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THE NON-UNIFORM RIEMANN APPROACH TO ITÔ'S INTEGRAL

Abstract

In this paper, we shall consider two generalized Riemann approaches to the Itô integral; namely, the Itô-Henstock and the Itô-McShane approaches, and by establishing the equivalence of the Itô-Henstock integral with the classical Itô integral, prove the equivalence of all the three integrals.

1 Introduction

It is well-known and often emphasized in texts that it is impossible to define stochastic integrals using the Riemann approach, since the integrators have paths of unbounded variation while the integrands are usually highly oscillating. As it is known in the field of Henstock integration, a generalized Riemann approach is designed to integrate functions which are highly oscillating which the usual Riemann approach fails to handle.

The generalized Riemann approach was introduced by J. Kurzweil and R. Henstock independently in 1950s. They used non-uniform meshes (that vary from point to point) instead of uniform meshes as in the usual Riemann approach. This technically minor but conceptually important modification of the classical Riemann approach leads to integrals which are more general than both the Riemann-Stieltjes and even the Lebesgue-Stieltjes integral.

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The generalized Riemann approach has been used to study Itô's stochastic integral, see [4, 7, 8, 10, 11, 12, 14]. In [4, 7], Henstock and Lee T. W. used a *full* division of $[0, 1]$ to define a stochastic integral for deterministic integrands, which was a direct generalisation of the classical non-stochastic case.

In [8], McShane used a belated *partial* division to define his stochastic integral. Protter in [10] proved that McShane's integral is equivalent to the classical Itô's integral. In Section 5 of this note, by using the mean convergence theorem, we shall give another proof of the equivalence.

In [12], Xu and Lee defined a stochastic integral by using a *full* belated division and proved the equivalence of this integral and the classical Itô integral. They used a full-belated division which was obtained from Henstock's full division in [4]. The construction of this belated full division is technically involved. A variational approach was introduced in [11] to study the Itô integral, which is a modification of [8] by using *partial* divisions.

In this paper, we shall consider two generalized Riemann approaches to the Itô integral; namely, the Itô-Henstock and the Itô-McShane approaches, and by establishing the equivalence of the Itô-Henstock integral with the classical Itô's integral, prove the equivalence of all the three integrals. This offers an alternative proof of equivalence given in [11]. For the convenience of our presentation, we shall consider the integral over the interval $[0, 1]$ throughout our discussion, although the theory holds over any compact interval $[a, b]$.

2 Itô's Stochastic Integral

Let Ω denote the set of all real-valued continuous functions on $[0, 1]$ and $\mathbb{R}$ the set of all real numbers.

The class of all Borel cylindrical sets B in Ω, denoted by $\mathcal{C}$, is a collection of all the sets B in Ω of the form

$$B = \{w : (w(t_1), w(t_2), \ldots, w(t_n)) \in E\}$$

where $0 \le t_1 < t_2 < \cdots < t_n \le 1$ and E is any Borel set in $\mathbb{R}^n$ (n is not fixed). The Borel σ-field of $\mathcal{C}$ is denoted by $\mathcal{F}$; i.e., it is the smallest σ-field which contains $\mathcal{C}$. Let P be the Wiener measure defined on $(\Omega, \mathcal{F})$. Then $(\Omega, \mathcal{F}, P)$ is a probability space; that is, a measure space with $P(\Omega) = 1$.

Let $\{\mathcal{F}_t\}$ be an increasing family of σ-subfields of $\mathcal{F}$ for $t \in [0, 1]$; that is, $\mathcal{F}_r \subset \mathcal{F}_s$ for $0 \le r < s \le 1$ with $\mathcal{F}_1 = \mathcal{F}$. The probability space together with its family of increasing σ-subfields is called a *standard filtering space* and denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

A process $\{\varphi(t, \omega) : t \in [0, 1]\}$ on $(\Omega, \mathcal{F}, P)$ is a family of $\mathcal{F}$-measurable functions (which are called random variables) on $(\Omega, \mathcal{F}, P)$. Very often, $\varphi(t, \omega)$

is denoted by $\varphi_t(\omega)$. Now we shall consider a very special and important process; namely, the Brownian motion denoted by W.

Let $W = \{W_t(\omega)\}_{0\le t\le 1}$ be a canonical Brownian motion; that is, it satisfies the following properties:

1. $W_0(\omega) = 0$ for all $\omega \in \Omega$;
2. it has **Normal Increments**; that is, $W_t - W_s$ has a Normal distribution with mean 0 and variance $t - s$ for all $t > s$ (which naturally implies that W_t has a Normal distribution with mean 0 and variance t);
3. it has **Independent Increments**; that is, $W_t - W_s$ is independent of its past; that is, W_u, $0 \le u < s < t$; and
4. its sample paths are continuous; i.e., for each $\omega \in \Omega$, $W_t(\omega)$ as a function of t is continuous on $[0, 1]$.

A process $\{\varphi_t(\omega) : t \in [0, 1]\}$ is said to be adapted to the filtering $\{\mathcal{F}_t\}$ if for each $t \in [0, 1]$, φ_t is $\mathcal{F}_t$-measurable. We always assume that $W = \{W_t(\omega)\}$ is adapted to $\{\mathcal{F}_t\}$. For example, if $\{\mathcal{F}_t\}$ is the smallest σ-field generated by $\{W_s(\omega) : s \le t\}$. Then $W = \{W_t(\omega)\}$ is adapted to $\{\mathcal{F}_t\}$.

A process $X = \{X_t(\omega) : t \in [0, 1]\}$ on the standard filtering space is said to be a *Martingale* if

1. X is adapted to $\{\mathcal{F}_t\}$; that is, X_t is $\mathcal{F}_t$-measurable for each $t \in [0, 1]$;
2. $\int_\Omega |X_t| dP$ is finite for almost all $t \in [0, 1]$, and
3. $E(X_t|\mathcal{F}_s) = X_s$ for all $t \ge s$, where $E(X_t|\mathcal{F}_s)$ is the conditional expectation of X_t given $\mathcal{F}_s$, which is defined to be a random variable such that $E(X_t|\mathcal{F}_s)$ is $\mathcal{F}_s$-measurable and $\int_A E(X_t|\mathcal{F}_s)\, dP = \int_A X_t\, dP$ for each $A \in \mathcal{F}_s$. By Radon-Nikodym Theorem, $E(X_t|\mathcal{F}_s)$ is well-defined.

If in addition we have $\sup_{t\in[0,1]} \int_\Omega |X_t|^2\, dP$ is finite, we say that X is a L_2-martingale.

In the following we define $E(f)$ to be $\displaystyle\int_\Omega f\, dP$ for any random variable f.

It is well-known, see [9, p. 239] for example, that the following properties hold (details are given for the convenience of readers who are not familiar with stochastic analysis):

(i) $E[X_s] = E\left[E[X_t|\mathcal{F}_s]\right] = E[X_t]$ for any $t \ge s$; that is, $E[X_s]$ is a constant for all $s \in [0, 1]$.

(ii) For any $0 \le u < v \le s < t \le 1$, we have

$$E[(X_t - X_s)(X_v - X_u)] = E\left\{E[(X_t - X_s)(X_v - X_u)|\mathcal{F}_s]\right\}$$

$$=E\{(X_v - X_u)E[X_t - X_s|\mathcal{F}_s]\}$$
$$=E\{(X_v - X_u)[E[X_t|\mathcal{F}_s] - X_s]\} = 0,$$

that is, a martingale has orthogonal increments.

(iii) From (ii) we get $E\left|(D)\sum(X_v - X_u)\right|^2 = (D)\sum E(X_v - X_u)^2$ for any partial partition $D = \{[u,v]\}$ of $[0,1]$.

(iv) For any $u < v$ we have

$$E[X_v X_u] = E\left[E[X_v X_u|\mathcal{F}_u]\right] = E\left[X_u E[X_v|\mathcal{F}_u]\right] = E[X_u^2]$$

and then $E(X_v - X_u)^2 = E(X_v^2 - X_u^2)$.

It is also well-known, see [9, p. 28], that a canonical Brownian motion is a martingale. In fact, it is an L_2-martingale with $E(W_t^2) = t$, see property 2 of a Brownian motion.

Let $L_2(\Omega, \mathcal{F}, P)$ be the space of all real-valued $\mathcal{F}$-measurable functions q on Ω (which are called random variables) such that $||q||_{L_2}^2 = \int_\Omega |q(w)|^2\,dP < \infty$. Since Ω is separable, L_2 is separable.

We denote by $\mathcal{L}_2$ the space of all processes $\{\varphi(t,w)\}_{0\le t\le 1}$ defined on $(\Omega, \mathcal{F}, P)$ such that $||\varphi||_{\mathcal{L}_2}^2 = \int_0^1\int_\Omega |\varphi(t,w)|^2\,dP\,dt < \infty$.

We further assume that $\{\varphi_t(\omega) : t \in [0,1]\}$ is adapted to the filtration $\{\mathcal{F}_t\}$; that is, φ_t is $\mathcal{F}_t$-adapted for each $t \in [0,1]$.

Recall that $E[\cdot]$ denotes $\int_\Omega \cdot\, dP$ so that we write $||\varphi||_{\mathcal{L}_2}^2 = \int_0^1 E|\varphi(t,w)|^2\,dt$.

Let $\mathcal{L}_0$ be the set of all step processes $\varphi(t,w)$ satisfying the following conditions:

1. there exists $M > 0$ such that $|\varphi(t,w)| \le M$ for all $t \in [0,1]$ and all $w \in \Omega$;

2. there are a finite sequence of points $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ and a finite sequence of random variables $f_i(w), i = 0,1,2,\ldots,n$, such that each f_i is $\mathcal{F}_{t_i}$-measurable for $i = 0,1,2,\ldots,n$, and that $\varphi(t,w) = f_0(\omega)\chi_{\{0\}}(t) + \sum_{i=1}^n f_{i-1}(w)\chi_{(t_{i-1},t_i]}(t)$ for $t \in [0,1]$ where $\chi_{(t_i,t_{i+1}]}$ denotes the characteristic function of $(t_i, t_{i+1}]$.

Then $\mathcal{L}_0$ is dense in $\mathcal{L}_2$ (see, for example, [9, p. 22–25]); i.e., for every $\varphi \in \mathcal{L}_2$ there is a sequence $\{\varphi_1, \varphi_2, \ldots\}$ in $\mathcal{L}_0$ such that $||\varphi_m - \varphi||_{\mathcal{L}_2} \to 0$ as $m \to \infty$.

For each $\varphi \in \mathcal{L}_0$, define the Itô integral of φ to be

$$I(\varphi)(\omega) = \sum_{i=1}^n f_{i-1}(\omega)[W(t_i,\omega) - W(t_{i-1},\omega)].$$

Note that $I(\varphi) \in L_2$. Since L_2 is complete and using the Itô isometry, see [9, p. 23], or see the last equality of this section, for a general $\varphi \in \mathcal{L}_2$, we can define its Itô integral $I(\varphi)$ to be $I(\varphi) = \lim_{n\to\infty} I(\varphi_n)$ in L_2, where $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{L}_0$ converging to φ, see [9, p. 22–26]. The Itô integral is uniquely determined in L_2. In this paper, the classical Itô integral of φ with respect to W is written as $(I)\displaystyle\int_0^1 \varphi_t dW_t$. It is well-known that the Itô isometry property holds. $E\left((I)\int_0^1 \varphi_t dW_t\right)^2 = E\left(\int_0^1 \varphi_t^2\, dt\right)$, see [8, p. 23].

3 Non-Uniform Mesh

In this section we shall discuss three ways of constructing non-uniform meshes in defining the stochastic integrals.

Type a. McShane's Full Division.

Let $\delta(x) > 0$ be a function defined on all $x \in [0,1]$. A finite collection D of interval-point pairs $\{(I_i, x_i) : i = 1, 2, 3, \ldots, n\}$ is said to be a McShane δ-fine full division of $[0,1]$ if

1. $I_i, i = 1, 2, 3, \ldots, n$, are disjoint left-open subintervals of $[0,1]$,
2. $\displaystyle\bigcup_{i=1}^{n} \overline{I_i} = [0,1]$, where $\overline{I_i}$, for each $i = 1, 2, 3, \ldots, n$, is the closure of I_i, and
3. each (I_i, x_i) is McShane δ-fine; that is, $I_i \subset [x_i - \delta(x_i), x_i + \delta(x_i)]$.

Note that in defining McShane's δ-fine full division above we do not assume that x_i must be contained in I_i.

In our definition of stochastic integral we do not use McShane's full division as above since the integral defined does not reflect its adaptedness property. Instead, we consider McShane's δ-fine *belated* division of $[0,1]$ where each x_i is to the left of the interval I_i. However, such a δ-fine belated *full* division may not exist, for example, take $\delta(x) = (1-x)/2$. We thus consider a δ-fine McShane's belated partial division that covers *almost* the entire interval $[0,1]$ in Type b. Such a division always exists by virtue of Vitali's Covering Theorem.

Type b. McShane's Belated Partial Division.

A finite collection D of interval-point pairs $\{(I_i, x_i) : i = 1, 2, 3, \ldots, n\}$ is a McShane's (δ, η)-fine belated partial division of $[0,1]$ if

1. $I_i, i = 1, 2, 3, \ldots n$, are disjoint left-open subintervals of $[0,1]$,
2. each I_i is belated δ-fine; that is, $I_i \subset [x_i, x_i + \delta(x_i)]$, and
3. $\left| [0,1] \backslash \bigcup_{i=1}^{n} I_i \right| < \eta$, where $|J|$ denotes the length of the interval J .

Type c. Belated Partial Division.

If the collection of intervals $\{(I_i, x_i) : i = 1, 2, 3, \ldots, n\}$ only satisfies conditions 1 and 2 but not 3 above, we call it a δ-fine belated partial division.

4 Itô-Henstock's Approach to Itô's Integral

In this section we shall discuss a non-uniform Riemann approach to Itô's integral using belated partial divisions of Type c in Section 3 above. This is motivated by the Henstock Lemma for the classical Henstock integration theory, (see for example [5]).

Definition 1. Let $\mathcal{I}$ be a family of all left-open subintervals of $[0,1]$, and $F(I,\omega)$ be a real-valued function defined on $\mathcal{I} \times \Omega$. Then F is an additive function on $\mathcal{I}$ if for every pair of disjoint intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$, we have $F(I \cup J, \omega) = F(I,\omega) + F(J,\omega)$ for all $\omega \in \Omega$.

An example of an additive function is the primitive function of Itô's stochastic integral; that is,

$$F(J,\omega) = (I) \int_a^b \varphi_t(\omega) dW_t(\omega)$$

where $J = (a, b]$.

We are now ready to give the definition of Itô-Henstock integral.

Definition 2. [11]. Let $\varphi = \{\varphi_t : t \in [0,1]\}$ be a processes adapted to the standard filtering space. Then φ is said to be Itô-Henstock (IH) integrable on $[0,1]$ with respect to Brownian motion W if for any $\varepsilon > 0$ there exists an additive function $F : \mathcal{I} \times \Omega \to \mathbb{R}$ and a $\delta(x) > 0$ on $[0,1]$ such that for any δ-fine belated partial division $D = \{((u_i, v_i], \xi_i) : i = 1, 2, 3, \ldots, n\}$ of $[0,1]$ we have

$$E\left(\sum_{i=1}^{n} \{\varphi(\xi_i,\omega)[W(v_i,\omega) - W(u_i,\omega)] - F((u_i,v_i],\omega)\}\right)^2 < \varepsilon.$$

We denote $F((0,1],\cdot)$ by $(IH)\int_0^1 \varphi_t dW_t$. The function F is unique up to zero variation, (see [3, p. 76–77]). Very often, we shall impose some conditions on F; for example, absolute continuity in this note, such that each equivalence class is small. For a non-stochastic case, see [1; 8].

Theorem 3. (Basic Properties of IH-integral)

1. *Let f and g be adapted processes on $[0,1]$ which are IH integrable with respect to W on $[0,1]$, and let $\alpha \in \mathbb{R}$. Then $f \pm g$, αf are IH integrable with respect to W and that*
$$\begin{aligned}(IH)\int_0^1 (f_t \pm g_t) dW_t =& (IH)\int_0^1 f_t dW_t \pm (IH)\int_0^1 g_t dW_t \\ (IH)\int_0^1 \alpha f_t dW_t =& \alpha (IH)\int_0^1 f_t dW_t\end{aligned}$$
2. *Let the adapted process f be IH integrable on $[0,a]$ and $[a,1]$ with respect to W. Then f is IH integrable on $[0,1]$ and further*
$$(IH)\int_0^1 f_t dW_t = (IH)\int_0^a f_t dW_t + (IH)\int_a^1 f_t dW_t.$$
3. *If f is IH integrable on $[0,1]$ with respect to W, then f is IH integrable on any subinterval $[c,d]$ of $[0,1]$.*

Theorem 3 consists of standard results of classical theory of Henstock's integration see for example [2,3,5,6], hence omitted.

Definition 4. Let A and $A^{(n)}$, $n = 1,2,3,\ldots$, be real-valued functions on $\mathcal{I} \times \Omega$. Then $A^{(n)}$ is said to *converge variationally* to A if given $\varepsilon > 0$ there exists a positive integer N such that for any finite collection of disjoint intervals $\{(u_i,v_i] : i = 1,2,3,\ldots,q\}$ for all $n \geq N$
$$E\left(\sum_{i=1}^{q}\left\{A^{(n)}((u_i,v_i],\omega) - A((u_i,v_i],\omega)\right\}\right)^2 < \varepsilon.$$

We remark that the convergence in Definition 4 above is also called the *Mean Convergence* for deterministic case, (see [5, p. 17] for example).

Proposition 5. *Let $\varphi \in \mathcal{L}_2$ and $\{\varphi_n\}$ a sequence of step processes in $\mathcal{L}_0$ such that $\varphi_n \to \varphi$ in $\mathcal{L}_2$. Let F denote the primitive function of the classical Itô stochastic integral of φ and F^n of φ_n for each $n = 1,2,3,\ldots$. Then F^n converges to F variationally.*

PROOF. Let $\varepsilon > 0$ be given. If $\varphi \in \mathcal{L}_2$, let $F_t = (I)\int_0^t \varphi_t dW_t$ be the primitive function. Then F_t is an L_2 -martingale, (see [9, Corollary 3.14, p. 30; [13], Remark 12.15, p. 241]). Similarly for each $n \in \mathbb{N}$, if $F_t^n = (I)\int_0^t \varphi_t^n dX_t$, then each F_t^n is also an L_2-martingale. Choose an integer $N > 0$ such that whenever $n \geq N$

$$E\left|(I)\int_0^1 \varphi_t^n dX_t - (I)\int_0^1 \varphi_t dX_t\right|^2 < \varepsilon.$$

Given a finite collection of disjoint subintervals $D = \{(u, v]\}$ from $[0, 1]$, let $\sum_1$ denote the summation over those intervals included in D and $\sum_2$ be the summation over the collection of subintervals of $[0, 1]$ of D^c which consists of all left-open subintervals of $(0, 1]$ not included in D and $D \cup D^c = (0, 1]$. Let $\sum_{1+2} = \sum_1 + \sum_2$. Then

$$\begin{aligned}
E\Big|\sum_1 (I)\int_u^v \varphi_t^n dX_t - \sum_1 (I)\int_u^v \varphi_t dX_t\Big|^2 \\
&= E\Big|\sum_1 (F_v^n - F_v^n) - \sum_1 (F_v - F_u)\Big|^2 \\
&= E\Big|\sum_1 \{(F_v^n - F_v) - (F_u^n - F_u)\}\Big|^2 = E\Big|\sum_1 (\Psi_v^n - \Psi_u^n)\Big|^2
\end{aligned}$$

where $\Psi_t^n = F_t^n - F_t$, which is also an L_2-martingale since the difference of two martingales is again a martingale. From the orthogonal increment of martingales, we thus have

$$\begin{aligned}
E\left|\sum_1 (\Psi_v^n - \Psi_u^n)\right|^2 &= \sum_1 E(\Psi_v^n - \Psi_u^n)^2 \\
&\leq \sum_1 E(\Psi_v^n - \Psi_u^n)^2 + \sum_2 E(\Psi_v^n - \Psi_u^n)^2 \\
&= E\left|\sum_{1+2} (\Psi_v^n - \Psi_u^n)\right|^2 = E\left|\Psi_1^n - \Psi_0^n\right|^2 \\
&= E\left|(I)\int_0^1 \varphi_t^n dX_t - (I)\int_0^1 \varphi_t dX_t\right|^2 < \varepsilon.
\end{aligned} \tag{1}$$

We remark that (1) is due to the orthogonal increment of martingales, (see (ii) and (iii) in Section 2.) □

We shall next establish a series of results, ultimately leading to the establishment of the equivalence of the IH integral and the classical Itô integral at the end of this section. As mentioned in the introduction section of this note, the equivalence was proved in [11]. Here we offer an alternative proof by using convergence theorem.

Proposition 5a. *Suppose $D = \{((u_i, v_i], \xi_i)\}$ is a δ-fine belated partial division and $h(\xi_i, \omega)$ is a function which is $\mathcal{F}_{\xi_i}$-measurable. Then*

$$E\Big((D)\sum_i h(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]\Big)^2 = E\Big((D)\sum_i h^2(\xi_i,\omega)(v_i-u_i)\Big).$$

Proof. Note that

$$\begin{aligned}
&E\Big((D)\sum_i h(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]\Big)^2\\
=&E\Big\{\sum_i \Big(h(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]\Big)^2\\
&+\sum_{i\neq j}\Big(h(\xi_i,\omega)h(\xi_j,\omega)[W(v_i,\omega)-W(u_i,\omega)][W(v_j,\omega)-W(u_j,\omega)]\Big)\Big\}\\
=&\sum_i\Big\{E\Big([h(\xi_i,\omega)]^2[W(v_i,\omega)-W(u_i,\omega)]^2\Big)\Big\}\\
&+\sum_{i\neq j}\Big\{E\Big(h(\xi_i,\omega)h(\xi_j,\omega)[W(v_i,\omega)-W(u_i,\omega)][W(v_j,\omega)-W(u_j,\omega)]\Big)\Big\}\\
=&\sum_i\Big\{E\Big([h(\xi_i,\omega)]^2(v_i-u_i)\Big)\Big\} = E\Big((D)\sum_i h^2(\xi_i,\omega)(v_i-u_i)\Big).
\end{aligned}$$

We remark that the equality above is due to the orthogonal increment of Brownian motion or a martingale and $E(W_{v_i}-W_{u_i})^2 = v_i-u_i$, (see property (ii) and property of Normal increments of W in Section 2) .

Theorem 6. *Let $\varphi, \varphi^{(n)}$, $n = 1, 2, \ldots$, be adapted processes such that for each $\xi \in [0,1], E\Big(f^{(n)}(\xi,\omega) - f(\xi,\omega)\Big)^2 \to 0$ as $n \to \infty$. Suppose that each $\varphi^{(n)}$ is IH-integrable to $A^{(n)}$ on $[0,1]$ with respect to the Brownian motion W and $A^{(n)}$ variationally converges to A. Then f is IH-integrable to A on $[0,1]$.*

Proof. The idea of the proof is standard in the theory of Henstock integration. Let $\varepsilon > 0$ be given. For any $\xi \in [0,1]$ there exists $n(\xi) > 0$ such that

$E\Big(\varphi^{n(\xi)}(\xi,\omega)-\varphi(\xi,\omega)\Big)^2 < \frac{\varepsilon}{3^2}$. Since $\varphi^{(n)}$ is IH-integrable to $A^{(n)}$ on $[0,1]$ for each $n = 1,2,3,\ldots$, there exists $\delta^{(n)}(\xi) > 0$ on $[0,1]$ such that for any $\delta^{(n)}(\xi)$-fine belated partial divisions $D_n = \{((u,v],\xi)\}$ of $[0,1]$, we have

$$E\bigg\{(D_n)\sum_i\Big(\varphi^{(n)}(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]-A^{(n)}((u_i,v_i],\omega)\Big)\bigg\}^2 < \frac{\varepsilon}{(2^n3)^2}.$$

Since $A^{(n)}$ variationally converges to A, there exists a positive integer N such that for any finite collection of disjoint intervals $\{(u_i,v_i]; i=1,\ldots,q\}$, we have

$$E\bigg\{\sum_{i=1}^{q}\Big[A^{(n)}((u_i,v_i],\omega)-A((u_i,v_i],\omega)\Big]\bigg\}^2 < \frac{\varepsilon}{3^2\cdot(2^n)^2}$$

whenever $n \ge N$. Choose a subsequence $\{A^{(n_k)}\}$ of $\{A^{(n)}\}$, for $k = 1,2,\ldots$, such that

$$E\bigg\{\sum_{i=1}^{q}\Big[A^{(n_k)}((u_i,v_i],\omega)-A((u_i,v_i],\omega)\Big]\bigg\}^2 < \frac{\epsilon}{3^2\cdot(2^k)^2}.$$

In the proof that follows, we shall use the subsequence $\{A^{(n_k)}\}$ and $\{f^{(n_k)}\}$. However, for the convenience of our presentation, we denote $\{A^{(n_k)}\}$ and $\{f^{(n_k)}\}$ by $\{A^{(k)}\}$ and $\{f^{(k)}\}$ respectively. Now, let $\delta(\xi) = \delta^{n(\xi)}(\xi)$ and $D = \{((u,v],\xi)\}$ be any δ-fine belated partial division of $[0,1]$. Thus, we have

$$\begin{aligned}
&E\bigg\{(D)\sum_i\Big(\varphi(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]-A((u_i,v_i],\omega)\Big)\bigg\}^2\\
=&E\bigg\{(D)\sum_i\Big(\big[\varphi(\xi_i,\omega)-\varphi^{n(\xi_i)}(\xi_i,\omega)\big][W(v_i,\omega)-W(u_i,\omega)]\Big)\\
&+(D)\sum_i\Big(A^{n(\xi_i)}((u_i,v_i],\omega)-A((u_i,v_i],\omega)\Big)\\
&+(D)\sum_i\Big(\varphi^{n(\xi_i)}(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]-A^{n(\xi_i)}((u_i,v_i],\omega)\Big)\bigg\}^2\\
\le&3E\bigg\{(D)\sum_i\Big(\big[\varphi(\xi_i,\omega)-\varphi^{n(\xi_i)}(\xi_i,\omega)\big][W(v_i,\omega)-W(u_i,\omega)]\Big)\bigg\}^2\\
&+3E\bigg\{(D)\sum_i\Big(A^{n(\xi_i)}((u_i,v_i],\omega)-A((u_i,v_i],\omega)\Big)\bigg\}^2
\end{aligned}$$

$$+ 3E\bigg\{(D)\sum_i\Big(\varphi^{n(\xi_i)}(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)] - A^{n(\xi_i)}((u_i,v_i],\omega)\Big)\bigg\}^2$$
$$=3X+3Y+3Z.$$

Now,

$$\begin{aligned}3X =&3E\bigg\{(D)\sum_i\left[\varphi(\xi_i,\omega)-\varphi^{n(\xi_i)}(\xi_i,\omega)\right]\left[W(v_i,\omega)-W(u_i,\omega)\right]\bigg\}^2\\ =&3E\Big((D)\sum_i\left[\varphi(\xi_i,\omega)-\varphi^{n(\xi_i)}(\xi_i,\omega)\right]^2(v_i-u_i)\Big)\end{aligned}$$

since $\varphi(\xi_i,\omega)-\varphi^{n(\xi_i)}(\xi_i,\omega)$ is $\mathcal{F}_{\xi_i}$-measurable, and thus $\mathcal{F}_{u_i}$-measurable, Proposition 5a can be applied. Therefore,

$$\begin{aligned}3X =&3\sum_i\bigg\{(v_i-u_i)E\Big[\varphi^{n(\xi_i)}(\xi_i,\omega)-\varphi(\xi_i,\omega)\Big]^2\bigg\}\\ <&3\cdot\frac{\varepsilon}{3^2}\cdot\sum_i(v_i-u_i)\le\frac{\varepsilon}{3}(1-0)=\frac{\varepsilon}{3}.\end{aligned}$$

We have

$$\begin{aligned}Y^{\frac{1}{2}} =&\bigg\{E\Big((D)\sum_i\Big[A^{n(\xi_i)}\big((u_i,v_i],\omega\big)-A\big((u_i,v_i],\omega\big)\Big]\Big)^2\bigg\}^{\frac{1}{2}}\\ \le&\sum_j\bigg\{E\Big((D_j)\sum_i\Big[A^{(n_j)}\big((u_i,v_i],\omega\big)-A\big((u_i,v_i],\omega\big)\Big]\Big)^2\bigg\}^{\frac{1}{2}}\\ <&\sum_j\sqrt{\frac{\varepsilon}{3^2\cdot(2^j)^2}}=\frac{\sqrt{\varepsilon}}{3},\end{aligned}$$

where $D_j=\{((u_i,v_i],\xi_i):n(\xi_i)=n_j\}$.

Similarly,

$$\begin{aligned}Z^{\frac{1}{2}} =&\bigg\{E\Big((D)\sum_i\Big[\varphi^{n(\xi_i)}(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]-A^{n(\xi_i)}((u_i,v_i],\omega)\Big]\Big)^2\bigg\}^{\frac{1}{2}}\\ \le&\sum_j\bigg\{E\Big[(D_j)\sum_i\Big(\varphi^{(n_j)}(\xi_i,\omega)[W(v_i,\omega)-W(u_i,\omega)]-A^{(n_j)}((u_i,v_i],\omega)\Big)\Big]^2\bigg\}^{\frac{1}{2}}\\ <&\sum_j\sqrt{\frac{\varepsilon}{3^2\cdot(2^j)^2}}=\frac{\sqrt{\varepsilon}}{3}.\end{aligned}$$

Thus $3X + 3Y + 3Z < \varepsilon$, showing that φ is IH-integrable to A on $[0, 1]$. $\square$

Lemma 7. *Let $h(\omega)$ be a bounded random variable on $(\Omega, \mathcal{F}, P_\Omega)$ with $|h(\omega)| \leq M$, for all $\omega \in \Omega$. Let $s \in [0, 1]$ be fixed. Suppose that $f(t, \omega) = h(\omega)$, if $t = s$ and $f(t, \omega) = 0$, if $t \neq s$. Then f is IH-integrable to zero on $[0, 1]$.*

PROOF. For any $\varepsilon > 0$, let $\delta(\xi) = \varepsilon/M^2$. Consider any δ-fine belated partial divisions $D = \{((u, v], \xi)\}$ of $[0, 1]$. Assume that $s = \xi_j$, for some j; otherwise, it is trivial since $f(\xi_i, \omega) = 0$, for all i. Thus,

$$\begin{aligned}
&E\biggl((D)\sum_i \Bigl(f(\xi_i, \omega)\bigl[W(v_i, \omega) - W(u_i, \omega)\bigr] - 0\Bigr)\biggr)^2 \\
=&E\biggl(f(s, \omega)\bigl[W(v_j, \omega) - W(u_j, \omega)\bigr]\biggr)^2 \\
\leq&M^2 E\Bigl(W(v_j, \omega) - W(u_j, \omega)\Bigr)^2 \\
=&M^2 E(v_j - u_j) = M^2(v_j - u_j) < M^2\left(\frac{\varepsilon}{M^2}\right) = \varepsilon.
\end{aligned}$$

Hence, f is IH-integrable to zero on $[0, 1]$. $\square$

From a bounded random variable, we now move on to an adapted bounded step process in $\mathcal{L}_0$. We will show that this process is also IH-integrable. Let $W((a, b], \omega)$ denote $W(b, \omega) - W(a, \omega)$, for any $a < b$, where $a, b \in \mathbb{R}$ for succinctness of our presentation.

Lemma 8. *Let $f \in \mathcal{L}_0$. Then f is IH-integrable on $[0, 1]$ and its integral is equal to the classical Itô-integral of f with respect to W.*

PROOF. Let $f \in \mathcal{L}_0$ be expressed as

$$f(t, \omega) = f_0(\omega)\chi_{\{0\}}(t) + \sum_{i=1}^{n} f_{i-1}(\omega)\chi_{(t_{i-1}, t_i]}(t)$$

be as given in the definition of $\mathcal{L}_0$. By Lemma 7, we may ignore the values of $f(t, \omega)$ at $t_0, t_1, t_2, \ldots, t_n$. Let $\xi \in [0, 1]$. We only need to consider $\xi \neq t_i$ for all $i = 0, 1, 2, \ldots, n$. Assume $\xi \in (t_{i-1}, t_i)$. Define $\delta(\xi)$ such that $[\xi, \xi + \delta(\xi)] \subset (t_{i-1}, t_i)$. Hence $f(\xi, \omega) = f_{i-1}(\omega)$ and

$$f(\xi, \omega)[W_v(\omega) - W_u(\omega)] = f_{i-1}(\omega)[W_v(\omega) - W_u(\omega)]$$

whenever $(u,v] \subset [\xi, \xi+\delta(\xi)]$. Recall that for such interval $(u,v]$,

$$(I)\int_u^v f(t,\omega)dW_t(\omega) = f_{i-1}(\omega)[W_v(\omega) - W_u(\omega)].$$

Let $D = \{((u,v],\xi)\}$ be a δ-fine belated partial division of $[0,1]$ with $\xi \neq t_i$ for all i. Then

$$E\left(\left|(D)\sum\left\{f(\xi,\omega)[W_v(\omega)-W_u(\omega)] - (I)\int_u^v f(t,\omega)dW_t\right\}\right|^2\right) = 0.$$

Thus f is IH-integrable on $[0,1]$ and $(IH)\int_0^1 f_t dW_t = (I)\int_0^1 f_t dW_t$, thereby completing the proof. □

Theorem 9. *If $f \in \mathcal{L}_2$, then f is IH-integrable. Furthermore, the IH-integral of f is equal to the classical Itô-integral of f.*

PROOF. For every $f \in \mathcal{L}_2$, there exists a sequence $\{f^{(n)}\}$ in $\mathcal{L}_0$ such that

$$\|f^{(m)} - f\|_{\mathcal{L}_2} \to 0 \text{ as } m \to \infty.$$

By Lemma 8, the IH-integral of $f^{(m)}$ is equal to the classical Itô-integral $I(f^{(m)})$ of f.

On the other hand, we have $E\left\{\left[I(f^{(m)})(w) - I(f)(w)\right]^2\right\} \to 0$ as $m \to \infty$. We shall next use Theorem 6 to prove that f is IH-integrable to $I(f)(w)$. Denote $I(f^{(m)})(w)$ by $A^{(m)}(w)$ and $I(f)(w)$ by $A(w)$. Since

$$\|f^{(n)} - f\|^2_{\mathcal{L}_2} = \int_0^1\int_\Omega \left(f^{(n)}(t,\omega) - f(t,\omega)\right)^2 \, dP\,dt \to 0 \text{ as } n \to \infty,$$

for every $\xi \in [0,1]$, except possibly on a set of measure zero, there exists a subsequence $f^{n(\xi)}$ such that $\int_\Omega \left(f^{n(\xi)}(\xi,\omega) - f(\xi,\omega)\right)^2 \, dP \to 0$, as $n(\xi) \to \infty$. We may assume that $\int_\Omega \left(f^{n(\xi)}(\xi,\omega) - f(\xi,\omega)\right)^2 \, dP \to 0$, for any $\xi \in [0,1]$. Denote $\{f^{n(\xi)}\}$ by $\{f^{(n)}\}$ and let $D = \{(u,v]\}$ by any finite collection of disjoint intervals. Thus, we have

$$\begin{aligned} &E\left\{(D)\sum_i \left[A^{(n)}((u_i,v_i],\omega) - A((u_i,v_i],\omega)\right]^2\right\} \\ =&(D)\sum_i\left\{E\left[\int_{u_i}^{v_i}\left(f^{(n)}(t,\omega) - f(t,\omega)\right)^2 \, dt\right]\right\} \text{ by isometric property} \end{aligned}$$

$$=(D)\sum_i\left[\int_{u_i}^{v_i}\int_\Omega\Big(f^{(n)}(t,\omega)-f(t,\omega)\Big)^2\,dP\,dt\right]$$
$$\le\int_0^1\int_\Omega\Big(f^{(n)}(t,\omega)-f(t,\omega)\Big)^2\,dP\,dt=\|f^{(n)}-f\|^2_{\mathcal{L}_2},$$

for any positive integer n.

Since $\|f^{(n)}-f\|^2_{\mathcal{L}_2}\to 0$ as $n\to\infty$, thus $A^{(n)}$ variationally converges to A. Since for each $\xi\in[0,1]$, we have $E\Big(f^{n(\xi)}(\xi,\omega)-f(\xi,\omega)\Big)^2\to 0$ with each $f^{(n)}$ being IH-integrable to $A^{(n)}$ on $[0,1]$ and $A^{(n)}$ variationally converging to A, therefore, f is IH-integrable to A on $[0,1]$, where A is the classical Itô-integral of f. Hence the IH-integral of f in our sense agrees with the classical Itô-integral of f. □

5 Itô-McShane's Approach to Stochastic Integral

McShane applied the generalized Riemann approach (with non-uniform meshes) to study stochastic integrals in 1969 and developed the theory of stochastic integrals when the integrator X satisfies some form of Lipschitz conditions, see [8]. In this section, we only consider the case when $X=W$ is a Brownian motion, over which the Lipschitz conditions are automatically satisfied.

Note that the interval used by McShane is right-open, see [8]. There is no difference between using right-open or left-open intervals when the integrator is continuous.

In this section, we shall modify the definition of McShane's stochastic integral. Instead of using right-open interval $[u,v)$ in the interval-point pair $\{[u,v),\xi\}$, we replace by left-open interval $(u,v]$ in $\{(u,v],\xi\}$, and we call this approach with the modification of using left-open intervals the Itô-McShane's (IM) approach, in line with the classical construction of the classical Itô integral.

Definition 10. Let φ and X be adapted processes defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$. Then φ is said to be *Itô-McShane* (IM) *integrable* to $A\in L_2(\Omega)$ on $[0,1]$ with respect to X if for every $\varepsilon>0$, there exist $\delta(\xi)>0$ on $[0,1]$ and $\eta>0$ such that

$$E\left(\left|(D)\sum\varphi_\xi[W_v-W_u]-A\right|\right)^2<\varepsilon$$

for any $(\delta(\xi),\eta)$-fine belated partial division $D=\{((u,v],\xi)\}$ of $[0,1]$. Of course, the interval $[0,1]$ can be replaced by any other interval $[a,b]$ in the above definition. In the above, $\sum\varphi_\xi[W_v-W_u]-A$ denotes $\sum\varphi_\xi(\omega)[W_v(\omega)-W_u(\omega)]-A(\omega)$ for each $\omega\in\Omega$.

It can be seen that the IM-integral is uniquely determined up to a set of P-measure zero. We denote the IM-integral of φ with respect to W by $(IM)\int_0^1 \varphi_t dW_t$.

Some standard properties of integration theory are appended in the following theorem. The proofs are omitted.

Theorem 11. (Cauchy's Criterion). *The process φ is IM-integrable with respect to X on $[0,1]$ if and only if there exist $\delta(\xi) > 0$ and $\eta > 0$ such that*

$$E\left(\left|(D_1)\sum \varphi_\xi[W_v - W_u] - (D_2)\sum \varphi_{\xi'}[W_{v'} - W_{u'}]\right|\right)^2 < \varepsilon$$

for any two $(\delta(\xi),\eta)$-fine McShane's belated divisions of $[0,1]$ denoted by $D_1 = \{((u,v],\xi)\}$ and $D_2 = \{((u',v'],\xi')\}$.

Theorem 12. 1. *If φ is IM-integrable on $[0,1]$ with respect to W, then φ is also IM-integrable on any $[a,b] \subset [0,1]$.*

2. *Let φ be IM-integrable on $[0,1]$ with respect to X. Then $(IM)\int_a^c \varphi_t dW_t = (IM)\int_a^b \varphi_t dW_t + (IM)\int_b^c \varphi_t dW_t$ for any subintervals $[a,b]$ and $[b,c]$ in $[0,1]$.*

It is a classical result from the non-stochastic integration theory that the primitive function of a Lebesgue integral is absolutely continuous. Next, we shall establish the corresponding results on absolute continuity for stochastic integrals. The ideas of the following proofs are standard in the classical theory of Henstock integration.

Definition 13. The real-valued function $F : \mathcal{I} \times \Omega \to \mathbb{R}$ is said to be $AC^2[0,1]$ if given $\varepsilon > 0$ there exists $\eta > 0$ such that $E\left|\sum_i F(I_i)\right|^2 < \varepsilon$ for any finite collection of disjoint left-open intervals $\{I_i : i = 1,2,3,\ldots,n\}$ for which $\sum_i |I_i| < \eta$.

Example 14. It is easy to see from Itô's isometry, see the last part of Section 2 or [9, p. 23] for example, that if $F(a,b] = (I)\int_a^b \varphi_t dW_t$, then F is $AC^2[0,1]$.

Lemma 15. *Let φ be IM-integrable on $[0,1]$. Given $\varepsilon > 0$, there exist $\delta(\xi) > 0$ and $\eta > 0$ such that $E\left(|(D)\sum \varphi_\xi[W_v - W_u]|\right)^2 < \varepsilon$ for any δ-fine belated partial division $D = \{((u,v],\xi)\}$ (type c of Sect 3) with $(D)\sum |v-u| < \eta$.*

PROOF. Given $\varepsilon > 0$ there exist $\delta(\xi) > 0$ and $\eta > 0$ such that for any McShane's $(\delta(\xi), \eta)$-fine belated division D_0,

$$E\left(\left|(D_0)\sum \varphi_\xi[W_v - W_u] - (IM)\int_0^1 \varphi_t dW_t\right|\right)^2 < \frac{\varepsilon}{4}.$$

Let $D = \{((u,v],\xi)\}$ be δ-fine belated partial division of $[0,1]$ such that $(D)\sum|v-u| < \eta$. Define a McShane's $(\delta(\xi), \eta)$-fine belated division D_1 of $[0,1]$ such that $D \cup D_1$ is a McShane's $(\delta(\xi), \eta)$-fine belated division of $[0,1]$. Hence

$$E\left(\left|(D\cup D_1)\sum \varphi_\xi[W_v - W_u] - (IM)\int_0^1 \varphi_t dW_t\right|\right)^2 < \frac{\varepsilon}{4}.$$

Consequently

$$\begin{aligned}
E\left|(D)\sum \varphi_\xi[W_v - W_u]\right|^2 =& E\Big|(D\cup D_1)\sum \varphi_\xi[W_v - W_u] - (IM)\int_0^1 \varphi_t dW_t \\
&+ (IM)\int_0^1 \varphi_t dW_t - (D_1)\sum \varphi_\xi[W_v - W_u]\Big|^2 \\
\le& 2E\Big|(D\cup D_1)\sum \varphi_\xi[W_v - W_u] - (IM)\int_0^1 \varphi_t dW_t\Big|^2 \\
&+ 2E\Big|(IM)\int_0^1 \varphi_t dW_t - (D_1)\sum \varphi_\xi[W_v - W_u]\Big|^2 \\
\le& 2\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) = \varepsilon
\end{aligned}$$

thereby completing the proof. □

Theorem 16. *Let φ be IM-integrable with respect on $[0,1]$. Then Φ is $AC^2[0,1]$, where $\Phi(J) = (IM)\int_J \varphi_t dW_t$ for any left-open subinterval J of $[0,1]$.*

PROOF. Let $\varepsilon > 0$ be given. By Lemma 15 there exist η and $\delta(\xi) > 0$ such that whenever $D_1 = \{((u,v],\xi)\}$ is a δ-fine belated partial division of $[0,1]$ with $(D_1)\sum|v-u| < \eta$ we have $E\left(|(D_1)\sum \varphi_\xi[W_v - W_u]|^2\right) < \varepsilon$. Let $\{(a_i, b_i]\}_{i=1}^N$ be a finite collection of disjoint subintervals from $[0,1]$, where $\sum_{i=1}^N |b_i - a_i| < \eta$. Then φ is IM-integrable on each $[a_i, b_i], i = 1, 2, 3, \ldots, N$. On each $[a_i, b_i]$

there exist $\delta_i(\xi) > 0$ and $\eta_i > 0$ such that

$$E\left(\left|(D_i)\sum \varphi_\xi[W_v - W_u] - (M)\int_{a_i}^{b_i} \varphi_t dW_t\right|^2\right) < \frac{\varepsilon}{2^{2i}}$$

whenever $D_i = \{((u,v],\xi)\}$ is a δ_i-fine belated partial division of $[a_i, b_i]$ with $(D_i)\sum |v-u| < \eta_i$. We may assume that $\delta_i(\xi) < \delta(\xi)$ for each $i = 1, 2, \dots, N$. Now $D = \bigcup_{i=1}^{N} D_i$ is a δ-fine belated partial division of $[0,1]$, with

$$(\cup D_i)\sum |v-u| \leq \sum |b_i - a_i| < \eta;$$

so that we have $E\left(\left|(\bigcup_{i=1}^N D_i)\sum \varphi_\xi[W_v - W_u]\right|\right)^2 < \varepsilon$. Consequently

$$\begin{aligned}
&E|\sum_i (IM)\int_{a_i}^{b_i} \varphi_t dW_t|^2 \\
\leq& 2E\left|\sum_i \left\{(IM)\int_{a_i}^{b_i} \varphi_t dW_t - (D_i)\sum \varphi_\xi[W_v - W_u]\right\}\right|^2 \\
&+ 2E\left|\sum_i (D_i)\sum \varphi_\xi[W_v - W_u]\right|^2 \\
\leq& 2\left\{\sum_i \sqrt{E\left|(IM)\int_{a_i}^{b_i} \varphi_t dW_t - (D_i)\sum \varphi_\xi[W_v - W_u]\right|^2}\right\}^2 + 2\varepsilon \\
\leq& 2\left\{\sum_{i=1}^{\infty} \frac{\sqrt{\varepsilon}}{2^i}\right\}^2 + 2\varepsilon \leq 4\varepsilon
\end{aligned}$$

showing that the primitive of IM-integrable function is $AC^2[0,1]$. □

We shall next establish the relation between IM-integral and IH-integral.

Theorem 17. *Let φ be IM-integrable. Then φ is IH-integrable and that*

$$(IM)\int_0^1 \varphi_t dW_t = (IH)\int_0^1 \varphi_t dW_t.$$

PROOF. Given $\varepsilon > 0$ there exist $\delta(\xi) > 0$ on $[0,1]$ and $\eta > 0$ such that for any $(\delta(\xi), \eta)$-fine McShane's belated partial division $D = \{(u,v], \xi\}$ we have

$$E\left((D)\sum \varphi_\xi[W_v - W_u] - (IM)\int_0^1 \varphi_t dW_t\right)^2 < \varepsilon.$$

Let $D_1 = \{(u,v], \xi\}$ be any $\delta(\xi)$-fine belated partial division (of Type c in Section 3) and define $F(u,v] = (IM)\int_u^v \varphi_t dW_t$. Suppose that

$$\overline{[0,1]\backslash\{\bigcup(u,v] : (u,v] \in D_1\}} = \bigcup_{i=2}^m [a_i, b_i].$$

Then by Theorem 12, φ is IM-integrable on each $[a_i, b_i]$, $i = 2, 3, \ldots, m$. Let $\delta_i(\xi) > 0$ and $\eta_i > 0$ be defined for each $[a_i, b_i]$ such that for any $(\delta_i(\xi), \eta_i)$-fine McShane's belated partial division of $[a_i, b_i]$, denoted by D_i, we have

$$E\left((D_i)\sum \varphi_\xi[W_v - W_u] - (IM)\int_u^v \varphi_t dW_t\right)^2 < \frac{\varepsilon}{2^{2i}}.$$

We may assume that $\sum_i \eta_i < \eta$ and that $\delta_i(\xi) < \delta(\xi)$ for all $i = 2, 3, 4, \ldots, m$. Consider the division $D = \bigcup_{i=1}^m D_i$, which is $(\delta(\xi), \eta)$-fine. Then

$$\begin{aligned}
S =& E\left((D_1)\sum\{\varphi_\xi[W_v - W_u] - F(u,v]\}\right)^2 \\
\leq& 2E\left((D)\sum \varphi_\xi[W_v - W_u] - (IM)\int_0^1 \varphi_t dW_t\right)^2 \\
&+ 2E\left(\sum_{i=2}^m (D_i)\left\{\sum \varphi_\xi[W_v - W_u] - (IM)\int_{a_i}^{b_i} \varphi_t dW_t\right\}\right)^2 \leq 2\varepsilon + 2J
\end{aligned}$$

where

$$\begin{aligned}
J &\leq \left\{\sum_{i=2}^m \sqrt{E\left\{(D_i)\sum \varphi_\xi[W_v - W_u] - (IM)\int_{a_i}^{b_i} \varphi_t dW_t\right\}^2}\right\}^2 \\
&\leq \left\{\sum_{i=2}^m \frac{\sqrt{\varepsilon}}{2^i}\right\}^2 \leq \varepsilon
\end{aligned}$$

thereby forcing $S \leq 4\varepsilon$ showing φ is IH-integrable to the same primitive as its IM-integral. □

Theorem 18. *Let φ be IH-integrable with primitive function $F : \mathcal{I} \times \Omega \to \mathbb{R}$. If F is $AC^2[0,1]$, then φ is IM-integrable with the same primitive.*

PROOF. Given $\varepsilon > 0$ there exists a $\delta(\xi) > 0$ such that for any δ-fine belated partial division $D = \{(u,v], \xi\}$ we have

$$E\left((D)\sum\{\varphi_\xi[W_v - W_u] - F(u,v]\}\right)^2 < \varepsilon.$$

Choose $\eta > 0$ such that $|\sum F(I)|^2 < \varepsilon$ for any finite collection of disjoint left-open subintervals of $(0,1]$ such that $\sum |I| < \eta$. Let $D_1 = \{(u,v], \xi\}$ be any $(\delta(\xi), \eta)$-fine McShane's belated partial division. Let the collection of the subintervals in $(0,1]$ not covered by D_1 be denoted by $\{J\}$, whose total length is less than η. Then

$$\begin{aligned} &E\left((D)\sum \varphi_\xi[W_v - W_u] - F(0,1]\right)^2 \\ \leq &2E\left((D)\sum\{\varphi_\xi[W_v - W_u] - F(u,v]\}\right)^2 + 2E\left|\sum F(J)\right|^2 \leq 4\varepsilon, \end{aligned}$$

thereby completing our proof. □

So that we have the result: if f is IM-integrable, then it is IH-integrable. Conversely, if f is IH-integrable with primitive which is $AC^2[0,1]$, then f is IM-integrable.

Corollary 19. *Let $\varphi \in \mathcal{L}_2$. Then it is both IM-integrable and IH-integrable. Moreover, the two integrals agree with each other and also with the classical Itô integral.*

PROOF. This follows directly from Theorem 9, Example 14 and Theorem 18. □

We remark that alternatively Corollary 19 can be proved directly as in [10], [11] or [12].

6 Further Remark

The use of generalized Riemann approach (with non-uniform mesh) to define stochastic integral can be further extended to the case when the integrators are semimartingales, the details of which will appear in a paper elsewhere. Here, we mention that when the integrators are more general, the mesh $\delta(\xi) > 0$ will have to be replaced by a *random* mesh $\delta(\xi, \omega)$ and the proofs are further generalizations of Section 4.

References

[1] Chew T. S., Lee P. Y., *Nonabsolute Integration using Vitali Covers*, New Zealand Journal of Mathematics **23**, (1994) 25–36.

[2] R. Henstock, *Lectures on the theory of integration*, World Scientific, Singapore, 1988.

[3] R. Henstock, *The general theory of integration*, Oxford University Press, Oxford, 1991.

[4] R. Henstock, *Stochastic and other functional integrals*, Real Analysis Exchange, **16** (1990-91), 460–470.

[5] Lee P. Y., *Lanzhou lectures on Henstock integration*, World Scientific, Singapore, 1989.

[6] Lee P. Y., R. Vyborny, *The integral, an easy approach after Kurzweil and Henstock*, Cambridge University Press, 2000.

[7] Lee T. W.,*On the generalized Riemann integral and stochastic integral*, Journal Australian Math. Soc., **21**(Series A) (1976), 64–71.

[8] E. J. McShane, *Stochastic Calculus and Stochastic Models*, Academic Press, New York, 1974.

[9] B. Oksendal, *Stochastic differential equations: An introduction with applications*, 4th edition, Springer, 1996.

[10] P. Protter, *A comparison of stochastic integrals*, Annals of Probability, **7** (1979), 276–289.

[11] Toh T. L., Chew T. S., *A variational approach to Itô's integral*, Proceedings of SAP's, **98** 1999, Taiwan 291–299, World Scientific, Singapore.

[12] Xu J. G., Lee P. Y., *Stochastic integrals of Itô and Henstock*, Real Analysis Exchange, **18** (1992-93), 352–366.

[13] Yeh, J., *Martingales and stochastic analysis*, World Scientific. Singapore, 1995.

[14] P. Muldowney, *The Henstock integral and the Black-Scholes theory of derivative asset pricing*, Real Analysis Exchange, **26** (2000-2001), 117–132.