RESEARCH

Real Analysis Exchange
Vol. 27(2), 2001/2002, pp. 441–456

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IRRATIONAL TWIST SYSTEMS FOR INTERVAL MAPS

Abstract

Let I be a compact real interval and $f\colon\ I \to I$ continuous. We describe a special infinite minimal subsystem - we call it irrational twist system - of dynamical system (I, f). We show that any twist system has an extremely regular behavior and it can be considered as an interval analogy of the irrational circle rotation.

1 Introduction

Consider a continuous function which maps a closed interval of the real line into itself. A fixed point divides this interval into two parts and we can measure these two parts using an ergodic invariant measure. Or more precisely, by an **eccentricity** (**rotation number**) of that measure which is the ratio not less than one where the numerator and denominator are the measures of the parts. One can ask the following question. If that ratio is maximal, what can we say about the dynamics on the support of corresponding 'maximal' measure? As it is often in mathematics it turned out that to solve this question it is necessary to distinguish rational and irrational cases. The rational one leads to special cycles and measures on periodic orbits [2], [9], [10]; the second one gives some view to the world of minimal sets and ergodic measures supported by them.

Key Words: Interval map, twist system, invariant measure
Mathematical Reviews subject classification: 26A18, 37A05, 37E05, 37E45
Received by the editors December 14, 2000
*Research supported by the Grant Agency of the Czech Republic, contract no. 201/00/0859.

In the theory of discrete dynamical systems the question of coexistence of different types of invariant sets arises. The aim of this paper is to describe infinite minimal sets (We call them **irrational twist systems**.) of an extremely regular behavior that can be considered as an interval analogy of the **irrational circle rotation**. They are defined - roughly speaking - as follows. If some continuous map acts minimally on a Cantor set with an ergodic invariant measure of an eccentricity β, then they together create a **twist β-system** if that map can be extended to the interval map such that all its invariant measures have eccentricities at most β.

Our main results characterize the twist β-system of prescribed irrational eccentricity (Section 2, Theorem A) and describe some situation when dynamical systems given by interval map have such a system as a subsystem (Section 2, Theorem B). Now it is already known that each interval map with positive topological entropy realized an irrational twist β-system for every irrational eccentricity β from some interval [8].

The paper is organized as follows. In Section 2 we give some basic notation and definitions. After introducing necessary notion we state the main results here. Section 3 is devoted to the lemmas used throughout the paper. In Section 4 we prove our main results - Theorems A and B.

Remark 1.1. Similar results to those obtained in this paper were announced (without proofs) in [3]. As we know from our private communication they have been obtained by different methods.

Acknowledement. The authors express their sincere thanks to the referee for his valuable comments.

2 Notation and Definitions

By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ we denote the sets of real, rational and positive integer numbers respectively. Let $\mathcal{T}$ be the set of all compact subsets of $\mathbb{R}$, where each $T \in \mathcal{T}$ is equipped with the Euclidean metric. For $T \in \mathcal{T}$ let $C(T)$ denote the set of all continuous functions which map T into itself. We consider the space $C(\mathcal{T}) = \bigcup\{C(T) : T \in \mathcal{T}\}$ and for $g \in C(\mathcal{T})$ we define g^n inductively by $g^0 = id$ and (for $n \geq 1$) $g^n = g \circ g^{n-1}$. As usually, g^n is called the n-th iterate of g. A map $f \in C(I)$ is said to be an interval map if $I \in \mathcal{T}$ is an interval.

Let $g \in C(T)$. For $x \in T$ the orbit of x under g is $\operatorname{orb}(g,x) = \{g^n(x)\}_{n=0}^{\infty}$. The ω-limit set $\omega(g,x)$ of $x \in T$ consists of all the limit points of $\operatorname{orb}(g,x)$. A point x is said to be periodic (of period n) if $g^n(x) = x$ and $g^i(x) \neq x$ for $0 < i < n$. The set of all periodic points is denoted by $\operatorname{Per}(g)$. A fixed point is a periodic point with period 1 and $\mathrm{Fix}(g)$ is the set of all fixed points of g. If $g \in C(T)$, then we say that (T, g) is a (dynamical) **system**.

For a system (T, g), a map $\tilde{g} \in C(\operatorname{conv} T)$ is said to be (T, g)-monotone if $\tilde{g}|T = g$ and $\tilde{g}|J$ is monotone (not necessarily strictly) for any interval $J \subset \operatorname{conv} T$ such that $J \cap T = \emptyset$. We will use the notation $C(T, g)$ for the set of all (T, g)-monotone maps. In particular, the (T, g)-monotone map which is affine on each component of $\operatorname{conv} T \setminus T$ will be denoted g_T.

An interval map $f \in C(I)$ is **piecewise monotone** if there are $k \in \mathbb{N}$ and points $\min I = c_0 < c_1 < \cdots < c_k < c_{k+1} = \max I$ such that f is monotone on each $[c_i, c_{i+1}]$, $i = 0, \dots, k$. The minimal such k will be called the **modality of** f.

A system (T, g) is said to be **minimal**, resp. **transitive** if for each $x \in T$, resp. for some $x \in T$ the ω-limit set $\omega(g, x)$ is equal to T. Such a point will be called transitive. A **cycle** is a minimal system (T, g) such that T is finite. A function $f \in C(\mathcal{T})$ has a cycle (T, g) if $f|\, T = g$. In this case we will often write (T, f) instead of (T, g).

Let μ be a normalized Borel measure on $T \in \mathcal{T}$. From now if we say "measure", then we in fact mean "normalized Borel measure" and if we measure some set, then we assume that it is Borel measurable. We will say that $g \in C(T)$ preserves the measure μ (or that μ is preserved by g) if $\mu(g^{-1}(S)) = \mu(S)$ for any $S \subset T$. Let $\mathcal{M}(g)$ be the set of measures preserved by g. We have the following simple inequality.

$$\text{If } \mu \in \mathcal{M}(g), \text{ then } \mu(g(S)) \geq \mu(S) \text{ for any } S \subset T.$$

In particular if $\mu(\{p\}) > 0$, then p must be a periodic point and $\mu(\{x\}) = \mu(\{p\})$ for any $x \in \mathrm{orb}(g, p)$. In this case we speak about the atomic measure μ on a periodic orbit $\mathrm{orb}(g, p)$.

We say that $S \subset T$ is g-invariant if $g(S) \subset S$. The measure $\mu \in \mathcal{M}(g)$ is called ergodic if for any g-invariant set $S \subset T$ either $\mu(S) = 0$ or $\mu(S) = 1$. We denote the set of all g-invariant ergodic measures by $\mathcal{M}_e(g)$. The support of a measure μ, denoted by $\operatorname{supp} \mu$, is the smallest closed set $S \subset T$ such that $\mu(S) = 1$. If $\mu \in \mathcal{M}(g)$, then $f(\operatorname{supp} \mu) = \operatorname{supp} \mu$ and if μ is ergodic, then either $\operatorname{supp} \mu = \mathrm{orb}(g, p)$ for some $p \in \mathrm{Per}(g)$ or $\operatorname{supp} \mu$ is a perfect set. Moreover we have following ergodic decomposition.

Theorem 2.1. *([13]) Let $g \in C(\mathcal{T})$ and $\mu \in \mathcal{M}(g)$. Then there is a measure m on $\mathcal{M}_e(g)$ such that $\mu(S) = \int_{\mathcal{M}_e(g)} \lambda(S)\, dm$ for any measurable set S.*

Definition 2.1. Let (T, g) be a system, $c \in \operatorname{conv} T$ and μ be a measure on T. Let $\beta = \mu([\min T, c])/\mu([c, \max T])$ ($\beta = \infty$ if $\mu([c, \max T]) = 0$). We define

$E(\mu, c) = \max\{\beta, 1/\beta\}$, $E(\mu) = \sup\{E(\mu, c) : c \in \mathrm{Fix}(g_T)\}$ and

$$E(T, g) = \sup\{E(\mu) : \mu \in \mathcal{M}_e(g)\}.$$

The value $E(T, g)$ will be called the **eccentricity of** (T, g).

Remark 2.1. (i) By virtue of Theorem 2.1 we can verify that for a system (T, g)

$$E(T, g) = \sup\{E(\mu) : \mu \in \mathcal{M}(g)\}.$$

(ii) If $I \in \mathcal{T}$ is an interval and $f \in C(I)$, then we often write $E(f)$ instead of $E(I, f)$. (iii) As it was shown in [10] if for $g \in C(T)$ the number of fixed points of g_T is greater than one, then $E(g_T) = \infty$. In this paper our interest is focused on maps with finite eccentricities. So in the sequel we will not consider any map g_T with more fixed points.

Definition 2.2. A system (T, g) is said to be a **unisystem** if g_T is piecewise monotone and $\#\,\mathrm{Fix}(g_T) = 1$. The set of all unisystems will be denoted by $\mathcal{U}$.

Remark 2.2. (i) Let $(T, g) \in \mathcal{U}$. In what follows we always use the letter c to denote a unique fixed point of g_T. (ii) Our definition of eccentricity gives $E(g_T) \geq E(T, g)$. It is well known that for some unisystems (T, g) we have $E(g_T) > E(T, g)$ [10].

Lemma 2.1. *([4]) Let $f \in C(I)$ be a piecewise monotone interval map and $\#\,\mathrm{Fix}(f) = 1$. The following hold:*

(i) $E(f) < \infty$.

(ii) There is a measure $\mu \in \mathcal{M}_e(f)$ such that $E(\mu) = E(f)$.

(iii) In particular, if $(T, g) \in \mathcal{U}$, then $E(g_T) < \infty$ and there is a measure $\mu \in \mathcal{M}_e(g_T)$ such that $E(\mu) = E(g_T)$. In general, $\operatorname{supp} \mu \not\subseteq T$.

The key definition follows. It 'works' with eccentricities of ergodic measures.

Definition 2.3. Let $\beta \in (1, \infty)$. A system $(T, g) \in \mathcal{U}$ is said to be a **twist β-system** if $T = \operatorname{supp} \mu$ for some $\mu \in \mathcal{M}_e(g)$ with $E(\mu) = \beta$, $E(g_T) = \beta$ and

$$\forall\ \nu \in \mathcal{M}_e(g_T) : E(\nu) = \beta \implies \operatorname{supp} \nu \subset T. \tag{$*$}$$

Remark 2.3. (i) If (T, g) is a twist β-system, then from Theorem 2.1 follows that $\forall\ \nu \in \mathcal{M}(g_T) : E(\nu) = \beta \implies \operatorname{supp} \nu \subset T$. (ii) In [10] we have studied a particular case of twist systems (we called them X-minimal). Namely, we supposed there that the set T is finite; then (T, g) is a cycle and $E(g_T) \in \mathbb{Q}$. It can be shown that a twist system (T, g) with a rational eccentricity need not satisfy $\#T < \infty$. In this paper we will deal mainly with irrational eccentricities. (iii) We have seen in Lemma 2.1 that for $(T, g) \in \mathcal{U}$ with $E(g_T) = \beta \in [1, \infty)$ there is some $\mu \in \mathcal{M}_e(g_T)$ such that $E(\mu) = \beta$. In the sequel we will always assume that $E(\mu) = \mu([\min T, c])/\mu([c, \max T])$. Otherwise we would use instead of g_T the map $h \circ g_T \circ h^{-1}$ with $h(x) = -x + \min T + \max T$, $x \in [\min T, \max T]$.

Definition 2.4. Let $(T, g) \in \mathcal{U}$, $\beta \in (1, \infty)$. For $x \in T \setminus \{c\}$ a function $K_x : \mathrm{orb}(g, x) \to \mathbb{R}$ is a β**-code of** $\mathrm{orb}(g, x)$ if for each $i \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} K_x(g^{i+1}(x)) &= K_x(g^i(x)) + \frac{1}{1+\beta} \text{ if } g^i(x) < c \\ K_x(g^{i+1}(x)) &= K_x(g^i(x)) - \frac{\beta}{1+\beta} \text{ if } g^i(x) > c. \end{aligned} \tag{1}$$

We say that the β-code of $\mathrm{orb}(g, x)$ is **monotone**, resp. **strictly monotone** if for any $y, z \in \mathrm{orb}(g, x)$ the relation $y \in \mathrm{conv}\{z, c\}$ implies $K_x(y) \geq K_x(z)$, resp. $K_x(y) > K_x(z)$. A continuous function $K : T \to \mathbb{R}$ is said to be a β**-coding of** (T, g) if for each $x \in T \setminus \{c\}$ the function $K|\,\mathrm{orb}(g, x) = K_x$ is a β**-code of** $\mathrm{orb}(g, x)$.

Remark 2.4. We note that any β-code of c is not defined. By definition, the β-code K_x of $\mathrm{orb}(g, x)$, resp. the β-coding K of (T, g) is determined (if it exists) uniquely up to an additive constant.

Our main results are the following.

Theorem A. *Let $(T, g) \in \mathcal{U}$, $\beta \in (1, \infty)$ irrational. The following statements are equivalent:*

(i) *(T, g) is a twist β-system.*

(ii) *$T = \mathrm{supp}\,\mu$ for some $\mu \in \mathcal{M}_e(g)$ with $E(\mu) = \beta$ and there is a map $\tilde{g} \in C(T, g)$ such that $E(\tilde{g}) = \beta$.*

(iii) *There is a transitive point $x \in T$ such that the β-code K_x of $\mathrm{orb}(g, x)$ is monotone.*

(iv) *(T, g) is minimal and there is the β-coding $K : T \to \mathbb{R}$ such that for each $x \in T$, the β-code $K_x = K|\,\mathrm{orb}(g, x)$ of $\mathrm{orb}(g, x)$ is strictly monotone.*

(v) *$T = \mathrm{supp}\,\mu$ for some $\mu \in \mathcal{M}_e(g)$ with $E(\mu) = \beta$, $E(\tilde{g}) = \beta$ for any $\tilde{g} \in C(T, g)$ and $\forall\ \nu \in \mathcal{M}(\tilde{g}) : E(\nu) = \beta \implies \mathrm{supp}\,\nu = T$.*

Remark 2.5. By Theorem A, the statements (i) and (iii) are equivalent for β irrational. Unfortunately, it is not the case for β rational. In order to hold on the compatibility with the definition of X-minimal cycle from [10] we prefer to include the condition $(*)$ defining a twist β-system for $\beta \in (1, \infty)$.

Theorem B. *Let $f \in C(\mathcal{T})$ be an interval map such that $E(f) \in (1, \infty)$ is irrational. If there is a measure $\mu \in \mathcal{M}_e(f)$ for which $E(\mu) = E(f)$ and $f_{\operatorname{supp} \mu}$ is piecewise monotone, then $(\operatorname{supp} \mu, f)$ is a twist $E(f)$-system.*

3 Lemmas

This section is devoted to developing preliminary results for proving Theorems A and B in Section 4. Statements 3.1–3.3 deal with the code of some transitive point and show useful consequences of its monotonicity. The main result of this section is Lemma 3.6 that explains what happens when the code is not monotone.

Lemma 3.1. *Let $(T, g) \in \mathcal{U}$, $\beta \in (1, \infty)$. Suppose that for a transitive point $x \in T$ a β-code $K_x : \operatorname{orb}(g, x) \to \mathbb{R}$ is monotone. Then $K_x : \operatorname{orb}(g, x) \to \mathbb{R}$ is continuous and bounded.*

Proof. By our assumptions if T is finite, then (T, g) is a cycle and any β-code K_x of $x \in T$ is continuous if and only if it exists. Thus the conclusion is valid in this case. Let us suppose that T is infinite and denote $S = \operatorname{orb}(g, x)$. First we will prove the continuity. Assume to the contrary that there is a right limit point $u \in S$ of S, $u < c$ and $L > 0$ such that

$$\lim_{v \to u^+, v \in S} K_x(v) = K_x(u) + L. \tag{2}$$

Obviously the limit in (2) exists because of the monotonicity of K_x. Choose $\delta > 0$ arbitrarily. There is a $k \in \mathbb{N}$ such that $g^k(u) \in U_\delta^+(u)$ and $K_x(g^k(u)) \geq K_x(u) + L$ (see (2). Moreover, for some sufficiently small $\varepsilon > 0$

(a) $g^k(U_\varepsilon(u)) \subset U_\delta^+(u)$ and

(b) for any $y \in U_\varepsilon(u) \cap S$ and $i \in \{0, \dots, k\}$ we have $c \notin [g^i(y), g^i(u)]$.

Hence it is clear that for each y satisfying (b), $K_x(g^k(y)) \geq K_x(y) + L$. Finally because u is a right limit point of S, there is an $m \in \mathbb{N}$ such that $g^m(u) \in U_\varepsilon^+(u)$. From (2),$(a)$,$(b)$ we obtain $K_x(g^m(u)) \geq K_x(u) + L$ and $K_x(g^{m+k}(u)) \geq K_x(u) + 2L$. But $g^{m+k}(u) \in U_\delta^+(u)$. Since δ was arbitrary, we have a contradiction to (2).

The other cases when either $u < c$ is a left limit point of S or $u > c$ is a limit point of S are similar. Thus, the β-code K_x of $S = \mathrm{orb}(g, x)$ is continuous. Let us prove that K_x is bounded from above. Divide the orbit S into the parts $S_L < c$, $S_R > c$ and the left part S_L into the sets S_{LL}, S_{LR} such that $g(S_{LL}) < c,\ g(S_{LR}) > c$. Notice that by monotonicity of K_x we get

$$g(S_R) \subset S_L. \tag{3}$$

Our system (T, g) is from $\mathcal{U}$; i.e., g_T is piecewise monotone. In particular, g_T is monotone on some left neighborhood of c. Since the orbit S is given by a transitive point $x \in T$ and $\#\,\mathrm{Fix}(g_T) = 1$, we immediately obtain

$$\sup S_{LL} = y < c,\ y < g_T(y),\ (y, g_T(y)) \cap S \neq \emptyset. \tag{4}$$

Moreover, we can see that $g_T(y) \leq \sup S_{LR}$. In fact $g_T(y) = \sup S_{LR}$ holds. To verify the last equality notice that by (3) for each $u \in S_{LR}$ we have ($\beta \in (1, \infty)$)

$$K_x(g^2(u)) = K_x(u) + \frac{1}{1+\beta} - \frac{\beta}{1+\beta} < K_x(u).$$

Hence $g^2(u) < u$. It means the set

$$\Big(S \cap [\min T, g_T(y)]\Big) \cup (S_R \cap g(S \cap [\min T, g_T(y)])$$

is g-invariant. Hence $g_T(y) = \sup S_{LR}$. Summarizing, from (4) for some $v \in (y, g_T(y)) \cap S$ we obtain

$$\sup_{u \in S_R} K_x(u) = \frac{1}{1+\beta} + \sup_{u \in S_L} K_x(u) = \frac{2}{1+\beta} + \sup_{u \in S_{LL}} K_x(u) < \frac{2}{1+\beta} + K_x(v).$$

It means the β-code K_x of $S = \mathrm{orb}(g, x)$ is bounded from above.

Finally let us show that K_x is bounded from below. Let $z = \min T$. Because g_T has the unique fixed point, we have that $g_T(z) > z$ and there is $u \in S$ such that $u < g_T(z)$. Now there is an $\varepsilon > 0$ such that for any $y \in U_\varepsilon(z)$ we have $f(y) > u$. Hence if $y \in S \cap U_\varepsilon(z)$, from the definition of K_x and its monotonicity we have that $K_x(f(y)) = K_x(y) + \frac{1}{1+\beta} \geq K_x(u)$ and clearly $K_x(y) \geq K_x(u) - \frac{1}{1+\beta}$. Since $\inf_{u \in S_L} K_x(u) + \frac{\beta}{1+\beta} = \inf_{u \in S_R} K_x(u)$, the β-code K_x of $S = \mathrm{orb}(g, x)$ is bounded from below. □

The following fact is an easy consequence of the above lemma.

Corollary 3.1. *Under the assumptions of Lemma 3.1 we have $T \cap \mathrm{Fix}(g_T) = \emptyset$.*

PROOF. We use the notation from the proof of the previous lemma. Put $A = \sup_{u\in S} K_x(u)$, $B = \inf_{u\in S} K_x(u)$ and choose a positive integer n such that $A + n(\frac{1}{1+\beta} - \frac{\beta}{1+\beta}) < B$. Suppose $c \in T$. Since g_T is piecewise monotone, $S \cap \{c\} = \emptyset$ and $c \in \overline{S}$ we can find $u \in S$ such that for each $i \in \{0, 1, \dots, n\}$ we have $g^{2i}(u) < c$ and $g^{2i-1}(u) > c$ for $i \in \{1, \dots, n\}$. Then

$$K_x(g^{2n}(u)) = K_x(u) + \frac{n}{1+\beta} - \frac{n\beta}{1+\beta} < A + n(\frac{1}{1+\beta} - \frac{\beta}{1+\beta}) < B$$

- a contradiction to our choice of B. Thus $T \cap \mathrm{Fix}(g_T) = \emptyset$. □

Lemma 3.2. *Let $(T, g) \in \mathcal{U}$, $\beta \in (1, \infty)$ irrational. Suppose that for a transitive point $x \in T$ a β-code $K_x : \mathrm{orb}(g, x) \to \mathbb{R}$ is monotone. Then T does not contain any periodic point of g_T.*

PROOF. Supposing to the contrary that $P \subset T$ for some cycle (P, g) of g_T, we have $E(P, g) \in \mathbb{Q}$ and so $E(P, g) > \beta$, resp. $E(P, g) < \beta$. Then using a sufficiently long block of $\mathrm{orb}(g, x)$ that is close to P it can be shown as in the proof of Corollary 3.1 that the β-code K_x from that corollary is not bounded by the value A, resp. B - a contradiction. □

Corollary 3.2. *([1]) It is known that for a transitive system $(T, g) \in \mathcal{U}$ exactly one of the following three possibilities is satisfied:*

(i) T is finite and then a cycle.

(ii) T is a Cantor set.

(iii) T is a union of finitely many closed intervals with disjoint interiors. First and third possibilities correspond to the presence of periodic points in T.

Thus, under the assumptions of Lemma 3.2 the set T is a Cantor set.

Lemma 3.3. *Let $(T, g) \in \mathcal{U}$, $\beta \in (1, \infty)$ irrational. Suppose that for a transitive point $x \in T$ a β-code $K_x : \mathrm{orb}(g, x) \to \mathbb{R}$ is monotone. Then there is a β-coding $K : T \to \mathbb{R}$.*

PROOF. Since β is irrational, the set T is infinite. Put $S = \mathrm{orb}(g, x)$. In what follows we show that the function $K : T \to \mathbb{R}$ defined as $K(u) = \lim_{v\to u, v\in S} K_x(v)$, $u \in T$ is a β-coding. Obviously it is sufficient to show that K is continuous. Divide the set T into the parts $T_L < c$ and $T_R > c$ and take $u \in T_L$ (case $u \in T_R$ is similar). We already know from Lemma 3.1 that

K is continuous if $u \in S \cup \{\min T_L, \max T_L\}$. Choose $u \in (\min T_L, \max T_L) \setminus S$ and put

$$L_t(u) = \inf\{K_x(v) : v \in S \cap T_L \text{ and } v > u\}$$

and analogously $L_b(u)$ the left limit of K_x at the point u. Since by Corollary 3.1 the set T does not contain a fixed point c of g_T, we have

$$0 \le L = L_t(u) - L_b(u) = L_t(g^n(u)) - L_b(g^n(u)) \text{ for each } n \in \mathbb{N}.$$

Moreover, by Lemma 3.2 the orbit of u is infinite and Lemma 3.1 says that the function K_x is monotone and bounded. Without loss of generality we may assume that for an increasing sequence $\{n(j)\}_{j\ge 0}$ of positive integers $\{u_j = g^{n(j)}(u)\}_{j\ge 0} \subset T_L$ and $u_0 < u_1 < \cdots < u_j < \cdots$. Then for each $j \ge 0$ we get $L_b(u_j) + L = L_t(u_j) \le L_b(u_{j+1})$. Hence also for each $j \ge 0$

$$L_b(u_0) + (j+1)L \le L_t(u_j) \le K(\max T_L).$$

Therefore $L = 0$; i.e., the function K is continuous at the point u. □

When proving the key Lemma 3.6 we will need the notion of a semicycle.

Definition 3.1. Let $f \in C(\mathcal{T})$ have a unique fixed point $c \in \mathrm{Fix}(f)$. A sequence $Q = \langle q_i \rangle_{i=0}^{a}$ will be called an ***f*-semicycle** if

$$f(q_{i-1}) = q_i \quad \text{for } 1 \le i \le a \ , \ q_0 \ne q_a \text{ and } q_0 \in \mathrm{conv}\{q_a, c\}.$$

Let $\beta = \#\{i > 0; q_i < c\}/\#\{i > 0; q_i > c\}$. The eccentricity of the f-semicycle Q will be $E(Q) = \max\{\beta, 1/\beta\}$.

Lemma 3.4. *([10]) Let f have a unique fixed point and Q be an f-semicycle. Then f has a cycle (P, f) such that $P \cap Q = \emptyset$ and $E(P, f) = E(Q)$.*

The following result will be useful in proving Lemma 3.6 and Theorem B.

Lemma 3.5. *([9] Let $f \in C(\mathcal{T})$ be an interval map. Suppose that $f(S) \subset S$ for a closed $S \subset \mathbb{R}$. Then for any $p \in \mathrm{Per}(f_S)$ there is $p^* \in \mathrm{Per}(f)$ such that $f_S|\,\mathrm{orb}(f_S, p) \circ h = h \circ f|\,\mathrm{orb}(f, p^*)$ where $h : \mathrm{orb}(f, p^*) \to \mathrm{orb}(f_S, p)$ is an order preserving bijection.*

A crucial role of a monotone code is shown by the following result.

Lemma 3.6. *Let $(T, g) \in \mathcal{U}$, $\beta \in (1, \infty)$. Suppose that for some measure $\mu \in \mathcal{M}_e(g)$ with $T = \mathrm{supp}\,\mu$, $E(\mu) = \beta$ and a generic point $x \in G(\mu)$ a β-code $K_x : orb(g, x) \to \mathbb{R}$ is not monotone. If $f \in C(I)$ is an interval map satisfying $f|T = g$, then f has a cycle (P, f) such that $E(P, f) > \beta$.*

Remark 3.1. In this lemma we assume again that

$$E(\mu) = \mu([\min T, c])/\mu([c, \max T]).$$

Otherwise we would use the map $h \circ g_T \circ h^{-1}$ where $h(x) = -x + \min T + \max T$, $x \in [\min T, \max T]$.

PROOF. It was proved in [10] for T finite. Thus let us suppose that T is infinite and put $f = g_T$. We know that then the measure μ is nonatomic. Take $x \in G(\mu)$. Since x is generic and $\operatorname{supp} \mu = T$, it is also a transitive point. Moreover, for each open set $U \subset T$ (in the relative topology) such that $\mu(\operatorname{bd} U) = 0$ [11]

$$\lim_n \frac{\#\{i \le n : \ g^i(x) \in U\}}{n} = \mu(U). \tag{5}$$

Let $S = \operatorname{orb}(g, x)$. If $K_x : S \to \mathbb{R}$ is not monotone, then one of the following two possibilities has to be satisfied:

(a) There are two points $u, v \in S$ such that $u \in \operatorname{conv}\{v, c\}$, $K_x(v) > K_x(u)$ and $v = g^k(u)$ for some $k \in \mathbb{N}$.

(b) There are two points $u, v \in S$ such that $u \in \operatorname{conv}\{v, c\}$, $K_x(v) > K_x(u)$ and $u = g^k(v)$ for some $k \in \mathbb{N}$.

(a) In this case we will show that g_T has a cycle with some eccentricity greater than β. Really, $\langle g^0(u), g^1(u), \dots, g^k(u)\rangle$ is a semicycle with the eccentricity $\frac{m}{n}$, m, n coprime. Since we know that $K_x(g^k(u)) = K_x(g^0(u)) + m_1 \frac{1}{1+\beta} - n_1 \frac{\beta}{1+\beta}$, where $m_1, n_1 \in \mathbb{N}$ and $\frac{m_1}{n_1} = \frac{m}{n}$, from $K_x(g^k(u)) > K_x(g^0(u))$ we obtain $\frac{m}{n} > \beta$. By Lemma 3.4 the map g_T has a cycle (P, g_T) such that $E(P, g_T) = \frac{m}{n}$. Thus, the conclusion holds in this case.

(b) Supposing that (a) does not hold. We will show that (b) is impossible for any two points from S.

Since g_T is continuous, we can find an $\varepsilon > 0$ such that the sets $U_\varepsilon(g^i(v))$, $U_\varepsilon(c)$, $i \in \{0, \dots, k\}$ are pairwise disjoint. Moreover there is a $\delta > 0$ and $U_\delta(v)$ such that $\mu(\operatorname{bd} U_\delta(v)) = 0$ and for any $z \in U_\delta(v)$ and $i \in \{0, 1, \dots, k\}$ we have

$$g^i(z) \in U_\varepsilon(g^i(v)). \tag{6}$$

Therefore for any $y, \tilde{y} \in U_\delta(v) \cap S$ the β-codes satisfy

$$K_x(g^k(y)) - K_x(y) = K_x(g^k(\tilde{y})) - K_x(\tilde{y}). \tag{7}$$

Let $M = \{a_i\}_{i=0}^{\infty}$ such that $a_i < a_{i+1}$ and $g^j(v) \in U_\delta(v)$ if and only if $j \in M$. Since $v \in G(\mu)$ (as the image of x), the set M is well defined. Notice that from

(6) we have $a_i - a_{i-1} > k$. Because case (a) is not possible, we can see that $K_x(g^{a_i}(v)) \le K_x(g^{a_{i-1}+k}(v))$. Using (7) we get $K_x(g^{a_i+k}(v)) - K_x(g^{a_i}(v)) = K_x(g^k(v)) - K_x(v)$. Hence $K_x(g^{a_i+k}(v)) - K_x(g^{a_{i-1}+k}(v)) \le K_x(g^k(v)) - K_x(v) < 0$ for $i \in \mathbb{N}$ and inductively

$$K_x(g^{a_i+k}(v)) - K_x(g^{a_0+k}(v)) \le i(K_x(g^k(v)) - K_x(v)) < 0. \tag{8}$$

We can write $a_i = m_i + n_i$ where $m_i = \#\{j \le a_i :\ g^{j+k}(v) < c\}$. Then we have

$$K_x(g^{a_i+k}(v)) = K_x(g^k(v)) + m_i \frac{1}{1+\beta} - n_i \frac{\beta}{1+\beta}. \tag{9}$$

Further because v is a generic point, by (5) and the equalities $\mu(\operatorname{bd} U_\delta(v)) = \mu(\{c\}) = 0$ we have the limits

$$\begin{aligned} \lim_{i\to\infty} \frac{m_i}{m_i+n_i} &= \mu([\min T, c]) = \frac{\beta}{1+\beta} \\ \lim_{i\to\infty} \frac{n_i}{m_i+n_i} &= \mu([c, \max T]) = \frac{1}{1+\beta} \\ \lim_{i\to\infty} \frac{i}{m_i+n_i} &= \mu(U_\delta(v)) > 0. \end{aligned} \tag{10}$$

Putting (8), (9) and (10) together we get

$$0 = \frac{\beta}{(1+\beta)^2} - \frac{\beta}{(1+\beta)^2} \le \mu(U_\delta(v))(K_x(u) - K_x(v)) < 0$$

- a contradiction. This excludes the case (b).

We have shown that under the condition of this theorem the possibility (a) has to be satisfied. Thus, the conclusion for $f = g_T$ follows.

If $f \in C(I)$ is an interval map satisfying $f|T = g$, then the conclusion is an immediate consequence of Lemma 3.5 and the fact that it is true for g_T. $\square$

4 Proof of Theorems A and B

The following definition and lemma apply to any dynamical system (X, f), where X is a compact metric space and $f : X \to X$ is continuous. The definition of ω-limit set and minimality is analogous as for systems with $X \subset \mathbb{R}$ - compare with Section 2.

Definition 4.1. (cf. [1] [Proposition 5, p. 93]) A point $x \in X$ is called **strongly recurrent** if $x \in \omega(f, x)$ and the system $(\omega(f, x), f)$ is minimal.

The following lemma is a special consequence of stronger statement from [12].

Lemma 4.1. *([12] [Proposition 8.6]) Let (X,ρ) be a compact metric space and $f : X \to X$ be continuous. Then for each $y \in X$ there is a strongly recurrent $z \in X$ such that $\liminf_{i\to\infty} \rho(f^i(y), f^i(z)) = 0$.*

Theorem A. *Let $(T,g) \in \mathcal{U}$, $\beta \in (1,\infty)$ irrational. The following statements are equivalent:*

(i) (T,g) *is a twist β-system.*

(ii) $T = \operatorname{supp}\mu$ *for some $\mu \in \mathcal{M}_e(g)$ with $E(\mu) = \beta$ and there is a map $\tilde{g} \in C(T,g)$ such that $E(\tilde{g}) = \beta$.*

(iii) There is a transitive point $x \in T$ such that the β-code K_x of $\operatorname{orb}(g,x)$ *is monotone.*

(iv) (T,g) *is minimal and there is the β-coding $K : T \to \mathbb{R}$ such that for each $x \in T$, the β-code $K_x = K|\ \operatorname{orb}(g,x)$ of $\operatorname{orb}(g,x)$ is strictly monotone.*

(v) $T = \operatorname{supp}\mu$ *for some $\mu \in \mathcal{M}_e(g)$ with $E(\mu) = \beta$ and for any $\tilde{g} \in C(T,g)$ it holds $E(\tilde{g}) = \beta$ and $\forall\ \nu \in \mathcal{M}(\tilde{g}) :\ E(\nu) = \beta \implies \operatorname{supp}\nu = T$.*

PROOF. The implication (i) $\Longrightarrow$ (ii) is clear.
(ii) $\Longrightarrow$ (iii).

Let $\mu \in \mathcal{M}_e(g)$ be a measure such that $\operatorname{supp}\mu = T$ and $E(\mu) = \beta$. Take a generic point $x \in G(\mu)$. This point is also transitive. Since $E(\tilde{g}) = \beta$ by Lemma 3.6 the β-code K_x of $\operatorname{orb}(g,x)$ is monotone. So (ii) implies (iii).
(iii) $\Longrightarrow$ (iv) Using Lemma 3.3 we obtain that the required β-coding exists. Notice that since β is irrational, for each $x \in T$ the β-code K_x is monotone if and only if it is strictly monotone. Thus, it remains to prove that (T,g) is minimal. Obviously, for the β-coding K, $z \in T$ and $i \in \mathbb{N}$ we have

$$K(g^i(z)) \equiv K(z) + \frac{i}{1+\beta} \pmod 1. \tag{11}$$

We know from Corollary 3.2 that T is a Cantor set. Let us denote G, resp. F the set of all transitive points, resp. all endpoints of T-contiguous intervals from T. Obviously, G is dense of type G_δ and F is countable. By the standard category arguments we obtain that $G \setminus \bigcup_{i\geq 0} g^{-i}(F) \neq \emptyset$; i.e., we can consider a transitive point $y \in G$ for which $\operatorname{orb}(g,y) \cap F = \emptyset$. Now let us apply Lemma 4.1. By this lemma there is a strongly recurrent point $z \in T$ such that

$$\liminf_{i\to\infty} |f^i(y) - f^i(z))| = 0. \tag{12}$$

It is a consequence of (11) that if $K(y) \not\equiv K(z) \pmod 1$, then since K is continuous and monotone, (12) cannot hold. Hence $K(y) \equiv K(z) \pmod 1$ and for the same reason $K(g^n(y)) = K(g^n(z))$ for some $n \in \mathbb{N}$. By our choice of y, $g^n(y) \notin F$; i.e., $g^n(y)$ is a two-sided limit point of $\operatorname{orb}(g, y)$ and $K|\operatorname{orb}(g, y)$ is strictly monotone. Thus $g^n(y) = g^n(z)$; i.e., y is strongly recurrent. Summarizing, the system $(T = \omega(g, y), g)$ is minimal.
(iv) $\Longrightarrow$ (v)

Obviously for $x \in T$ we can write

$$\frac{1}{n}\, K(g^n(x)) = \frac{1}{n}\, K(x) + \frac{1}{1+\beta}\, \frac{1}{n} \sum_{j=0}^{n-1} \chi_L(g^j(x)) - \frac{\beta}{1+\beta}\, \frac{1}{n} \sum_{j=0}^{n-1} \chi_R(g^j(x)),$$

where χ_L, resp. χ_R is an indicator function of $[\min T, c]$, resp. $[c, \max T]$. But the β-coding of T is bounded. Hence

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_L(g^j(x)) = \frac{\beta}{1+\beta} \text{ and } \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_R(g^j(x)) = \frac{1}{1+\beta}. \tag{13}$$

It is known [11] that (13) gives an existence of a measure $\mu \in \mathcal{M}_e(g)$ such that $E(\mu) = \beta$ and $\operatorname{supp} \mu = T$ ((T, g) is minimal). In fact we know more: any measure from $\mathcal{M}_e(g)$ with support equal to T has its eccentricity equal to β.

Since $(T, g) \in \mathcal{U}$, the map $\tilde{g} \in C(T, g)$ is piecewise monotone and $\# \operatorname{Fix}(\tilde{g}) = 1$. Hence by Lemma 2.1 there is a measure $\nu \in \mathcal{M}_e(\tilde{g})$ such that $E(\nu) = E(\tilde{g})$. Let us suppose that $S = \operatorname{supp} \nu \neq T$. Then since (T, g) is minimal we have either $S \supsetneqq T$ or $S \cap T = \emptyset$.

(1) Let $S \supsetneqq T$. Suppose that $\gamma = E(\nu) = E(\tilde{g}) > \beta$. We will use Lemma 3.6 for $(S, h = \tilde{g}|S) \in \mathcal{U}$, $\gamma \in (1, \infty)$ and $\nu \in \mathcal{M}_e(h)$. Take some generic point $x \in G(\nu)$. If some γ-code K_x of $\operatorname{orb}(h, x)$ is not monotone, we know from Lemma 3.6 that $\tilde{g}$ has a cycle with eccentricity greater than γ. But this is impossible since $E(\tilde{g}) = \gamma$. So we obtain that the γ-code K_x of $\operatorname{orb}(h, x)$ is monotone.

On the one hand Lemma 3.1 says that the γ-code $K_x : \operatorname{orb}(h, x) \to \mathbb{R}$ is continuous and bounded. On the other hand, if $\gamma > \beta$, then using the code of a sufficiently long block of $\operatorname{orb}(h, x)$ close to T it can be shown similarly as in the proof of Corollary 3.1 that the γ-code K_x of $\operatorname{orb}(h, x)$ is not bounded from below, a contradiction. It shows that $\gamma = \beta$; i.e., γ is irrational. Thus, we get that all conditions of part (iii) of this theorem are satisfied by a system (S, h). Since we have already proved (iii) $\Longrightarrow$ (iv), we can see that the system (S, h) is minimal. Hence $S = T$ and $h = g$. We have shown that (1) cannot hold.

(2) $S \cap T = \emptyset$. We will show that $\gamma = E(\nu) < \beta$ in this case. Since $\tilde{g}$ is (T, g)-monotone and K is monotone it holds $\tilde{g}([c, \max T]) = [\min T, c]$. It

implies for the measure ν that

$$\gamma = E(\nu) = \nu([\min T, c])/\nu([c, \max T]). \tag{14}$$

The sets S, T are closed and disjoint. Clearly the distance of S, T is positive. Hence S is a subset of finitely many T-contiguous intervals. Denote them by $\{J_i = (a_i, b_i)\}_{i=1}^{l}$ and consider their codes $K(J_i)$, where $K(J = (a, b))$ equals to $K(a)$ if $J < c$ and $K(b)$ for $J > c$. (If $c \in J = (a, b)$, take two intervals, $(a, c]$ and $[c, b)$.) Notice the following property given by the monotonicity of the β-coding K: if $\tilde{g}(J_i) \supset J_j$, then

$$K(J_j) \geq K(J_i) + \frac{1}{1+\beta} \text{ for } J_i < c \text{ and } K(J_j) \geq K(J_i) - \frac{\beta}{1+\beta} \text{ for } J_i > c. \tag{15}$$

Clearly the relation $\operatorname{int} \tilde{g}(J_i) \cap J_j \neq \emptyset$ implies $\tilde{g}(J_i) \supset J_j$. Hence one can verify that for a generic point $y \in S \cap G(\nu)$ there is a unique sequence $\{L_j\}_{j=0}^{\infty}$ of intervals from $\{J_i\}_{i=1}^{l}$ such that $y_j = \tilde{g}^j(y) \in L_j$ for each nonnegative integer j. Obviously $\tilde{g}(L_j) \supset L_{j+1}$. Using (15), if we put $K(L_{j+1}) = K(L_j) + \frac{1}{1+\beta} + \varepsilon(y_j)$ for $L_j < c$ and $K(L_{j+1}) = K(L_j) - \frac{\beta}{1+\beta} + \varepsilon(y_j)$ for $L_j > c$, then $\{\varepsilon_j = \varepsilon(y_j)\}_{j \geq 0}$ is a sequence of finitely many nonnegative values and we can write for each $n \in \mathbb{N}$

$$K(L_n) = K(L_0) + \sum_{j=0}^{n-1} (\frac{1}{1+\beta} + \varepsilon_j)\chi_L(y_j) - \sum_{j=0}^{n-1} (\frac{\beta}{1+\beta} - \varepsilon_j)\chi_R(y_j). \tag{16}$$

Here χ_L, resp. χ_R is an indicator function of $[\min T, c]$, resp. $[c, \max T]$.

The equality (16) divided by n can be rewritten in the form

$$\frac{1}{n}K(L_n) = \frac{1}{n}K(L_0) + \frac{1}{1+\beta}\ \frac{1}{n}\sum_{j=0}^{n-1}\chi_L(y_j) - \frac{\beta}{1+\beta}\ \frac{1}{n}\sum_{j=0}^{n-1}\chi_R(y_j) + \frac{1}{n}\ \sum_{j=0}^{n-1}\varepsilon_j. \tag{17}$$

Clearly $\lim_n \frac{1}{n}\ K(L_n) = \lim_n \frac{1}{n}\ K(L_0) = 0$. Now we use the fact that y is generic. We have (see(14))

$$\lim_n \frac{1}{n}\sum_{j=0}^{n-1}\chi_L(y_j) = \frac{\gamma}{1+\gamma} \text{ and } \lim_n \frac{1}{n}\sum_{j=0}^{n-1}\chi_R(y_j) = \frac{1}{1+\gamma}.$$

Putting the last limits together we get $0 \leq \lim_n \frac{1}{n}\ \sum_{j=0}^{n-1}\varepsilon_j = \frac{\beta-\gamma}{(1+\beta)(1+\gamma)}$. In what follows we show that in fact $0 < \lim_n \frac{1}{n}\ \sum_{i=0}^{n-1}\varepsilon_j$. We have seen that it is equivalent to the inequality $\gamma < \beta$. We will use again the fact that y is generic. Obviously there exists $n \in \mathbb{N}$ such that $K(L_n) = K(L_0)$. We suppose that β is irrational. Then (17) implies that for this n

$$0 \neq \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_j = \frac{\beta}{1+\beta} \frac{1}{n} \sum_{j=0}^{n-1} \chi_R(y_j) - \frac{1}{1+\beta} \frac{1}{n} \sum_{j=0}^{n-1} \chi_L(y_j);$$

i.e., some $\varepsilon(y_i) \neq 0$ for $i \in \{0, \ldots, n-1\}$. We can find a neighborhood $U(y_i)$ of y_i satisfying $\tilde{g}(U(y_i)) \subset L_{i+1}$, $\nu(\operatorname{bd} U(y_i)) = 0$ and $\nu(U(y_i)) = \delta > 0$. It is clear that if for some $k \in \mathbb{N}$ we have $y_k \in U(y_i)$, then also $\varepsilon(y_k) = \varepsilon(y_i)$. Since by (5) $\lim_n \frac{\#\{j \le n-1:\ y_j \in U(y_i)\}}{n} = \delta$, we also have $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon_j \geq \delta \cdot \varepsilon(y_i) > 0$, hence $\gamma < \beta$.

We have proved that $E(\nu) < \beta$ whenever $\nu \in \mathcal{M}_e(\tilde{g})$ and $\operatorname{supp} \nu \neq T$. Now the conclusion is an easy consequence of Theorem 2.1. This finishes the proof of this part.

The implication (v) $\Longrightarrow$ (i) follows directly from our definitions. □

Theorem B. *Let $f \in C(\mathcal{T})$ be an interval map such that $E(f) \in (1, \infty)$ is irrational. If there is a measure $\mu \in \mathcal{M}_e(f)$ for which $E(\mu) = E(f)$ and $f_{\operatorname{supp} \mu}$ is piecewise monotone, then* $(\operatorname{supp} \mu, f)$ *is a twist $E(f)$-system.*

PROOF. By our assumptions $\# \operatorname{Fix}(f) = 1$ (cf. Remark 2.1 in Section 2). Let $S = \operatorname{supp} \mu$. Then $f(S) = S$ and since f_S is piecewise monotone, (S, f) is a unisystem; i.e., $(S, f) \in \mathcal{U}$. By Theorem A(ii) we need to show that there is a transitive point $x \in S$ such that the $E(f)$-code K_x of $\operatorname{orb}(f, x)$ is monotone. Take an arbitrary point $x \in S$ generic for μ and suppose to the contrary that K_x is not monotone. By Lemma 3.6 there is a periodic point $p \in \operatorname{Per}(f_S)$ such that $E(\operatorname{orb}(f_S, p), f_S) > E(f)$. Then from Lemma 3.5 we obtain that there is a periodic point $p^* \in \operatorname{Per}(f)$ for which $E(\operatorname{orb}(f, p^*), f) = E(\operatorname{orb}(f_S, p), f_S) > E(f)$ - a contradiction. This proves the theorem. □

In the following remark we are speaking about the topological entropy of a twist system $(T, g) \in \mathcal{U}$. It is defined as the entropy of g_T - (for entropy see [11]).

Remark 4.1. It was shown (see [5], [6], [7]) that for twist systems with real eccentricities various behaviors are possible. Namely, it was proved that when modality increases, entropy may stay bounded, but it may also increase to infinity (independently of the eccentricities).

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