B_i \Big).$$

The last inequality contradicts $(*)$. So by induction each set $\bigcup_{i=0}^{k} B_i$ is 1-straight. Since therefore each member of the increasing sequence $B_0 \subseteq B_0 \cup B_1 \subseteq B_0 \cup B_1 \cup B_2 \subseteq B_0 \cup B_1 \cup B_2 \cup B_3 \subseteq \cdots$ is 1-straight, by Theorem 2.7 the union $B = \bigcup_{k=0}^{\infty} B_k$ is 1-straight. (For an easy example, choose $a_k = \frac{1}{4} \cdot \frac{1}{2^k}$.) □

In Example 3.18, observe that the midpoints of the line segments are a set of isolated points $\{x_i\}_{i=0}^{\infty}$. This is necessary if the countable union of line segments is 1-straight as Theorem 3.19 shows.

Theorem 3.19. *Let $E = \{x_i\}_{i=0}^{\infty} \subseteq \mathbb{R}^2$ be the set of midpoints of a countable collection of non-overlapping line segments $\{B_i\}_{i=0}^{\infty} \subseteq \mathbb{R}^2$. Let $B = \bigcup_{i=0}^{\infty} B_i$ and assume $\mathcal{H}^1(B) < \infty$. If B is 1-straight, then E is a set of isolated points.*

PROOF. Suppose $x_k \in E$ is a limit point of E. Let B_k be the line segment with length $2a_k > 0$ having x_k as its midpoint. Since $\infty > \mathcal{H}^1(B) = \sum_{i=0}^{\infty}(2a_i) = 2\sum_{i=0}^{\infty} a_i$, it follows that $\lim_{i \to \infty} a_i = 0$. Then there exists $n > k$, corresponding to a line segment B_n with length $2a_n > 0$, such that both $a_n < \frac{1}{2}a_k$ and $d(x_k, x_n) < \frac{1}{2}a_k$. So, the maximal distance of an endpoint of B_n from x_k is $d(x_k, x_n) + a_n < \frac{1}{2}a_k + \frac{1}{2}a_k = a_k$. Thus B_n is contained in the open disk with center x_k and radius a_k. So, $|B_k \cup B_n| \leq 2a_k = |B_k| < |B_k| + |B_n|$. Hence by Theorem 3.14 it follows that $B_k \cup B_n \subseteq B$ is not 1-straight. By Corollary 2.4 this contradicts the assumption that B is 1-straight. So, E contains no limit points, and is thus a set of isolated points. □

4　The Circle Is σ1-Straight

4.1　The Quarter Circle Contains a Perfect 1-Straight 1-Set

In [8, p. 735], Foran describes the construction of a perfect subset P of positive $\mathcal{H}^1$-measure of a quarter circle C and suggests that P is 1-straight. We prove that result here, providing a specific construction of P. Note that in $\mathbb{R}^2$ the diameter of any closed circular arc of central angle $0 \leq \theta \leq \pi$ is the length of the chord between its two endpoints. The $\mathcal{H}^1$-measure of a closed circular arc is just its arc length, by Theorem 3.5. We will also need the elementary fact

that on the unit circle the sine of an angle is half the length of the chord of twice the angle.

Lemma 4.1. *Let $C = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x^2 + y^2 = r^2\} \subseteq \mathbb{R}^2$ be the first quadrant quarter circle of radius $r > 0$. If θ is the angle between two radii to points p_1 and p_2 on C, then $d(p_1, p_2) = 2r\sin(\frac{\theta}{2})$. In particular on the unit circle, $d(p_1, p_2) = 2\sin(\frac{\theta}{2})$.*

PROOF. Without loss of generality the angle θ has one ray along the x-axis and $p_1 = (r, 0)$. Then, $p_2 = (r\cos\theta, r\sin\theta)$. By the Pythagorean identity and a half angle identity, we have $d(p_1, p_2) = \sqrt{r^2\sin^2\theta + (r - r\cos\theta)^2} = r\sqrt{2}\sqrt{1 - \cos\theta} = r\sqrt{2}\sqrt{2\sin^2\left(\frac{\theta}{2}\right)} = 2r\sin\left(\frac{\theta}{2}\right).$ $\qquad\square$

Theorem 4.2. *The first quadrant unit quarter circle $C = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ is not 1-straight, but contains a perfect 1-straight 1-set P such that $\mathcal{H}^1(P) = \frac{\pi}{2} - 1$ and $|P| = \sqrt{2}$.*

PROOF. The set C is not 1-straight since $\mathcal{H}^1(C) = \frac{\pi}{2} > \frac{3}{2} > \sqrt{2} = |C|$. We now construct a particular subset P of C. At stage 0 of the construction remove an open arc of length $\frac{1}{2}$ from the middle of C, centered about the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then at stage 1 remove two open arcs of length $\frac{1}{2} \cdot \frac{1}{4}$ from the middle of the two remaining equal length closed arcs. In general at stage n of the construction remove 2^n open arcs of length $\frac{1}{2} \cdot \frac{1}{4^n}$ from each of the remaining 2^n equal closed arcs of C. Repeat this process for each n, and call the perfect set which remains, P. The total arc length of the removed open arcs is $\sum_{n=0}^{\infty} \left(2^n \cdot \frac{1}{2} \cdot \frac{1}{4^n}\right) = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{2^n} = 1$. So $\mathcal{H}^1(P) = \frac{\pi}{2} - 1 > \frac{3}{2} - 1 = \frac{1}{2} > 0$, and by the construction, $|P| = \sqrt{2}$. We claim that P is 1-straight. Let $p_1, p_2 \in P$, and let $C(p_1, p_2)$ represent the closed arc of C between p_1 and p_2. Let $\beta(p_1, p_2) = \beta = \mathcal{H}^1\left[P \cap C(p_1, p_2)\right]$. Let $d(p_1, p_2) = d$. Then showing that $d \geq \beta$ for every choice of $p_1, p_2 \in P$ will imply that P is 1-straight, since for any subset $A \subseteq C$, it is true that $|A| = \sup_{q_1, q_2 \in A} d(q_1, q_2)$ equals the diameter of the smallest arc of C containing A. Let m be the least nonnegative integer such that p_1 and p_2 are first separated by the removal of the stage m open arc of $\mathcal{H}^1$-measure $\frac{1}{2} \cdot \frac{1}{4^m} = \frac{1}{2^{2m+1}}$. For $m = 1, 2, \ldots$, let a_{m-1} represent the length of each of the remaining equal closed arcs at stage $m-1$ of the construction. Then $a_0 = \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\right)$, $a_1 = \frac{1}{2^2}\left(\frac{\pi}{2} - \frac{1}{2} - 2 \cdot \frac{1}{8}\right) = \frac{1}{2^2}\left(\frac{\pi}{2} - 1 + \frac{1}{2^2}\right)$, $\ldots$, and in general, $a_{m-1} = \frac{1}{2^m}\left[\frac{\pi}{2} - \frac{1}{2}\sum_{n=0}^{m-1}\frac{1}{2^n}\right] = \frac{1}{2^m}\left[\frac{\pi}{2} - 1 + \frac{1}{2^m}\right]$. Let a_{m-1}^P represent the $\mathcal{H}^1$-measure of the intersection of P with any one of the remaining equal arcs at stage $m - 1$ of length a_{m-1}. Since by construction the partial sum in the formula for a_{m-1} is that of a geometric series which sums to 2, it follows that

$a_{m-1}^P = \frac{1}{2^m}\left[\frac{\pi}{2}-1\right]$. Now, consider the construction within the arc containing p_1 and p_2 of length a_{m-1}, at a subsequent stage $m+k$, for $k=0,1,\ldots$. At stage $m+k$, on each side of the arc removed at stage m, there remain 2^k arcs. Due to symmetry, there is a total of $1+2+3+\cdots+2^k = \frac{2^k(2^k+1)}{2}$ representative chords, meaning we count one chord for each pair of remaining arcs (on opposite sides) where the separated points p_1 and p_2 could possibly be located. The measure of the radial angle θ between p_1 and p_2 is the sum of the lengths of the arcs removed by the construction plus the sum of the lengths of the remaining arcs, all contained between the endpoints of the chord connecting p_1 and p_2. Thus, we distinguish between remaining arcs and removed arcs. Let j represent the number of such remaining arcs corresponding to a chord of minimum length, that is, one connecting the nearest endpoints of the two remaining arcs which contain p_1 and p_2. This minimum length chord will not in general be the chord connecting p_1 and p_2, but is always less than or equal in length. Note that j depends on (p_1, p_2, m, k) and j has a minimum value of 0. At stage $m+k$, the total number of arcs remaining on both sides of the arc removed at stage m is $2 \cdot 2^k$. Since a minimum length chord will always miss at least one remaining arc on each side, j has a maximum value of $2 \cdot 2^k - 2 = 2(2^k - 1)$. Next, by construction the arcs removed from between the endpoints of the chord connecting p_1 and p_2 will always include the arc of length $\frac{1}{2^{2m+1}}$ removed at stage m. The additional arcs removed at stage $m+k$ will have a combined length which is a multiple of $\frac{1}{4^k} \cdot \frac{1}{2^{2m+1}}$. Let i represent that multiple. Note that i also depends on (p_1, p_2, m, k) and i has a minimum value of 0. Since the number of additional removed arcs doubles at each stage up to stage $m+k$, while each of their lengths is quartered, the maximum combined additional length that can be removed is $\left[2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{4^2} + \cdots + 2^k \cdot \frac{1}{4^k}\right]\frac{1}{2^{2m+1}} = \left[\sum_{n=1}^{k}\frac{1}{2^n}\right]\frac{1}{2^{2m+1}} = \left[1 - \frac{1}{2^k}\right]\frac{1}{2^{2m+1}} = \left(4^k - 2^k\right) \cdot \frac{1}{4^k} \cdot \frac{1}{2^{2m+1}}$. So i has a maximum value of $4^k - 2^k$. Thus, in general at stage $m+k$, by the calculations above and Lemma 4.1, the minimum length chord connecting the two remaining arcs which contain p_1 and p_2 has length

$$d_k(p_1, p_2) = d_k = 2\sin\left[\frac{1}{2}\left(j \cdot a_{m+k} + \left(1 + \frac{i}{4^k}\right)\cdot\frac{1}{2^{2m+1}}\right)\right].$$

At stage $m+k$, the maximum number of remaining arcs associated with the chord connecting p_1 and p_2 is $j+2$ (the number of those corresponding to a minimum length chord plus the 2 remaining arcs which contain p_1 and p_2) out of a total of $2 \cdot 2^k$ such arcs. Thus, at stage $m+k$, the maximum possible $\mathcal{H}^1$-measure of P between p_1 and p_2 is

$$\beta_k(p_1, p_2) = \beta_k = \left(\frac{j+2}{2\cdot 2^k}\right)a_{m-1}^P = \left(\frac{1}{2}\frac{j}{2^k} + \frac{1}{2^k}\right)\left(\frac{1}{2^m}\left[\frac{\pi}{2}-1\right]\right).$$

Note that for each $k \geq 0$, by construction $d_k \leq d$ and $\beta_k \geq \beta$, so that $d - \beta \geq d_k - \beta_k$. If we show for any fixed m determined by p_1 and p_2, where i and j are dependent on k, that $\lim_{k \to \infty}(d_k - \beta_k) \geq 0$, then we will have shown that $d \geq \beta$, as desired. To see this, first note that $a_{m+k} = \frac{1}{2^k}\frac{1}{2^{m+1}}\left[\frac{\pi}{2} - 1 + \frac{1}{2^{m+k+1}}\right]$. Then

$$d_k - \beta_k = 2\sin\left[\frac{1}{2}\left(\frac{j}{2^k}\frac{1}{2^{m+1}}\left[\frac{\pi}{2} - 1 + \frac{1}{2^{m+k+1}}\right] + \frac{1}{2^{2m+1}}\left(1 + \frac{i}{4^k}\right)\right)\right]$$
$$- \left(\frac{1}{2}\frac{j}{2^k} + \frac{1}{2^k}\right)\left(\frac{1}{2^m}\left[\frac{\pi}{2} - 1\right]\right),$$

where $0 \leq j \leq 2(2^k - 1)$, $0 \leq i < 4^k - 2^k$ and $k = 0, 1, 2, \ldots$. Observe that, for any p_1 and p_2,

$$0 \leq \frac{j}{2^k} \leq \frac{2(2^k - 1)}{2^k} = 2\left(1 - \frac{1}{2^k}\right) \leq 2 \cdot \lim_{k \to \infty}\left(1 - \frac{1}{2^k}\right) = 2, \text{ and likewise}$$

$$0 \leq \frac{i}{4^k} \leq \frac{4^k - 2^k}{4^k} \leq \lim_{k \to \infty}\left(1 - \frac{1}{2^k}\right) = 1.$$

Replace $\frac{j}{2^k}$ with b, where $0 \leq b \leq 2$, and replace $\frac{i}{4^k}$ with c, where $0 \leq c \leq 1$. Then

$$d - \beta = \lim_{k \to \infty}(d_k - \beta_k)$$
$$= \lim_{k \to \infty}\left(2\sin\left[\frac{1}{2}\left(\frac{b}{2^{m+1}}\left[\frac{\pi}{2} - 1\right] + \frac{1+c}{2^{2m+1}}\right)\right] - \frac{b}{2}\frac{1}{2^m}\left[\frac{\pi}{2} - 1\right]\right). \quad (*)$$

If $b = 0$, then $d - \beta = \lim_{k \to \infty}2\sin\left[\frac{1+c}{2^{2m+2}}\right] \geq 2\sin\left[\frac{1}{2^{2m+2}}\right] > 0$, since $m \geq 0$. So, assume $b > 0$. To show that $d - \beta = (*) \geq 0$, it suffices to show for $m \geq 0$, independently of the values of $b > 0$ and c, that $f(m, b, c)/g(m, b) \geq 1$, where

$$f(m, b, c) = 2\sin\left[\frac{1}{2}\left(\frac{b}{2^{m+1}}\left[\frac{\pi}{2} - 1\right] + \frac{1+c}{2^{2m+1}}\right)\right] = 2\sin\left[\frac{\theta_m}{2}\right],$$

and $g(m, b) = \frac{b}{2}\frac{1}{2^m}\left[\frac{\pi}{2} - 1\right]$. Here $\theta_m = \frac{b}{2^{m+1}}\left[\frac{\pi}{2} - 1\right] + \frac{1+c}{2^{2m+1}}$. By the geometry of the construction, the angle θ_m satisfies $0 < \theta_m < \frac{\pi}{2}\frac{1}{2^m}$. (In fact, although it is not needed here, the geometry provides a more precise positive lower bound of $\frac{1}{2}\frac{1}{4^m}$.) For $u \in (0, \frac{1}{2}\frac{\pi}{2}\frac{1}{2^m}) = (0, \frac{\pi}{4}\frac{1}{2^m})$, it is elementary that $\sin u > \frac{4 \cdot 2^m}{\pi}\sin(\frac{\pi}{4}\frac{1}{2^m}) \cdot u$. The inequalities below are clear except perhaps for the last

inequality which uses the fact that for $u > 0$ we have $\sin u > u - \frac{u^3}{3!}$, so that $\frac{\sin u}{u} > 1 - \frac{u^2}{6}$. Thus, since $c \geq 0$ and $b \leq 2$,

$$f(m, b, c)/g(m, b)$$

$$= 2 \sin \left[\frac{1}{2} \left(\frac{b}{2^{m+1}} \left[\frac{\pi}{2} - 1 \right] + \frac{1+c}{2^{2m+1}} \right) \right] / \left(\frac{b}{2} \frac{1}{2^m} \left[\frac{\pi}{2} - 1 \right] \right)$$

$$\geq \sin \left[\frac{b}{4} \frac{1}{2^m} \left[\frac{\pi}{2} - 1 \right] + \frac{1}{4} \left(\frac{1}{2^m} \right)^2 \right] / \left(\frac{b}{4} \frac{1}{2^m} \left[\frac{\pi}{2} - 1 \right] \right)$$

$$> \frac{4 \cdot 2^m}{\pi} \sin \left(\frac{\pi}{4} \frac{1}{2^m} \right) \cdot \left[\frac{b}{4} \frac{1}{2^m} \left[\frac{\pi}{2} - 1 \right] + \frac{1}{4} \left(\frac{1}{2^m} \right)^2 \right] / \left(\frac{b}{4} \frac{1}{2^m} \left[\frac{\pi}{2} - 1 \right] \right)$$

$$= \frac{4 \cdot 2^m}{\pi} \sin \left(\frac{\pi}{4} \frac{1}{2^m} \right) \cdot \left[1 + \frac{2}{b(\pi - 2)} \frac{1}{2^m} \right]$$

$$\geq \frac{4 \cdot 2^m}{\pi} \sin \left(\frac{\pi}{4} \frac{1}{2^m} \right) \cdot \left[1 + \frac{1}{\pi - 2} \frac{1}{2^m} \right]$$

$$> \left[1 - \frac{1}{6} \left(\frac{\pi}{4} \frac{1}{2^m} \right)^2 \right] \cdot \left[1 + \frac{1}{\pi - 2} \frac{1}{2^m} \right] .$$

Let $x = \frac{1}{2^m}$, and $p(x) = \left[1 - \frac{\pi^2}{96} x^2 \right] \cdot \left[1 + \frac{1}{\pi - 2} x \right]$. For $x > 0$ it follows that $p''(x) = -\frac{\pi^2}{48} - \frac{\pi^2}{16(\pi - 2)} x < 0$, so $p(x)$ is concave down there. Now $p(0) = 1$. Since $\pi < 3.2$, then $p(1) = \left[1 - \frac{\pi^2}{96} \right] \cdot \left[1 + \frac{1}{\pi - 2} \right] > \left[1 - \frac{(3.2)^2}{96} \right] \cdot \left[1 + \frac{1}{3.2 - 2} \right] = \frac{737}{450} > 1$. Hence $p(x) > 1$ for $x \in (0, 1]$. Therefore for all $m \geq 0$ it follows that $f(m, b, c)/g(m, b) > 1$. Hence $d - \beta = (*) \geq 0$, so that $d(p_1, p_2) \geq \beta(p_1, p_2)$ as desired. Thus, P is 1-straight. $\square$

Theorem 4.3. *The first quadrant unit quarter circle $C = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ is the countable union of perfect 1-straight 1-sets along with a set of $\mathcal{H}^1$-measure zero; that is, C is σ1-straight.*

PROOF. Let $A \subseteq C$ be an $\mathcal{H}^1$-measurable set with $\mathcal{H}^1(A) > 0$. By [6, p. 30], all the usual results on Lebesgue measure on the line $\mathbb{R}$ transfer to arcs such as C. Let $P \subseteq C$ represent the perfect 1-straight 1-set constructed in the proof of Theorem 4.2. Let $q \in A$ be a point of density of A and $p \in P$ a point of density of P. If P_q is a congruent copy of P contained in the unit circle and rotated so that p and q coincide, then $A \cap P_q \subseteq A$ has positive $\mathcal{H}^1$-measure (length). By Corollary 2.6, then P_q is 1-straight and by Corollary 2.4 it follows that $A \cap P_q \subseteq P_q$ is 1-straight, so A contains the 1-straight 1-set $A \cap P_q$. Since A is arbitrary, by Theorem 2.8 it follows that C is a σ1-straight 1-set. $\square$

Corollary 4.4. *A circle of any radius $r > 0$ is $\sigma 1$-straight.*

PROOF. By Theorem 4.3, the unit quarter circle is $\sigma 1$-straight. Hence, as the union of four unit quarter circles, the unit circle is also $\sigma 1$-straight. By Corollary 2.6 dilations of 1-straight 1-sets are 1-straight, from which the conclusion follows. □

4.2 Other Perfect 1-Straight 1-Sets of the Circle

In Theorem 4.2, the perfect set P constructed as a subset of the quarter circle $C = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x^2 + y^2 = 1\}$ satisfies

$$0 < \mathcal{H}^1(P) = \frac{\pi}{2} - 1 \leq \sqrt{2} = |P|.$$

It is natural to ask whether given any $t \in (0, \sqrt{2}]$, a perfect 1-straight 1-set $P' \subseteq C$ can be similarly constructed with $\mathcal{H}^1(P') = t$. Theorem 4.6 answers this question, in particular providing a construction for the specific maximal case $t = \sqrt{2}$. We will use the following fact.

Lemma 4.5. *If $k, u > 0$ such that $0 < u + k < \frac{\pi}{2}$, then $\frac{\sin(u+k)}{u}$ is strictly decreasing as a function of u.*

PROOF. First $\frac{d}{du}\left(\frac{\sin(u+k)}{u}\right) = \frac{1}{u^2}\left[u \cdot \cos(u + k) - \sin(u + k)\right] < 0$ if and only if $\tan(u + k) - u > 0$. But $\frac{d}{du}(\tan(u + k) - u) = \sec^2(u + k) - 1 > 0$ so that here $\tan(u+k) - u$ is strictly increasing as a function of u. Since $u = 0$ implies $0 < k < \frac{\pi}{2}$ and $\tan(0 + k) - 0 = \tan(k) > 0$, it follows that $\tan(u + k) - u > 0$. Hence $\frac{\sin(u+k)}{u}$ is strictly decreasing as a function of $u > 0$. □

Theorem 4.6. *Let $C = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ be the first quadrant unit quarter circle. Let $r = \frac{1}{4}$ and $a = \frac{1}{2}(\frac{\pi}{2} - \sqrt{2})$. Construct a set $P_{a,r} \subseteq C$ in the same manner as the set P in Theorem 4.2; that is, at stage 0 of the construction remove an open arc of length a from the middle of C, centered about the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then at stage 1 remove two open arcs of length ar from the middle of the two remaining equal length closed arcs. In general at stage n of the construction remove 2^n open arcs of length ar^n from each of the remaining 2^n equal closed arcs of C. Repeat this process for each n. Call the perfect 1-set which remains, $P_{a,r}$. Then $P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}$ is 1-straight, and is maximal in the sense that*

$$\mathcal{H}^1(P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}) = \sqrt{2} = |P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}|.$$

Also, by the continuity of $\mathcal{H}^1$-measure, for each $t \in (0, \sqrt{2}]$ there exists a perfect 1-straight 1-set $P' \subseteq P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}$ such that $\mathcal{H}^1(P') = t$ and $|P'| = \sqrt{2}$.

Proof. Suppose $P_{a,r}$ is defined as above. (We will substitute for a and r later.) By construction, $P_{a,r}$ is a perfect 1-set and $|P_{a,r}| = \sqrt{2}$. Notation here follows that of Theorem 4.2. Making the substitutions $r > 0$ and $a > 0$ into the expressions in the proof of Theorem 4.2, we have for any m, determined by $p_1, p_2 \in P_{a,r}$, and any choice of j and c^* (bounded as indicated below),

$$d_k - \beta_k$$
$$= 2\sin\left[\frac{1}{2}\left(\frac{j}{2^k}\frac{1}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r} + \frac{a(2r)^{m+k+1}}{1-2r}\right] + (1+c^*)ar^m\right)\right]$$
$$- \left(\frac{1}{2}\frac{j}{2^k} + \frac{1}{2^k}\right)\left(\frac{1}{2^m}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]\right)$$

where, $0 \le j \le 2(2^k - 1)$ and $0 \le c^* \le \sum_{n=1}^{k}(2r)^n = \frac{2r}{1-2r}(1 - (2r)^k)$, $k = 0, 1, 2, \ldots$. As in Theorem 4.2, since

$$0 \le \frac{j}{2^k} \le \frac{2(2^k - 1)}{2^k} = 2\left(1 - \frac{1}{2^k}\right) \le 2 \cdot \lim_{k \to \infty}\left(1 - \frac{1}{2^k}\right) = 2,$$

replace $\frac{j}{2^k}$ with b, where $0 \le b \le 2$. Observe that

$$0 \le c^* \le \frac{2r}{1-2r}(1 - (2r)^k) \le \lim_{k \to \infty}\frac{2r}{1-2r}(1 - (2r)^k) = \frac{2r}{1-2r}.$$

Then

$$d - \beta = \lim_{k \to \infty}(d_k - \beta_k)$$
$$= \lim_{k \to \infty}\left(2\sin\left[\frac{1}{2}\left(\frac{b}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right] + (1+c^*)ar^m\right)\right]\right.$$
$$\left. -\frac{b}{2}\frac{1}{2^m}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]\right). \tag{$*$}$$

If $b = 0$, then $d - \beta = \lim_{k \to \infty} 2\sin\left[\frac{1}{2}(1+c^*)ar^m\right] \ge 2\sin\left[\frac{a}{2}r^m\right] > 0$, since $a, r > 0$ and $m \ge 0$. So, assume $b > 0$. To show that $d - \beta = (*) \ge 0$, it suffices to show for $m \ge 0$, independently of the values of $b > 0$ and c^*, that $F(m, b, c^*)/G(m, b) \ge 1$, where

$$F(m, b, c^*) = 2\sin\left[\frac{1}{2}\left(\frac{b}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right] + (1+c^*)ar^m\right)\right] = 2\sin\left[\frac{\theta_m}{2}\right],$$

and $G(m, b) = \frac{b}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]$. Here $\theta_m = \frac{b}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right] + (1+c^*)ar^m$. By the geometry of the construction, the angle θ_m satisfies $0 < \theta_m < \frac{\pi}{2}\frac{1}{2^m}$.

(In fact, as in the proof of Theorem 4.2, the geometry provides a more precise positive lower bound of ar^m.) For $u \in (0, \frac{1}{2}\frac{\pi}{2}\frac{1}{2^m}) = (0, \frac{\pi}{4}\frac{1}{2^m})$, it is elementary that $\sin u > \frac{4 \cdot 2^m}{\pi} \sin(\frac{\pi}{4}\frac{1}{2^m}) \cdot u$. As in the proof of Theorem 4.2, the last inequality below uses the fact that for $u > 0$ we have $\sin u > u - \frac{u^3}{3!}$, so that $\frac{\sin u}{u} > 1 - \frac{u^2}{6}$. We also write $r^m = \left(\frac{1}{2^m}\right)^\alpha$, where $\alpha = \frac{-\ln r}{\ln 2} > 1$. Thus, since $c^* \geq 0$ and $b \leq 2$,

$F(m, b, c^*)/G(m, b)$

$$= 2\sin\left[\frac{1}{2}\left(\frac{b}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right] + (1+c^*)ar^m\right)\right] \Big/ \left(\frac{b}{2^{m+1}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]\right)$$

$$\geq \sin\left[\frac{b}{2^{m+2}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right] + \frac{a}{2}\frac{1}{2^{m\alpha}}\right] \Big/ \left(\frac{b}{2^{m+2}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]\right)$$

$$> \frac{4 \cdot 2^m}{\pi}\sin\left(\frac{\pi}{4}\frac{1}{2^m}\right)\left[\frac{b}{2^{m+2}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right] + \frac{a}{2}\frac{1}{2^{m\alpha}}\right] \Big/ \left(\frac{b}{2^{m+2}}\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]\right)$$

$$= \frac{4 \cdot 2^m}{\pi}\sin\left(\frac{\pi}{4}\frac{1}{2^m}\right)\left[1 + \frac{2a}{b}\frac{1}{\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]}\frac{1}{2^{m(\alpha-1)}}\right]$$

$$\geq \left[1 - \frac{1}{6}\left(\frac{\pi}{4}\frac{1}{2^m}\right)^2\right] \cdot \left[1 + \frac{a}{\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]}\left(\frac{1}{2^m}\right)^{\alpha-1}\right].$$

Let $x = \frac{1}{2^m}$. The previous expression then becomes

$$\left[1 - \frac{\pi^2}{96}x^2\right] \cdot \left[1 + \frac{a}{\left[\frac{\pi}{2} - \frac{a}{1-2r}\right]}x^{\alpha-1}\right]. \qquad (**)$$

Now substitute $r = \frac{1}{4}$ and $a = \frac{1}{2}(\frac{\pi}{2} - \sqrt{2})$. Then $(**)$ becomes $q(x) = \left[1 - \frac{\pi^2}{96}x^2\right] \cdot \left[1 + \frac{1}{2\sqrt{2}}(\frac{\pi}{2} - \sqrt{2})x\right]$. For $x > 0$ it follows that $q''(x) = -\frac{\pi^2}{48} - \frac{\pi^2}{32}\frac{1}{\sqrt{2}}(\frac{\pi}{2} - \sqrt{2})x < 0$, so $q(x)$ is concave down there. Note that below we freely use as needed the common approximations $3.1 < 3.14 < \pi < 3.15 < 3.2$ and $1.4 < \sqrt{2} < 1.415$. Since $q(0) = 1$ and

$$q\left(\frac{1}{2}\right) = \left[1 - \frac{\pi^2}{4 \cdot 96}\right] \cdot \left[1 + \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} - \sqrt{2}\right)\right]$$

$$> \left[1 - \frac{(3.15)^2}{4 \cdot 96}\right] \cdot \left[1 + \frac{1}{4 \cdot 1.415}\left(\frac{3.14}{2} - 1.415\right)\right] = \frac{58006951}{57958400} > 1,$$

then $q(x) > 1$ for $x \in (0, \frac{1}{2}]$, corresponding to $m = 1, 2, \dots$. However,

$$q(1) = \left[1 - \frac{\pi^2}{96}\right] \cdot \left[1 + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2} - \sqrt{2}\right)\right]$$

$$< \left[1 - \frac{(3.1)^2}{96}\right] \cdot \left[1 + \frac{1}{2 \cdot 1.4}\left(\frac{3.2}{2} - 1.4\right)\right] = \frac{8639}{8960} < 1.$$

So for the case $x = 1$, corresponding to $m = 0$, we use directly

$$F(0, b, c^*)/G(0, b) = \sin\left[\frac{b\sqrt{2}}{4} + (1 + c^*)\frac{1}{4}\left(\frac{\pi}{2} - \sqrt{2}\right)\right] \bigg/ \left(\frac{b\sqrt{2}}{4}\right).$$

Note that by definition $b \leq 2\left(1 - \frac{1}{2^k}\right) \leq 2 \cdot \lim_{k \to \infty}\left(1 - \frac{1}{2^k}\right) = 2$, and since $r = \frac{1}{4}$ here, by definition $c^* \leq \frac{2r}{1-2r}(1-(2r)^k) = 1 - \frac{1}{2^k} \leq \lim_{k \to \infty}\left(1 - \frac{1}{2^k}\right) = 1$. Thus, $b = 2$ if and only if $c^* = 1$. By Lemma 4.5, since $0 < b \leq 2$ implies $0 < \frac{b\sqrt{2}}{4} \leq \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < \frac{1}{1.4} = \frac{100}{140} < \frac{196}{140} = \frac{3}{2} - \frac{1}{10} < \frac{\pi}{2} - .1$, and so $0 = \frac{1}{4}(\frac{3}{2} - \frac{3}{2}) < \frac{b\sqrt{2}}{4} + (1+c^*)\frac{1}{4}(\frac{\pi}{2} - \sqrt{2}) < \frac{\pi}{2} - .1 + 2\frac{1}{4}(\frac{3.2}{2} - 1.4) = \frac{\pi}{2} - .1 + .1 = \frac{\pi}{2}$, we have that $\sin\left[\frac{b\sqrt{2}}{4} + (1+c^*)\frac{1}{4}(\frac{\pi}{2} - \sqrt{2})\right] \big/ \left(\frac{b\sqrt{2}}{4}\right)$ is strictly decreasing here as a function of b. So,

$$F(0, b, c^*)/G(0, b) = \sin\left[\frac{b\sqrt{2}}{4} + (1 + c^*)\frac{1}{4}\left(\frac{\pi}{2} - \sqrt{2}\right)\right] \bigg/ \left(\frac{b\sqrt{2}}{4}\right)$$

$$\geq \sin\left[\frac{2\sqrt{2}}{4} + (1 + 1)\frac{1}{4}\left(\frac{\pi}{2} - \sqrt{2}\right)\right] \bigg/ \left(\frac{2\sqrt{2}}{4}\right)$$

$$= \sqrt{2}\sin\left[\frac{\pi}{4}\right] = 1.$$

Therefore for all $m = 0, 1, 2, \dots$, it follows that $F(m, b, c^*)/G(m, b) \geq 1$. Hence, $d - \beta = (*) \geq 0$, so that $d(p_1, p_2) \geq \beta(p_1, p_2)$, as desired. Thus, the particular set $P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}$ is 1-straight and $\mathcal{H}^1(P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}) = \frac{\pi}{2} - 2\frac{1}{2}(\frac{\pi}{2} - \sqrt{2}) = \sqrt{2} = |P_{(1/2)[(\pi/2)-\sqrt{2}],1/4}|.$ $\square$

Using the result of Theorem 4.6, we construct a 1-straight 1-set E contained in the unit circle, which is maximal in the sense that $\mathcal{H}^1(E) = 2 = |E|$.

Example 4.7. Let $C_1 = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } x^2 + y^2 = 1\}$ and $C_2 = \{(x,y) \in \mathbb{R}^2 \mid x, y \leq 0 \text{ and } x^2 + y^2 = 1\}$. Let $P_1 \subseteq C_1$ be the perfect 1-straight 1-set constructed as in Theorem 4.6, for which $\mathcal{H}^1(P_1) = 1$ and $|P_1| = \sqrt{2}$. Let $P_2 \subseteq C_2$ be a congruent copy of P_1, symmetric to P_1 with

respect to the origin $(0,0)$, *for which* $\mathcal{H}^1(P_2) = 1$ *and* $|P_2| = \sqrt{2}$. *Then* $E = P_1 \cup P_2$ *is a* 1-*straight* 1-*set contained in the unit circle, which is maximal in the sense that* $\mathcal{H}^1(E) = 2 = |E|$.

PROOF. By construction it is immediate that $\mathcal{H}^1(E) = 2 = |E|$. The set $E = P_1 \cup P_2$ will be 1-straight if for each $\mathcal{H}^1$-measurable $A \subseteq E$ it follows that $\mathcal{H}^1(A) \leq |A|$. If A is a subset of either perfect 1-straight 1-set P_1, P_2, then A is 1-straight. Suppose $A = A_1 \cup A_2$, with $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$. It is clear by the geometry that $|A| \geq \sqrt{2} = d((1,0),(0,-1)) = d((-1,0),(0,1))$, where the notation (a,b) indicates a point in $\mathbb{R}^2$. So, if $\mathcal{H}^1(A) \leq \sqrt{2}$, then $\mathcal{H}^1(A) \leq |A|$. Therefore, suppose that $\mathcal{H}^1(A) = \mathcal{H}^1(A_1) + \mathcal{H}^1(A_2) > \sqrt{2}$. Then at least one of A_1, A_2, say A_1, satisfies $\mathcal{H}^1(A_1) > \frac{1}{2}\sqrt{2}$. Since $\mathcal{H}^1(A_1) \leq 1$, it is also necessary that $\mathcal{H}^1(A_2) > \sqrt{2} - 1$. Let $A_2' \subseteq P_1$ be a congruent copy of A_2, symmetric to A_2 with respect to the origin $(0,0)$. So also $\mathcal{H}^1(A_2') > \sqrt{2} - 1$, and $A_1 \cup A_2' \subseteq P_1$. A pair of diametrically opposite points of A will correspond to a single point in the intersection $A_1 \cap A_2'$. In fact it is not possible that $A_1 \cap A_2' = \emptyset$ because $\mathcal{H}^1(A_1 \cap A_2') = 0$ yields the contradiction

$$1 = \mathcal{H}^1(P_1) \geq \mathcal{H}^1(A_1 \cup A_2') = \mathcal{H}^1(A_1) + \mathcal{H}^1(A_2')$$
$$> \frac{1}{2}\sqrt{2} + \sqrt{2} - 1 > \frac{3}{2}(1.4) - 1 = 1.1.$$

So A contains a pair of diametrically opposite points, and since the diameter of the unit circle is 2,

$$\mathcal{H}^1(A) = \mathcal{H}^1(A_1) + \mathcal{H}^1(A_2) \leq \mathcal{H}^1(P_1) + \mathcal{H}^1(P_2) = 2 = |A|.$$

Thus, $\mathcal{H}^1(A) \leq |A|$ for arbitrary $A \subseteq E$. Hence, $E = P_1 \cup P_2$ is 1-straight. □

Example 4.8 finishes the proof of Theorem 2.11, showing why the converse implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ both fail in general. Note that by Theorem 3.15, such an example cannot be constructed using two line segments.

Example 4.8. *There exist* 1-*straight* 1-*sets* $E_1, E_2 \subseteq \mathbb{R}^2$ *such that* $E = E_1 \cup E_2$ *is* 1-*straight, but* E_1, E_2 *are neither* 1-*separated nor* 1-*aligned.*

PROOF. Let $E_1 = P_1$ and $E_2 = P_2$, where P_1, P_2 are the subsets of the unit circle described in Example 4.7. So, $|E| = |P_1 \cup P_2| = 2$ and $|P_1| = |P_2| = \sqrt{2}$. By Example 4.7, the union $E = P_1 \cup P_2$ is 1-straight. By Definition 2.9, the sets P_1, P_2 are not 1-separated because $d(P_1, P_2) = \sqrt{2} < 2\sqrt{2} = |P_1| + |P_2|$. By Definition 2.10, the sets P_1, P_2 are not 1-aligned because, choosing the bounded subset $A = E = P_1 \cup P_2$, we have $|A| = |P_1 \cup P_2| = 2 < 2\sqrt{2} = |P_1| + |P_2|$. □

5 Acknowledgements

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