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A CHARACTERIZATION OF $C^{k,1}$ FUNCTIONS

Abstract

In this work we provide a characterization of $C^{k,1}$ functions on $\mathbb{R}^n$ (that is, k times differentiable with locally Lipschitzian k-th derivatives) by means of $(k+1)$-th divided differences and Riemann derivatives. In particular we prove that the class of $C^{k,1}$ functions is equivalent to the class of functions with bounded $(k+1)$-th divided difference. From this result we deduce a Taylor's formula for this class of functions and a characterization through Riemann derivatives.

1 Introduction

In this paper we give some necessary and sufficient conditions for a real function on $\mathbb{R}^n$ to be of class $C^{k,1}$; that is, k times differentiable with locally Lipschitzian k-th derivatives. The relations mentioned above involve the boundedness of the $(k+1)$-th divided difference or of the $(k+1)$-th Riemann derivatives. The authors have studied these concepts in the case of real functions of one real variable in [18] and in [19] and a first generalization to the case of $C^{1,1}$ functions on $\mathbb{R}^n$ in [20]. In this paper we extend the results in [20] to the case of $C^{k,1}$ functions on $\mathbb{R}^n$. Furthermore we prove a Taylor's formula for this class of functions.

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The class of $C^{k,1}$ functions has been studied since the work of Hiriart-Urruty, Strodiot and Hien Nguyen [13] who introduced the concept of generalized Hessian matrix for $C^{1,1}$ functions proving also second-order optimality conditions for nonlinear constrained problems. Since that paper many works have been written on the applications of $C^{1,1}$ functions to optimization problems. In particular, Klatte and Tammer [14] obtained second-order necessary and sufficient optimality conditions for mathematical programming problems in terms of generalized Hessians and Cominetti and Correa [4] introduced a generalized second-order directional derivative. Thereafter, Yang and Jeyakumar [35], recalling a paper by Michel and Penot [27], introduced a different notion of second-order directional derivative and, later, Yang [36] used this notion to obtain optimality conditions in semi-infinite programming. Furthermore $C^{1,1}$ functions have revealed their importance in optimization methods as one can see in [3], [30], [31], [32], [33], [34].

Later, the more general case of $C^{k,1}$ functions were investigated by Luc [25] who extended Taylor's formula, proved higher order optimality conditions when derivatives of order greater than k do not exist and provided characterizations of generalized convex functions.

In this section we recall some concepts which are fundamental for understanding the proofs of the results. The second and the third sections are devoted to the main results.

1.1 Divided Differences, k-Convex Functions, Peano and Riemann Derivatives

Let us consider a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, Ω open and let $x \in \Omega$, $h \in \mathbb{R}$ and $d \in S^1 = \{v \in \mathbb{R}^n : \|v\| = 1\}$. We define

$$\Delta_k^d f(x;h) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ihd).$$

Definition 1.1. *The k-th Riemann derivative of f at a point $x \in \Omega$ in the direction d is defined as $D_k f(x;d) = \lim_{h\to 0} \dfrac{\Delta_k^d f(x;h)}{h^k}$, if this limit exists.*

The upper and lower k-th Riemann derivatives are respectively defined as

$$\overline{D}_k f(x;d) = \limsup_{h\to 0} \frac{\Delta_k^d f(x;h)}{h^k} \text{ and } \underline{D}_k f(x;d) = \liminf_{h\to 0} \frac{\Delta_k^d f(x;h)}{h^k}$$

Similarly we can define differences

$$\delta_k^d f(x;h) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ihd - \frac{1}{2}khd).$$

The corresponding k-th Riemann-type derivative is defined by

$$\mathcal{D}_k f(x;d) = \lim_{h\to 0} \frac{\delta_k^d f(x;h)}{h^k}.$$

It is easy to see that $\Delta_k^d f(x;h) = \delta_k^d f(x + \frac{k}{2}hd; h)$.

Besides these expressions, in the proof of Theorem 2.1, we will consider differences $\tilde{\Delta}_k^d f(x;h)$ defined recursively by

$$\tilde{\Delta}_1^d f(x;h) = f(x+hd) - f(x),\ \tilde{\Delta}_k^d f(x;h) = \tilde{\Delta}_{k-1}^d f(x;2h) - 2^{k-1}\tilde{\Delta}_{k-1}^d f(x;h).$$

As observed in [26], we have

$$\tilde{\Delta}_k^d f(x;h;d) = a_k f(x+2^{k-1}hd)+a_{k-1}f(x+2^{k-2}hd)+\cdots+a_1 f(x+hd)+a_0 f(x),$$

where, for any fixed k, a_j depends only on j $(j = 1,\dots,k-1)$ and $a_k = 1$. The following lemma can be easily deduced from Lemma 2 in [26].

Lemma 1.1. *There are constants $C_0, C_1, \dots, C_{2^{k-1}-k}$ such that $\forall x, h, d$*

$$\tilde{\Delta}_k^d f(x;h) = \sum_{i=0}^{2^{k-1}-k} C_i \Delta_k^d f(x + ihd; h).$$

The proof of the following lemma is straightforward from the previous result.

Lemma 1.2. *If there exist neighborhoods $U \subset \mathbb{R}^n$ of the point x_0 and $V \subset \mathbb{R}$ of the origin such that $\dfrac{\Delta_k^d f(x;h)}{h^k}$ is bounded on $U \times V\backslash\{0\}$ uniformly with respect to $d \in S^1$, then there exist neighborhoods U_1 of x_0 and V_1 of the origin such that $\dfrac{\tilde{\Delta}_k^d f(x;h)}{h^k}$ is bounded on $U_1 \times V_1\backslash\{0\}$ uniformly with respect to $d \in S^1$.*

The proof of the following lemma is similar to that of Lemma 6 in [26] and we give it for the sake of completeness.

Lemma 1.3. *Assume that f is bounded in a neighborhood of the point x_0. If there exist neighborhoods U of the point x_0 and V of the origin such that $\dfrac{\tilde{\Delta}_k^d f(x;h)}{h^k}$ is bounded on $U \times V\backslash\{0\}$ uniformly with respect to $d \in S^1$, then also $\dfrac{\tilde{\Delta}_{k-1}^d f(x;h)}{h^{k-1}}$ is bounded on $U \times V\backslash\{0\}$ uniformly with respect to $d \in S^1$.*

PROOF. From the hypotheses we obtain that there exists a number $\delta > 0$, such that $\forall x \in U$, $\forall h$ with $|h| \leq \delta, h \neq 0$ and $\forall d \in S^1$

$$\left|\tilde{\Delta}^d_{k-1} f(x;h) - 2^{k-1}\tilde{\Delta}^d_{k-1} f(x;\frac{h}{2})\right| \leq M \left|\frac{h}{2}\right|^k,$$

$$\left|\tilde{\Delta}^d_{k-1} f(x;\frac{h}{2}) - 2^{k-1}\tilde{\Delta}^d_{k-1} f(x;\frac{h}{4})\right| \leq M \left|\frac{h}{4}\right|^k, \ldots$$

$$\left|\tilde{\Delta}^d_{k-1} f(x;\frac{h}{2^{n-1}}) - 2^{k-1}\tilde{\Delta}^d_{k-1} f(x;\frac{h}{2^n})\right| \leq M \left|\frac{h}{2^n}\right|^k.$$

Multiplying these inequalities by $1, 2^{k-1}, 2^{2(k-1)}, \ldots, 2^{(n-1)(k-1)}$ respectively, by addition we obtain

$$\left|\tilde{\Delta}^d_{k-1} f(x;h) - 2^{n(k-1)}\tilde{\Delta}^d_{k-1} f(x;\frac{h}{2^n})\right| \leq 2M \left|\frac{h}{2}\right|^k,$$

and hence

$$\left|\frac{2^{n(k-1)}\tilde{\Delta}^d_{k-1} f(x;\frac{h}{2^n})}{h^{k-1}}\right| \leq M'$$

for $\frac{1}{2}\delta \leq |h| \leq \delta$, by using the boundedness of f. Hence, writing $\xi = \dfrac{h}{2^n}$, we have

$$\left|\frac{\tilde{\Delta}^d_{k-1} f(x;\xi)}{\xi^{k-1}}\right| \leq M' \text{ for } \frac{\delta}{2^{n+1}} \leq |\xi| \leq \frac{\delta}{2^n},\ n = 0, 1, \ldots,$$

and the lemma is established, since n can be chosen arbitrarily. □

Definition 1.2. *If there exist numbers $f_1(x;d), \ldots, f_k(x;d)$ such that*

$$f(x+hd) = f(x) + hf_1(x;d) + \frac{h^2}{2} f_2(x;d) + \cdots + \frac{h^k}{k!} f_k(x;d) + r(x;h;d),$$

where $\dfrac{r(x;h;d)}{h^k} \to 0$ as $h \to 0$, then f is said to admit a k-th Peano derivative at x in the direction $d \in S^1$. The number $f_k(x;d)$ is called the k-th Peano derivative of f at x in the direction $d \in S^1$.

In the following we set

$$f'(x;d_1) = \lim_{h\to 0} \frac{f(x+hd_1) - f(x)}{h}$$

and recursively define $f^{(k)}(x; d_1, \ldots, d_k)$ to be

$$\lim_{h\to 0} \frac{f^{(k-1)}(x + hd_k; d_1, \ldots, d_{k-1}) - f^{(k-1)}(x; d_1, \ldots, d_{k-1})}{h},$$

where $d_1, \ldots, d_k \in S^1$.

In particular, we will set $f^{(k)}(x; d) = f^{(k)}(x; d, \ldots, d)$. It is well known that the existence of the ordinary k-th directional derivative of f at x in the direction d, $f^{(k)}(x; d)$ implies the existence of $f_k(x; d)$ and this in turn implies the existence of $D_k f(x; d)$.

Definition 1.3. *If there exist i-linear functions $\tilde{\alpha}_i(x; \cdot, \ldots, \cdot) : \mathbb{R}^{n\times i} \to \mathbb{R}$, $i = 1 \ldots k$ (where $\mathbb{R}^{n\times i} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ i times), such that*

$$f(x + u) = f(x) + \tilde{\alpha}_1(x; u) + \frac{1}{2}\tilde{\alpha}_2(x; u, u) + \cdots + \frac{1}{k!}\tilde{\alpha}_k(x; u, \ldots, u) + r(x; u),$$

where $\dfrac{r(x; d)}{\|u\|^k} \to 0$ as $u \to 0$, then f is said to admit a k-th uniform Peano differential at x.

Remark 1.1. *The previous definition is equivalent to*

$$f(x + hd) = f(x) + h\tilde{\alpha}_1(x; d) + \frac{h^2}{2}\tilde{\alpha}_2(x; d) + \cdots + \frac{h^k}{k!}\tilde{\alpha}_k(x; d) + r(x; h; d),$$

where $\dfrac{r(x; h; d)}{h^k} \to 0$ as $h \to 0$, uniformly with respect to $d \in S^1$ (or equivalently with respect to $d \in B(0, \delta) = \{d \in \mathbb{R}^n : \|d\| \le \delta\}$, whenever $\delta > 0$).

Lemma 1.4. *[26] If $f_k(x; d)$ exists, then so does $\lim_{h\to 0} \dfrac{\tilde{\Delta}^d_k f(x; h)}{h^k}$ and there exists a number λ_k, depending only on $k \in \mathbb{N}$, such that $\lambda_k \lim_{h\to 0} \dfrac{\tilde{\Delta}^d_k f(x; h)}{h^k} = f_k(x; d)$.*

For a survey on Riemann and Peano derivatives one can see for instance [7], [12], [28] and [38]. Further properties of Peano and Riemann derivatives are given in [9], [10] and [11].

Definition 1.4. *A continuous function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be locally k-convex at $x_0 \in \Omega$ when $\dfrac{\Delta^d_{k+1} f(x; h)}{h^{k+1}} \ge 0$, $\forall x$ in a neighborhood U of x_0, $\forall d \in S^1$ and $\forall h$ such that $x + (k + 1)hd \in U$.*

Remark 1.2. *If f is not continuous, the definition of k-convex function has to be given considering divided differences at arbitrary (not equally spaced) points (see for instance [2] and [8] for details).*

When $k = 1$ the previous definition reduces to that of convex function.

Remark 1.3. *Let $f : \Omega \to \mathbb{R}$ be a function such that $\frac{\Delta^d_{k+1}(x,h)}{h^{k+1}} \geq M$, for each x in a neighborhood U of $x_0 \in \Omega$, h in a neighborhood V of 0 and $d \in S^1$. If $M \geq 0$, then f is obviously k-convex. If $M < 0$, let*

$$p(x) = p(x_1, \ldots, x_n) = \sum_{j=0}^{k+1} \sum_{i_1+\ldots+i_n=j} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

be a polynomial of degree at most $k+1$ in the real variables $x_1, \ldots, x_n$. It is known that, letting $d = (d_1, \ldots, d_n)$, $d_i \in \mathbb{R}$,

$$\frac{\Delta^d_{k+1} p(x;h)}{h^{k+1}} = \sum_{i_1+\ldots+i_n=k+1} c_{i_1,\ldots,i_n} d_1^{i_1} \cdots d_n^{i_n},$$

so that one can always choose the coefficients of the polynomial so that

$$\inf_{d \in S^1} \frac{\Delta^d_{k+1} p(x;h)}{h^{k+1}} \geq -M$$

for every x and h and hence, with this choice of the coefficients, the function $f(x) + p(x)$ is locally k-convex at x_0.

Theorem 1.1. *[5] Let $f : \Omega \to \mathbb{R}$ be a continuous k-convex function, and let $x \in \Omega$. If f admits a k-th Peano differential at x, then f is k times Fréchet differentiable at the point x and the k-th Fréchet differential of f at x coincides with $\tilde{\alpha}_k(x; \cdot, \ldots, \cdot)$.*

1.2 Standard Mollifiers

The function

$$\phi(x) = \begin{cases} C \exp(\frac{1}{\|x\|^2 - 1}), & \text{if } \|x\| < 1 \\ 0, & \text{if } \|x\| \geq 1 \end{cases}$$

is $C^\infty(\mathbb{R}^n)$ and we can choose the constant $C \in \mathbb{R}$ such that $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$.

Definition 1.5. *Let $\varepsilon > 0$. The functions $\phi_\varepsilon(x) = \frac{\phi(\frac{x}{\varepsilon})}{\varepsilon^n}$ are called standard mollifiers.*

Definition 1.6. *Let $f : \Omega \to \mathbb{R}$. We say that $f \in C_0^k(\Omega)$ if $f \in C^k(\Omega)$ and*

$$spt_f = \overline{\{x \in \Omega : f(x) \neq 0\}} \subset \Omega.$$

Theorem 1.2. *[1] The functions ϕ_ε are $C^\infty(\mathbb{R}^n)$ and satisfy:*

i) $\int_{\mathbb{R}^n} \phi_\varepsilon(x)\,dx = 1$

ii) $spt_{\phi_\varepsilon} \subset B(0,\varepsilon)$.

For a bounded function $f : \Omega \to \mathbb{R}$, and $\varepsilon > 0$, we define functions $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by the convolution $f_\varepsilon(x) = \int_\Omega \phi_\varepsilon(y-x) f(y) dy$. Observe that $f_\varepsilon(x) = 0$ if $x \notin \Omega + B(0,\varepsilon)$ and that $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Theorem 1.3. *[1] Suppose that $f \in L^1_{loc}(\Omega)$. Then $f_\varepsilon(x) \to f(x)$ a.e. $x \in \Omega$, when $\varepsilon \to 0$. If $f \in C(\Omega)$, then the convergence is uniform on compact subsets of Ω.*

Theorem 1.4. *[1] Let K be a compact subset of Ω. Then $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \leq \varepsilon_0$ and $\forall x \in K$, the function $y \to \phi_\varepsilon(y - x)$ is $C_0^\infty(\Omega)$.*

2 A Characterization of $C^{k,1}$ Functions

Definition 2.1. *A function $f : \Omega \to \mathbb{R}$ is locally Lipschitz at x_0 when there exist a constant K and a neighborhood U of x_0 such that*

$$|f(x) - f(y)| \leq K \|x - y\|, \forall x, y \in U.$$

Definition 2.2. *A function $f : \Omega \to \mathbb{R}$ is said to be of class $C^{k,1}$ at x_0 when its k-th order partial derivatives exist in a neighborhood of x_0 and are locally Lipschitz at x_0.*

Theorem 2.1. *Assume that the function $f : \Omega \to \mathbb{R}$ is bounded in a neighborhood of the point $x_0 \in \Omega$. Then f is of class $C^{k,1}$ at x_0 if and only if there exist neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}$ of 0 such that $\dfrac{\Delta^d_{k+1} f(x;h)}{h^{k+1}}$ is bounded by a constant $M > 0$ on $U \times V \backslash \{0\}$ uniformly with respect to $d \in S^1$; that is, $\left|\dfrac{\Delta^d_{k+1} f(x;h)}{h^{k+1}}\right| \leq M \;\forall x \in U,\; \forall h \in V \backslash \{0\},\; \forall d \in S^1$.*

Proof. i) Sufficiency. From Lemmas 1.2 and 1.3, the uniform boundedness of $\dfrac{\Delta^d_{k+1} f(x;h)}{h^{k+1}}$ on $U \times V \backslash \{0\}$ implies the existence of neighborhoods U_1 of x_0 and

V_1 of 0 and a constant $M_1 > 0$ such that for $j = 1, \dots, k$, $\left|\frac{\tilde{\Delta}_j^d f(x;h)}{h^j}\right| \le M_1$, $\forall x \in U_1$, $\forall h \in V_1 \backslash \{0\}$ and $\forall d \in S^1$. Observe that $\left|\frac{\tilde{\Delta}_j^d f(x;h)}{h}\right| \le M_1$, $\forall x \in U_1$, $\forall h \in V_1 \backslash \{0\}$ and $\forall d \in S^1$ means that f is locally Lipschitz at the point x_0 and hence continuous in a neighborhood of x_0.

If we choose a neighborhood U_2 of x_0 such that $U_2 \subset U_1$, we can find a compact set K such that $U_2 \subset K \subset U_1$. By Theorem 1.4, there exists a number $\varepsilon_0 > 0$ such that the function $y \to \phi_\varepsilon(y - x)$ is of class $C_0^\infty(U_1)$, $\forall x \in U_2$, $\forall \varepsilon \le \varepsilon_0$. Hence, $\forall x \in U_2$, $\forall d \in S^1$ and $\forall \varepsilon \le \varepsilon_0$, by Lemma 1.4 and the Lebesgue convergence theorem, for $1 \le j \le k$ we have

$$\begin{aligned}
f_\varepsilon^{(j)}(x;d) &= (-1)^j \int_{\mathbb{R}^n} \phi_\varepsilon^{(j)}(y-x;d) f(y)\, dy \\
&= (-1)^j \int_{U_1} \phi_\varepsilon^{(j)}(y-x;d) f(y)\, dy \\
&= (-1)^j \lambda_j \int_{U_1} \lim_{h \to 0} \frac{\tilde{\Delta}_j^d \phi_\varepsilon(y-x;h)}{h^j} f(y)\, dy \\
&= (-1)^j \lambda_j \int_{U_1} \lim_{h \to 0} \frac{\sum_{i=1}^j a_i \phi_\varepsilon(y-x+2^{i-1}hd) + a_0 \phi_\varepsilon(y-x)}{h^j} f(y)\, dy \\
&= (-1)^j \lambda_j \lim_{h \to 0} \int_{U_1} \frac{\sum_{i=1}^j a_i \phi_\varepsilon(y-x+2^{i-1}hd) + a_0 \phi_\varepsilon(y-x)}{h^j} f(y)\, dy.
\end{aligned}$$

Now, putting $z = y + 2^{i-1}hd$, we obtain

$$\int_{U_1} \frac{a_i \phi_\varepsilon(y-x+2^{i-1}hd)}{h^j} f(y)\, dy = \int_{U_1+2^{i-1}hd} \frac{a_i f(z-2^{i-1}hd)\phi_\varepsilon(z-x)}{h^j}\, dz.$$

Thus

$$\begin{aligned}
&(-1)^j \lambda_j \int_{U_1} \frac{\sum_{i=1}^j a_i \phi_\varepsilon(y-x+2^{i-1}hd) + a_0 \phi_\varepsilon(y-x)}{h^j} f(y)\, dy \\
=&(-1)^j \lambda_j \sum_{i=1}^j \int_{U_1+2^{i-1}hd} \frac{a_i f(z-2^{i-1}hd)\phi_\varepsilon(z-x)}{h^j}\, dz \\
&+ (-1)^j \lambda_j \int_{U_1} \frac{a_0 f(z) \phi_\varepsilon(z-x)}{h^j}\, dz.
\end{aligned}$$

Since K is a compact subset of U_1, there exists a number $h_0 > 0$ such that $\forall h$ with $|h| \le h_0$ we have $K \subset U_1 + 2^{i-1}hd, \quad i = 1, \dots, k$. By Theorem 1.4 we get that $\forall\varepsilon < \varepsilon_0$ the previous equation equals

$$\begin{aligned}
&(-1)^j\lambda_j \sum_{i=1}^{j} \int_{U_1} \frac{a_i f(z-2^{i-1}hd)\phi_\varepsilon(z-x)}{h^j}\,dz \\
&\quad + (-1)^j\lambda_j \int_{U_1} \frac{a_0 f(z)\phi_\varepsilon(z-x)}{h^j}\,dz \\
=&(-1)^j\lambda_j \int_{U_1} \frac{\tilde{\Delta}_j^d f(z,-h)}{h^j}\phi_\varepsilon(z-x)\,dz = \lambda_j \int_{U_1} \frac{\tilde{\Delta}_j^d f(z,-h)}{(-h)^j}\phi_\varepsilon(z-x)\,dz.
\end{aligned}$$

Hence we get

$$f_\varepsilon^{(j)}(x;d) = \lambda_j \lim_{h\to 0} \int_{U_1} \frac{\tilde{\Delta}_j^d f(z,h)}{h^j}\phi_\varepsilon(z-x)\,dz.$$

Since $\left|\dfrac{\tilde{\Delta}_j^d f(x;h)}{h^j}\right| \leq M_1$, $\forall x \in U_1$, $\forall h \in V_1\backslash\{0\}$, $\forall d \in S^1$, by Theorem 1.2 we get $\left|f_\varepsilon^{(j)}(x;d)\right| \leq M_1$ $\forall\varepsilon \leq \varepsilon_0$, $\forall x \in U_2$ and $\forall d \in S^1$. In this way we established that for every $j = 1,\ldots,k$, $f_\varepsilon^{(j)}(\cdot;d)$ is bounded (uniformly with respect to $\varepsilon \leq \varepsilon_0$ and $d \in S^1$) on U_2. Similarly one can prove that $f_\varepsilon^{(k+1)}(\cdot;d)$ is bounded (uniformly with respect to $\varepsilon \leq \varepsilon_0$ and $d \in S^1$) on U_2. Hence, for any $x \in U_2$ and $d \in S^1$, there is a sequence ε_n converging to 0 such that $\forall j = 1\ldots k$, the sequence $f_{\varepsilon_n}^{(j)}(x;d)$, converges to a limit which we denote by $\alpha_j(x;d)$. Note that the functions $\alpha_j(x;d)$, $j = 1,\ldots,k$, are bounded on U_2, uniformly with respect to $d \in S^1$, by the constant M_1.

Let U_3 be a neighborhood of x_0 such that $U_3 \subset U_2$. If h is in a suitable neighborhood of 0, then $x + hd \in U_2$, whenever $x \in U_3$ and $d \in S^1$. The functions $f_{\varepsilon_n}(x)$ are of class C^∞, and hence, for any $x \in U_3$, $d \in S^1$ and h in a neighborhood of 0, we have

$$f_{\varepsilon_n}(x+hd) = f_{\varepsilon_n}(x) + hf'_{\varepsilon_n}(x;d) + \cdots + \frac{h^k}{k!}f_{\varepsilon_n}^{(k)}(x;d) + \frac{h^{k+1}}{(k+1)!}f_{\varepsilon_n}^{(k+1)}(\xi_n;d),$$

where $\xi_n \in (x, x+hd)$. By Theorem 1.3, taking the limit as $n \to +\infty$ it follows that $f_{\varepsilon_n}^{(k+1)}(\xi_n;d)$ converges to a limit which we denote by $\beta(x;h;d)$ and

$$f(x+hd) = f(x) + h\alpha_1(x;d) + \cdots + \frac{h^k}{k!}\alpha_k(x;d) + \frac{h^{k+1}}{(k+1)!}\beta(x;h;d).$$

We observe that $\beta(x;h;d)$ is bounded for each $x \in U_3$, h in a neighborhood of x_0 and $d \in S^1$; so that, putting $r(x;h;d) = \dfrac{h^{k+1}}{(k+1)!}\beta(x;h;d)$, we have that $\lim_{h\to 0} \dfrac{r(x;h;d)}{h^k} = \lim\limits_{x'\to x, h\to 0} \dfrac{r(x';h;d)}{h^k} = 0$, uniformly with respect to $d \in S^1$ (or equivalently with respect to $d \in B(0,\delta)$, whenever $\delta > 0$; see also Remark 1.1). It is easy to see that for each $j = 1,\dots,k$ we have $\alpha_j(x,su) = s^j \alpha_j(x,u)$ for every $s \in \mathbb{R}$ and $u \in \mathbb{R}^n$. Now we prove that, for each $x \in U_3$ there exist j-linear functions $\tilde{\alpha}_j(x;\cdot,\dots,\cdot) : \mathbb{R}^{n\times j} \to \mathbb{R}$, $j = 1,\dots,k$, such that $\tilde{\alpha}_j(x;u,\dots,u) = \alpha_j(x;u)$, $\forall u \in \mathbb{R}^n$; so that f admits a uniform Peano differential at x. We begin proving that $\alpha_1(x;u)$ is linear with respect to $u \in \mathbb{R}^n$. We have only to prove the additive property. In fact from the boundedness of $f'_{\varepsilon_n}(x;d) =< \nabla f_{\varepsilon_n}(x), d >$, for $x \in U_3$ and $d \in S^1$ (uniformly with respect to $\varepsilon \le \varepsilon_0$), it follows that there exists a constant K for which $\|\nabla f(x)\| \le K$ in U_3. Hence, without loss of generality we can think that $\nabla f_{\varepsilon_n}(x) \to a$; so that $\alpha_1(x,u) = \lim_{n\to+\infty} < \nabla f_{\epsilon_n}(x), u >=< a, u >$ whenever $u \in \mathbb{R}^n$ and hence is linear. Inductively suppose that there exist multilinear functions $\tilde{\alpha}_2(x;u_1,u_2)$, $\dots,\tilde{\alpha}_{k-1}(x;u_1,u_2,\dots,u_{k-1})$ such that $\tilde{\alpha}_j(x;u,\dots,u) = \alpha_j(x;u)$, $\forall u \in \mathbb{R}^n$, $j = 1\dots k-1$. Let $x \in U_3$, $v,w \in \mathbb{R}^n$, h and s in suitable neighborhoods of $0 \in \mathbb{R}$; so that $x + sw \in U_3$ and $x + sw + hd \in U_3$. We consider the following estimation of $f(x + sw + shv)$ in two ways, by expansion about x and by expansion about $x + sw$, obtaining

$$\begin{aligned} f(x+sw+shv) &= f(x) + \sum_{j=1}^{k} \frac{1}{j!}\alpha_j(x;sw+shv) + r_1(x;s;w+hv) \\ &= f(x+sw) + \sum_{j=1}^{k} \frac{1}{j!}\alpha_j(x+sw;shv) + r_2(x+sw;s;hv), \end{aligned}$$

where, by Remark 1.1, we have

$$\lim_{s\to 0} \frac{r_1(x;s;w+hv)}{s^k} = \lim_{s\to 0} \frac{r_2(x+sw;s;hv)}{s^k} = 0,$$

uniformly with respect to $v,w \in B(0,\delta)$, whenever $\delta > 0$ and h in a neighborhood of 0 (let us say h with $|h| < c$). Hence

$$\begin{aligned} \alpha_k(x;sw+shv) =& k!\big[f(x+sw) + \sum_{j=1}^{k} \frac{1}{j!}\alpha_j(x+sw;shv) - f(x) \\ & - \sum_{j=1}^{k-1} \frac{1}{j!}\alpha_j(x;sw+shv) + r_2(x+sw;s;hv) \\ & - r_1(x;s;w+hv)\big] \end{aligned}$$

$$\begin{aligned}=&k!\big[f(x+sw)+\sum_{j=1}^{k}\frac{1}{j!}\alpha_j(x+sw;shv)-f(x)\\&-\sum_{j=1}^{k-1}\frac{1}{j!}\tilde{\alpha}_j(x;sw+shv,\dots,sw+shv)+r_2(x+sw;s;hv)\\&-r_1(x;s;w+hv)\big].\end{aligned}$$

Then

$$\begin{aligned}\alpha_k(x;w+hv)=&\frac{k!}{s^k}[f(x+sw)+\sum_{j=1}^{k}\frac{1}{j!}\alpha_j(x+sw;shv)-f(x)\\&-\sum_{j=1}^{k-1}\frac{1}{j!}\tilde{\alpha}_j(x;sw+shv,\dots,sw+shv)\\&+r_2(x+sw;s;hv)-r_1(x;s;w+hv)\big].\end{aligned}$$

Since $\tilde{\alpha}_j(x,\cdot,\dots,\cdot)$ is multilinear for $j\le k-1$ and $\alpha_j(x+sw;\cdot)$ is homogeneous of degree j for $j\le k$, the terms on the right, except for r_2-r_1, form a polynomial in h of degree at most k. Since $\lim_{s\to 0}\dfrac{r_2-r_1}{s^k}=0$ and the limit is uniform for $|h|<c$ and w,v fixed vectors in any ball of center 0 and radius δ, it follows that $\alpha_k(x;w+hv)$ is a uniform limit of polynomials in h of degree less than or equal k if $|h|<c$. Hence $\alpha_k(x;w+hv)$ is itself a polynomial in h of degree less than or equal k if $|h|<c$. Now let $e_1,e_2,\dots,e_n$ be the vectors of the canonical basis of $\mathbb{R}^n$. Putting $h_1e_1+\dots+h_ne_n=w+hv$ where $w=h_1e_1+\dots+h_{m-1}e_{m-1}+h_{m+1}e_m+\dots+h_ne_n$, $h_m=h$ and $e_m=v$, for every $m=1,\dots,n$ the function defined by $\gamma(h_1,\dots,h_n)=\alpha_k(x;h_1e_1+\dots+h_ne_n)$ is a polynomial of degree less than or equal k in h_m if $|h_m|<c$. Hence $\gamma(h_1,\dots h_n)$ is a polynomial in $h_1,h_2,\dots,h_n$ of degree less than or equal nk. Since γ is homogeneous of degree k; that is, $\gamma(sh,\dots,sh)=s^k\gamma(h,\dots,h)$, $\forall s\in\mathbb{R}$, every non zero term in the expression of γ has degree exactly k. It follows that

$$\gamma(h_1,\dots,h_n)=\sum_{i_1+\dots+i_n=k}a_{i_1,i_2,\dots,i_n}h_1^{i_1}\dots h_n^{i_n}.$$

Now let $\underline{h_1}=(h_{1,1},\dots,h_{n,1})$, $\dots,\underline{h_k}=(h_{1,k},\dots,h_{n,k})$. Consider the term $h_j^{i_j}$, $j=1,\dots,n$, and in the previous sum substitute it with the product

$$\prod_{r=\left(\sum_{l=1}^{j-1}i_l\right)+1}^{\sum_{l=1}^{j}i_l}h_{j,r},$$

where we set $\sum_{l=1}^{0} i_l = 0$. In this way we construct a multilinear function $\tilde{\gamma}(\underline{h_1}, \ldots, \underline{h_k})$ on $\mathbb{R}^{n\times k}$ and $\tilde{\gamma}(\underline{h}, \ldots, \underline{h}) = \gamma(h_1, \ldots, h_n)$, where $\underline{h} = (h_1, \ldots, h_n)$. Putting $\tilde{\alpha}_k(x; \cdot, \ldots, \cdot) = \tilde{\gamma}(x; \cdot, \ldots, \cdot)$, it follows that f admits a uniform k-th Peano differential at each point $x \in U_3$. By Remark 1.3, without loss of generality (by eventually adding a polynomial of degree $k+1$) we can always assume that f is k-convex. Hence, by Theorem 1.1 it follows that f is k times differentiable for each $x \in U_3$ and the k-th differential of f at x coincides with $\tilde{\alpha}_k(x; \cdot, \ldots, \cdot)$.

Furthermore since the functions $f_\varepsilon^{(k+1)}(x; d)$ are bounded on U_3 (uniformly with respect to $\varepsilon \le \varepsilon_0$ and $d \in S^1$), the functions $f_{\varepsilon_n}^{(k)}(x; d)$ satisfy the uniform Lipschitz condition

$$\left| f_{\varepsilon_n}^{(k)}(y; d) - f_{\varepsilon_n}^{(k)}(x; d) \right| \le B\|y - x\|, \forall x, y \in U_3, \ \forall d \in S^1.$$

Eventually extracting subsequences we can assume that $f_{\varepsilon_n}^{(k)}(x; d)$ and $f_{\varepsilon_n}^{(k)}(y; d)$ converge to $f^{(k)}(x; d)$ and to $f^{(k)}(y; d)$. So, taking the limit for $n \to +\infty$ we obtain that $f^{(k)}(x; d)$ is Lipschitzian on U_3 whenever $d \in S^1$.

ii) Necessity. Assume that f is of class $C^{k,1}$ at x_0. Set

$$\overline{\Delta}_1[f; s_1; d_1](x) = f(x + s_1 d_1) - f(x),$$

and recursively define

$$\overline{\Delta}_{k+1}[f; s_1, \ldots, s_{k+1}; d_1, \ldots, d_{k+1}](x) =$$

$$\overline{\Delta}_k[f; s_1, \ldots, s_k; d_1, \ldots, d_k](x + s_{k+1} d_{k+1}) - \overline{\Delta}_k[f; s_1, \ldots, s_k; d_1, \ldots, d_k](x),$$

where x is is a neighborhood of x_0, $d_i \in S^1$ and $s_i \in \mathbb{R}$, s_i "small enough", $i = 1, \ldots, k+1$. Observe that, for any $d \in S^1$

$$\overline{\Delta}_1[f; s_1; d_1]'(x; d) = f'(x + s_1 d_1; d) - f'(x; d) = \overline{\Delta}_1[f'(\cdot; d); s_1; d_1](x)$$

and in general

$$\overline{\Delta}_k[f; s_1, \ldots, s_k; d_1, \ldots, d_k]'(x; d) = \overline{\Delta}_k[f'(\cdot; d); s_1, \ldots, s_k; d_1, \ldots, d_k](x).$$

Now, we have, using the mean value theorem and by the definition of $\overline{\Delta}_{k+1}$

$$\frac{\overline{\Delta}_{k+1}[f; s_1, \ldots, s_{k+1}; d_1, \ldots, d_{k+1}](x)}{s_1 \cdots s_{k+1}}$$
$$= \frac{\overline{\Delta}_k[f; s_1, \ldots, s_k; d_1, \ldots, d_k]'(x + \theta_{k+1} s_{k+1} d_{k+1}; d_{k+1})}{s_1 \cdots s_k}$$

$$= \frac{\overline{\Delta}_k[f'(\cdot; d_{k+1}); s_1, \dots, s_k; d_1, \dots, d_k](x + \theta_{k+1} s_{k+1} d_{k+1})}{s_1 \cdots s_k},$$

where $\theta_{k+1} \in (0,1)$. Now we prove by induction that $\forall k \geq 2$

$$\frac{\overline{\Delta}_k[f; s_1, \dots, s_k; d_1, \dots, d_k](x)}{s_1 \cdots s_k}$$
$$= \frac{\overline{\Delta}_1[f^{(k-1)}(\cdot; d_2, \dots d_k); s_1; d_1](x + \theta_2 d_2 + \cdots + \theta_k s_k d_k)}{s_1},$$

where $\theta_2, \dots, \theta_k \in (0,1)$. In fact, the previous equality holds for $k = 2$, as

$$\frac{\overline{\Delta}_2[f; s_1, s_2; d_1, d_2](x)}{s_1 s_2}$$
$$= \frac{f(x + s_1 d_1 + s_2 d_2) - f(x + s_1 d_1) - f(x + s_2 d_2) + f(x)}{s_1 s_2}$$

and setting $g(\tau) = f(x + s_1 d_1 + \tau d_2) - f(x + \tau d_2)$, we have

$$\begin{aligned} \frac{\overline{\Delta}_2[f; s_1, s_2; d_1; d_2]}{s_1 s_2} = \frac{g(s_2) - g(0)}{s_1 s_2} &= \frac{g'(\theta_2 s_2)}{s_1} \\ &= \frac{f'(x + s_1 d_1 + \theta_2 s_2 d_2; d_2) - f'(x + \theta_2 s_2 d_2; d_2)}{s_1} \\ &= \frac{\overline{\Delta}_1[f'(\cdot; d_2); s_1, d_1](x + \theta_2 s_2 d_2)}{s_1}, \end{aligned}$$

where $\theta_2 \in (0,1)$. Now we assume that the stated equality holds for $\overline{\Delta}_{k-1}$ and we prove that it holds also for $\overline{\Delta}_k$. In fact

$$\frac{\overline{\Delta}_k[f; s_1, \dots, s_k; d_1, \dots, d_k](x)}{s_1 \cdots s_k}$$
$$= \left\{ \begin{matrix} \overline{\Delta}_{k-1}[f; s_1, \dots, s_{k-1}; d_1, \dots, d_{k-1}](x + s_k d_k) - \\ \overline{\Delta}_{k-1}[f; s_1, \dots, s_{k-1}; d_1, \dots, d_{k-1}](x) \end{matrix} \right\} \Big/ (s_1 \cdots s_k)$$
$$= \frac{\overline{\Delta}_{k-1}[f; s_1, \dots, s_{k-1}; d_1, \dots, d_{k-1}]'(x + \theta_k s_k d_k; d_k)}{s_1 \cdots s_{k-1}}$$
$$= \frac{\overline{\Delta}_{k-1}[f'(\cdot, d_k); s_1, \dots, s_{k-1}; d_1, \dots, d_{k-1}](x + \theta_k s_k d_k)}{s_1 \cdots s_{k-1}}$$
$$= \frac{\overline{\Delta}_1[f^{(k-1)}(\cdot; d_2, \dots, d_{k-1}); s_1; d_1](x + \theta_2 s_2 d_2 + \cdots + \theta_k s_k d_k)}{s_1},$$

where $\theta_2, \ldots, \theta_k \in (0,1)$. Hence

$$\frac{\overline{\Delta}_{k+1}[f; s_1, \ldots, s_{k+1}; d_1, \ldots, d_{k+1}](x)}{s_1 \cdots s_{k+1}}$$
$$= \frac{\overline{\Delta}_1[f^{(k)}(\cdot; d_2, \ldots, d_k); s_1; d_1](x + \theta_2 s_2 d_2 + \cdots + \theta_{k+1} s_{k+1} d_{k+1})}{s_1},$$

where $\theta_2, \ldots, \theta_{k+1} \in (0,1)$. Since f is of class $C^{k,1}$ at x_0, there is a constant M, a neighborhood U of x_0 and a number $\delta > 0$ such that

$$\left| \frac{\overline{\Delta}_{k+1}[f; s_1, \ldots, s_{k+1}; d_1, \ldots, d_{k+1}](x)}{s_1 \cdots s_{k+1}} \right| \leq M,$$

$\forall x \in U$, $|s_i| < \delta$, $s_i \neq 0$, $d_i \in S^1$, $i = 1, \ldots, k+1$. Now the conclusion follows easily observing that if $s_1 = s_2 = \cdots = s_{k+1} = h$ and $d_1 = \cdots = d_{k+1} = d$, then $\overline{\Delta}_{k+1}[f; s_1, \ldots, s_{k+1}; d_1, \ldots, d_{k+1}](x) = \Delta^d_{k+1} f(x; h)$. □

Corollary 2.1. *Assume that the function f is bounded on a neighborhood of x_0. Then f is of class $C^{k,1}$ at x_0 if and only if there exist neighborhoods U of x_0 and V of 0 such that $\frac{\delta^d_{k+1} f(x; h)}{h^{k+1}}$ is bounded on $U \times V \backslash \{0\}$, uniformly with respect to $d \in S^1$.*

PROOF. The proof is straightforward recalling that

$$\Delta^d_{k+1} f(x; h) = \delta^d_{k+1} f(x + \frac{k+1}{2} h; h). \qquad \square$$

Theorem 2.2. *Assume that f is continuous and $\underline{D}_{k+1} f(x; d)$ exists on a neighborhood of the point x_0, $\forall d \in S^1$. Then f is of class $C^{k,1}$ at x_0 if and only if there exists a neighborhood U of x_0 and a function $g \in L^1(U)$ such that:*

i) $\exists M \geq 0$ such that $|D_{k+1} f(x; d)| \leq M$, $\forall x \in U$, $\forall d \in S^1$,

ii) $\left| \frac{\Delta^d_{k+1} f(x; h)}{h^{k+1}} \right| \leq g(x)$, for $|h|$ "small enough" ($h \neq 0$), $d \in S^1$ and a.e. $x \in U$.

PROOF. i) Sufficiency. Arguing in a fashion similar to that of the previous theorem and using Lebesgue's theorem, for ε "sufficiently small", for every x in a neighborhood of x_0 and $d \in S^1$ we obtain

$$f^{(k+1)}_{\varepsilon}(x; d) = \lim_{h \to 0} \int_{\Omega} \frac{\Delta^d_{k+1} f(z; h)}{h^{k+1}} \phi_{\varepsilon}(z - x)\, dz$$

$$= \int_\Omega \lim_{h\to 0} \frac{\Delta^d_{k+1} f(z;h)}{h^{k+1}} \phi_\varepsilon(z-x)\,dz = \int_\Omega D_{k+1} f(z;d)\phi_\varepsilon(z-x)\,dz.$$

For each $d \in S^1$ $f_\varepsilon^{k+1}(x,d)$ is bounded on U (uniformly with respect to ε). Using the integral representation of divided differences (see for instance [16], ch. 6, th. 2), we have

$$\frac{\Delta_{k+1} f_\varepsilon(x;h)}{h^{k+1}} = \int_0^1 dt_1 \int_0^{t_1} dt \ldots \int_0^{t_k} f_\varepsilon^{(k+1)}(x + t_{k+1}hd + \cdots + t_1 hd; d)\,dt_{k+1}.$$

For x and h in suitable neighborhoods of x_0 and 0 respectively, the left member in the previous inequality is bounded by a constant M (uniformly with respect to ε). Sending ε to 0 by Theorem 1.3, we get the existence of neighborhoods U of x_0 and V of $0 \in \mathbb{R}$ such that $\forall d \in S^1$, $\dfrac{\Delta^d_{k+1} f(x;h)}{h^{k+1}}$ is bounded on $U \times V\backslash\{0\}$. The conclusion now follows from Theorem 2.1.

ii) Necessity. The proof is similar to that of the necessary condition in Theorem 2.1. □

Remark 2.1. *Theorems 2.1 and 2.2 extend the elementary condition which relates the Lipschitz condition on $f^{(k)}(x;d)$ and the boundedness of $f^{(k+1)}(x;d)$. We generalize this relation without requiring any differentiability hypothesis and linking the existence and the Lipschitz behavior of $f^{(k)}(x;d)$ to the boundedness of $\dfrac{\Delta^d_{k+1} f(x;h)}{h^{k+1}}$ or of the upper and lower Riemann derivatives.*

Remark 2.2. *Conditions similar to those of Theorem 2.2, expressed in terms of $\mathcal{D}_{k+1} f(x;d)$ can be proved in an analogous way.*

3 Taylor's Formula for $C^{k,1}$ Functions

In this section we give a Taylor's formula for $C^{k,1}$ functions expressed by means of Riemann derivatives. The proof of the following lemma is included in that of Theorem 2.1

Lemma 3.1. *If f is of class $C^{k,1}$ at x_0, then for every $d \in S^1$ there exist sequences ε_n converging to 0 and $\xi_n \in (x_0, x_0 + hd)$ such that $f_{\varepsilon_n}^{(k+1)}(\xi_n; d)$ converges to a limit $\beta(x_0;h;d)$ and*

$$\begin{aligned} f(x_0 + hd) =& f(x_0) + hf'(x_0;d) + \frac{h^2}{2} f''(x_0;d) + \cdots \\ &+ \frac{h^k}{k!} f^{(k)}(x_0;d) + \frac{h^{k+1}}{(k+1)!}\beta(x_0;h;d). \end{aligned}$$

Theorem 3.1. *Let f be of class $C^{k,1}$ at x_0. If the function $x \to \overline{D}_{k+1}(x;d)$ is upper semicontinuous in a neighborhood of x_0, then there exists $\xi \in [x_0, x_0+td]$ such that*

$$f(x_0+hd) \leq f(x_0) + hf'(x_0;d) + \frac{h^2}{2}f''(x_0;d) + \cdots + \frac{h^{k+1}}{(k+1)!}\overline{D}_{k+1}f(\xi;d).$$

If the function $x \to \underline{D}_{k+1}(x;d)$ is lower semicontinuous in a neighborhood of x_0, then there exists $\xi \in [x_0, x_0+td]$ such that

$$f(x_0+hd) \geq f(x_0) + hf'(x_0;d) + \frac{h^2}{2}f''(x_0;d) + \cdots + \frac{h^{k+1}}{(k+1)!}\underline{D}_{k+1}f(\xi;d).$$

PROOF. Without loss of generality, the term $\beta(x_0;h;d)$ in the previous lemma can be expressed as $\beta(x_0;h;d) = \lim_{n\to+\infty} f_{\epsilon_n}^{(k+1)}(\xi_n;d)$ for some sequences $\xi_n \to \xi \in [x_0, x_0+hd]$ and $\epsilon_n \to 0$. As in the proof of Theorem 2.1, one can write that[1]

$$\begin{aligned}
f_{\varepsilon_n}^{(k+1)}(\xi_n,d) &= (-1)^{k+1}\int_\Omega \phi_{\varepsilon_n}^{(k+1)}(y-\xi_n;d)f(y)\,dy \\
&= (-1)^{k+1}\lim_{h\to 0}\int_{B(0,1)} \frac{\Delta_{k+1}^d \phi_{\varepsilon_n}(y-\xi_n;h)}{h^{k+1}}f(y)\,dy \\
&= (-1)^{k+1}\lim_{h\to 0}\int_{B(0,1)} \frac{\Delta_{k+1}^d f(\xi_n+\varepsilon_n y;h)}{h^{k+1}}\phi_{\varepsilon_n}(y)\,dy \\
&\leq (-1)^{k+1}\int_{B(0,1)} \limsup_{h\to 0}\frac{\Delta_{k+1}^d f(\xi_n+\epsilon_n y;h)}{h^{k+1}}\phi(y)\,dy \\
&= \int_{B(0,1)} \overline{D}_{k+1}f(\xi_n+\epsilon_n y;d)\phi(y)\,dy.
\end{aligned}$$

Now using the upper semicontinuity of $\overline{D}_{k+1}f(\cdot;d)$ we have

$$\begin{aligned}
\beta(x;h;d) &\leq \int_{B(0,1)} \limsup_{n\to+\infty}\overline{D}_{k+1}f(\xi_n+\epsilon_n y;d)\phi(y)\,dy \\
&\leq \int_{B(0,1)} \overline{D}_{k+1}f(\xi;d)\phi(y)\,dy = \overline{D}_{k+1}f(\xi;d)
\end{aligned}$$

and the proof of the first inequality is complete. The proof of the second inequality is analogous. $\square$

[1] In the proof of this theorem we will use the following generalized version of Fatou's lemma: *if f_n is a sequence of measurable functions, $f_n \geq M$ and $E \subset \mathbb{R}^n$ is a subset of finite measure, then* $\limsup_{n\to+\infty}\int_E f_n \leq \int_E \limsup_{n\to+\infty} f_n$

When $D_{k+1}f(\cdot;d)$ exists in a neighborhood of x_0 and is continuous, the previous theorem reduces to the classical Taylor's formula, as one can easily deduce from the following result.

Theorem 3.2. *Let $f:\Omega\to\mathbb{R}$ be a function of class $C^{k,1}$ at x_0. If $D_{k+1}f(\cdot;d)$ exists and is continuous in a neighborhood of x_0, then $f^{(k+1)}(x_0;d)$ exists and $D_{k+1}f(x_0;d)=f^{(k+1)}(x_0;d)$.*

PROOF. From the previous theorem it follows that

$$f(x_0+hd)=f(x_0)+hf'(x_0;d)+\cdots+\frac{h^k}{k!}f^{(k)}(x_0;d)+\frac{h^{k+1}}{(k+1)!}D_{k+1}f(\xi;d),$$

where $\xi\in[x_0,x_0+hd]$. Considering the expansion of the terms $f(x_0+ihd)$, $i=1,\dots,k$, about x_0 and about x_0+hd we obtain

$$\begin{aligned}\Delta_k^d f(x_0;h)=&\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}[f(x_0)+ihf'(x_0;d)+\cdots\\&+\frac{(ih)^k}{k!}f^{(k)}(x_0;d)+\frac{(ih)^{k+1}}{(k+1)!}D_{k+1}f(\xi_i;d)]\\=&\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}[f(x_0+hd)+(i-1)hf'(x_0+hd;d)+\cdots\\&+\frac{[(i-1)h]^k}{k!}f^{(k)}(x_0+hd;d)+\frac{[(i-1)h]^{k+1}}{(k+1)!}D_{k+1}f(\overline{\xi}_i;d)]\end{aligned}$$

where $\xi_i\in[x_0,x_0+ihd]$ and $\overline{\xi}_i\in[x_0+hd,x_0+ihd]$. From [6] we have $\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^j=0$ for $j=0,1,\dots,k-1$, and $\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^k=k!$, after simple calculations from the previous equalities we get

$$\begin{aligned}[f^{(k)}(x_0+hd;d)-f^{(k)}(x_0;d)]h^k=&\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{(ih)^{k+1}}{(k+1)!}D_{k+1}f(\xi_i;d)\\&-\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{[(i-1)h]^{k+1}}{(k+1)!}D_{k+1}f(\overline{\xi}_i;d).\end{aligned}$$

Dividing by h^{k+1} and taking the limit as $h\to 0$ using the continuity of $D_{k+1}f(\cdot;d)$ we obtain

$$f^{(k+1)}(x_0;d)=\lim_{h\to 0}\frac{f^{(k)}(x_0+hd;d)-f^{(k)}(x_0;d)}{h}$$

$$= D_{k+1}f(x_0;d)\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\left[\frac{(i)^{k+1}}{(k+1)!}-\frac{[(i-1)]^{k+1}}{(k+1)!}\right]$$
$$= D_{k+1}f(x_0;d). \qquad \square$$

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