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ON TOPOLOGIES CONNECTED WITH HAUSDORFF MEASURES

Abstract

Notions of density points and topologies associated with various set-theoretic ideals connected with Hausdorff measures are introduced and their properties investigated. Inclusions between the ideals and between the topologies are shown.

It is well known that there are several senses (categorical, measure-theoretic, etc.) in which a subset of the real line may be "small." In particular, for any $s \in (0,1)$, one can consider a natural family of small sets connected with Hausdorff s-dimensional measure: the family of null sets for this measure. These families depend on the chosen s and it is obvious that there are natural inclusions between them.

It is natural to ask whether, given a fixed s, as above, one can introduce other proper set-theoretic ideals related to the Hausdorff s-dimensional measure which will allow for a finer distinctions between various types of small sets. In this paper we show that this is indeed possible by considering the ideals of sets of σ-finite measure and of sets whose Hausdorff dimension is at most equal to s. We further analyze some properties of related density and Hashimoto-type topologies and observe that the so-defined variety of ideals allows us to construct different kinds of sets which are "dense" in their every point in the sense of Hausdorff measures but do not belong to the density topology connected with Lebesgue measure.

Recall the basic notions of the properties to be used [1].

If U is a non-empty subset of $\mathbb{R}^n$, the diameter of U is defined as $|U| = \sup\{d(x,y) : x, y \in U\}$, where d denotes the Euclidean metric in $\mathbb{R}^n$. Let $\delta > 0$. If $E \subset \bigcup_{i\in I} U_i$ and $0 < |U_i| \leq \delta$ for each $i \in I$, then $\{U_i\}_{i\in I}$ is said to be a δ-cover of E.

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Let $E \subset \mathbb{R}^n$ and $s \geq 0$. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s, \tag{1}$$

where the infimum is over all countable δ-covers $\{U_i\}_{i\in\mathbb{N}}$ of E. It is easy to see that $\mathcal{H}^s_\delta$ is an outer measure on $\mathbb{R}^n$.

Finally, the Hausdorff s-dimensional outer measure is defined by the formula $\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E)$ or, equivalently, by $\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E)$.

For a given set $E \subset \mathbb{R}^n$, there is a unique $s \in [0, \infty]$ such that $\mathcal{H}^t(E) = +\infty$ for all $t < s$ and $\mathcal{H}^t(E) = 0$ for all $t > s$. This value is called the Hausdorff dimension of the set E. We denote it by $\dim E$.

The definition of the Hausdorff measure may be generalized [3] by replacing $|U_i|^s$ by $h(|U_i|)$ in (1), where h is some function defined for all $t \geq 0$ (possibly taking the value $+\infty$ for some t), positive for $t > 0$, increasing and continuous on the right. The family of such functions will be denoted by $\mathcal{H}$, and the Hausdorff measure which is obtained using the function h by μ^h (so that if $f_r(t) = t^r$ for $t \in [0, +\infty)$, then $\mu^{f_r} = \mathcal{H}^r$).

Now we want to restrict our considerations to the real line. Since, for any $s > 1$, each subset of $\mathbb{R}$ has outer $\mathcal{H}^s$ measure equal to 0, for $s = 0$ we get counting measure and for $s = 1$ Lebesgue measure, we assume now that $s \in (0, 1)$.

For a given $s \in (0, 1)$ we can consider the following families of sets.

$$\begin{aligned}
\mathcal{N}_s &= \{A \subset \mathbb{R} : \mathcal{H}^s(A) = 0\}, \\
\mathcal{N}_{s-\dim} &= \{A \subset \mathbb{R} : \dim A \leq s\}, \\
\mathcal{N}_{s-\sigma}7 &= \{A \subset \mathbb{R} : A \text{ is of } \sigma\text{-finite measure } \mathcal{H}^s\}.
\end{aligned}$$

Each of these families is a σ-ideal. The following inclusions are easily seen: $\mathcal{N}_s \subset \mathcal{N}_{s-\sigma} \subset \mathcal{N}_{s-\dim}$. The first inclusion is proper [1]. We shall show that so is the second one. Before we do it, we need a few more facts.

First, following Rogers [3, p. 78], introduce a partial order into the family $\mathcal{H}$, by saying that g corresponds to a smaller generalized dimension than h, if $h(t)/g(t) \to 0$ as $t \to 0$ and denoting this by $g \prec h$. We also need the following theorem.

Theorem (Rogers [3] Cor. to Thm. 40 on p. 79)**.** *Let f, g, h be functions in $\mathcal{H}$ with $f \prec g \prec h$. If a subset E of a metric space has σ-finite positive μ^g-measure, then $\mu^h(E) = 0$ and E has non-σ-finite μ^f-measure.*

Now we are ready to formulate and prove the previously mentioned theorem.

Theorem 1. *For any given $s_0 \in (0,1)$, there exists a set $E \subset \mathbb{R}$ which has non-σ-finite measure $\mathcal{H}^{s_0}$, but whose Hausdorff dimension is equal to s_0.*

Proof. Our proof starts with the observation that, if there exists a function $g \in \mathcal{H}$ such that $f_{s_0} \prec g \prec h_r$, where $f_{s_0}(t) = t^{s_0}$ and $h_r(t) = t^r$, for all $r > s_0$, then a set E of positive σ-finite measure μ^g satisfies the conditions of our statement, since, by the theorem quoted above, we are able to conclude that E has non-σ-finite measure $\mathcal{H}^{s_0}$ and for all $r > s_0$, we get $\mathcal{H}^r(E) = 0$. Hence $\dim(E) = s_0$.

It remains to prove that such a function g exists, since the existence of a set of positive finite measure μ^g for a given function $g \in \mathcal{H}$ was established in [1].

Let $\{s_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of numbers in $(0,1)$, converging to s_0. Then, with the notation $f_n(t) = t^{s_n}$ for $n \in \mathbb{N} \cup \{0\}$, we have $f_0 \prec \cdots \prec f_n \prec f_{n-1} \prec \cdots \prec f_2 \prec f_1$. Indeed, $f_{n-1}(t)/f_n(t) = t^{s_{n-1}-s_n} \to 0$ as $t \to 0$, since $s_{n-1} - s_n > 0$. Similarly, we can see that $f_0 \prec f_n$ for every $n \in \mathbb{N}$.

We now define the sequence $\{t_n\}_{n\in\mathbb{N}}$. Since both $\lim_{t\to 0} \frac{f_1(t)}{f_2(t)} = 0$ and $\lim_{t\to 0} \frac{f_3(t)}{f_0(t)} = 0$, one can find a point t_1 such that $\frac{f_1(t_1)}{f_2(t_1)} < \frac{1}{2}$ and $\frac{f_3(t_1)}{f_0(t_1)} < \frac{1}{2}$. From now on we proceed by induction. Suppose we have already chosen points $t_1, \ldots, t_n$ with the following properties: $t_1 > t_2 > \cdots > t_n$, $\frac{f_k(t_k)}{f_{k+1}(t_k)} < \frac{1}{k+1}$, $\frac{f_{k+2}(t_k)}{f_0(t_k)} < \frac{1}{k+1}$ and $f_{k+1}(t_k) < f_k(t_{k-1})$ for $k \in \{2, \ldots, n\}$. Since $\lim_{t\to 0} \frac{f_{n+1}(t)}{f_{n+2}(t)} = 0$ and $\lim_{t\to 0} \frac{f_{n+3}(t)}{f_0(t)} = 0$, we can find a point t_{n+1} such that $t_{n+1} < t_n$, $\frac{f_{n+1}(t_{n+1})}{f_{n+2}(t_{n+1})} < \frac{1}{n+2}$ and $\frac{f_{n+3}(t_{n+1})}{f_0(t_{n+1})} < \frac{1}{n+2}$. Moreover, since $\lim_{t\to 0} f_{n+2}(t) = 0$ we can assume, by decreasing t_{n+1}, if necessary, that $f_{n+2}(t_{n+1}) < f_{n+1}(t_n)$.

Let g be defined by

$$g(t) = \begin{cases} 0 & \text{for } t = 0, \\ f_{n+1}(t_n) & \text{for } t = t_n, n \in \mathbb{N}, \\ \text{increasing and continuous} & \text{for } t \in [t_{n+1}, t_n], \\ \quad \text{and such that } f_{n+1}(t) \le g(t) \le f_{n+2}(t) & \text{on this interval.} \end{cases}$$

It is evident that the function g belongs to $\mathcal{H}$ and for $n \in \mathbb{N}$ the quotient $\frac{f_n(t)}{g(t)}$ is not greater that $\frac{f_n(t)}{f_{n+1}(t)}$ for $t < t_n$, which is smaller than $\frac{1}{n+1}$ and $\frac{g(t)}{f_0(t)}$ for $t \in [t_{n+1}, t_n]$ is not greater than $\frac{f_{n+2}(t)}{f_0(t)}$ which is smaller than $\frac{1}{n+1}$. Therefore $f_0 \prec g \prec f_n$ for $n \in \mathbb{N}$. Since for each $r > s_0$, there exists s_n such that $s_n < r$, it follows that $f_0 \prec g \prec f_n \prec h$, which completes the proof. $\square$

From the definition of $\mathcal{H}^s$ it follows that for arbitrary $E \subset \mathbb{R}$, if $s < t < 1$, then $\mathcal{H}^s(E) \ge \mathcal{H}^t(E) \ge \lambda(E)$. Moreover, if $\mathcal{H}^t(E) > 0$, then $\mathcal{H}^s(E) = \infty$

and for each $s \in (0,1)$ there exists a set $E \subset \mathbb{R}$ with positive and finite outer measure $\mathcal{H}^s$ [1]. Consequently, if $0 < s < t < 1$, then $\mathcal{N}_s \subset \mathcal{N}_t \subset \mathcal{N}$, and these inclusions are proper. It is also evident that $\mathcal{N}_{s-\dim} \subset \mathcal{N}_t$ and, taking a set $E \in \mathcal{N}_t \setminus \mathcal{N}_{\frac{s+t}{2}}$, one can see that the last inclusion is proper.

Summarizing, we have the following assertion.

Theorem 2. *If $0 < s < t < 1$, then*

$$\mathcal{N}_{0-\dim} = \bigcap_{0<r<1} \mathcal{N}_r \subset \mathcal{N}_s \subset \mathcal{N}_{s-\sigma} \subset \mathcal{N}_{s-\dim} \subset \mathcal{N}_t \subset \mathcal{N},$$

and all these inclusions are proper.

It is worth pointing out that all the σ-ideals mentioned above are invariant under multiplication by numbers. This is a simple consequence of the following property of the Hausdorff s-dimensional outer measure: $\mathcal{H}^s(\alpha \cdot A) = |\alpha|^s \mathcal{H}^s(A)$ for $\alpha \in \mathbb{R}$, $A \subset \mathbb{R}$, $s \in (0,1)$.

Using the σ-ideal $\mathcal{N}_s$, E. Wagner-Bojakowska and W. Wilczyński [4] introduced the notion of an $\mathcal{H}^s$-density point and defined an operation $\Phi_s : \mathcal{L} \to 2^{\mathbb{R}}$.

Definition 1 ([4])**.** The point 0 is an $\mathcal{H}^s$-density point of a set $A \in \mathcal{L}$ if, and only if, for each subsequence $\{n_k\}_{k\in\mathbb{N}}$ of the sequence of positive integers, there exists a subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ such that

$$\chi_{(n_{k_p}\cdot A)\cap[-1,1]} \xrightarrow[p\to\infty]{} \chi_{[-1,1]}$$

except on a set from $\mathcal{N}_s$. (Here $n \cdot A = \{na : a \in A\}$ and χ_A denotes the characteristic functions of A.) Clearly, the convergence above holds if and only if $\limsup_p (n_{k_p} \cdot A') \cap [-1,1] \in \mathcal{N}_s$, where $A' = \mathbb{R} \setminus A$.

A point x is said to be an $\mathcal{H}^s$-density point of $A \in \mathcal{L}$ if, and only if, 0 is an $\mathcal{H}^s$-density point of the set $A - x = \{a - x : a \in A\}$. Two sets A and B are called s-equivalent ($A \sim B$), if $A \triangle B \in \mathcal{N}_s$.

For $A \in \mathcal{L}$, $0 < s < 1$ let

$$\Phi_s(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{H}^s\text{-density point of } A\},$$

and let $\Phi(A)$ denote the set of all density points of A, for $A \in \mathcal{L}$.

Replacing the σ-ideal $\mathcal{N}_s$ by $\mathcal{N}_{s-\sigma}$ (or $\mathcal{N}_{s-\dim}$), we define below an operation $\Phi_{s-\sigma}$ (or, respectively, $\Phi_{s-\dim}$).

Definition 2. For any $A \in \mathcal{L}$ and $s \in (0,1)$, let

$$\Phi_{s-\sigma}(A) = \Big\{x \in \mathbb{R} : \forall_{\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}} \exists_{\{n_{k_p}\}_{p\in\mathbb{N}} \subset \{n_k\}_{k\in\mathbb{N}}} \limsup_p (n_{k_p} \cdot (A' - x)) \cap [-1,1] \in \mathcal{N}_{s-\sigma}\Big\}$$

and

$$\Phi_{s-\dim}(A) = \Big\{x \in \mathbb{R} : \forall_{\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}} \exists_{\{n_{k_p}\}_{p\in\mathbb{N}} \subset \{n_k\}_{k\in\mathbb{N}}} \limsup_p (n_{k_p} \cdot (A' - x)) \cap [-1, 1] \in \mathcal{N}_{s-\dim}\Big\}.$$

Theorem 2 now implies that for each s, t, $0 < s < t < 1$, we can find sets $A, B, C, D \in \mathcal{L}$ such that $\Phi_s(A) \subsetneq \Phi_{s-\sigma}(A)$, $\Phi_{s-\sigma}(B) \subsetneq \Phi_{s-\dim}(B)$, $\Phi_{s-\dim}(C) \subsetneq \Phi_t(C)$, and $\Phi_t(D) \subsetneq \Phi(D)$.

Theorem 3. *For each $A, B \in \mathcal{L}$ and $s \in (0, 1)$,*

1. *if $A \subset B$, then $\Phi_{s-\sigma}(A) \subset \Phi_{s-\sigma}(B)$;*
2. *if $A \sim_{s-\sigma} B$ (i.e. $A \triangle B \in \mathcal{N}_{s-\sigma}$), then $\Phi_{s-\sigma}(A) = \Phi_{s-\sigma}(B)$;*
3. *$\Phi_{s-\sigma}(\emptyset) = \emptyset$ and $\Phi_{s-\sigma}(\mathbb{R}) = \mathbb{R}$;*
4. *$\Phi_{s-\sigma}(A \cap B) = \Phi_{s-\sigma}(A) \cap \Phi_{s-\sigma}(B)$;*
5. *there exists a set $E \in \mathcal{L}$ such that $E \setminus \Phi_{s-\sigma}(E) \notin \mathcal{N}_{s-\sigma}$ (a natural analogue of the Lebesgue Density Theorem does not hold).*

The operations $\Phi_{s-\dim}$ and Φ_s also have properties (1)–(5). (For Φ_s, see [4].)

Proof. For (1)–(4) the proofs are obvious. In (5) it is enough to take a set E with positive and finite outer measure $\mathcal{H}^t$ for $t > s$. Then $\lambda(E) = 0$ and $\Phi(E) = \emptyset$. Consequently, $\Phi_s(E) = \Phi_{s-\sigma}(E) = \Phi_{s-\dim}(E) = \emptyset$, and E does not belong to $\mathcal{N}_t$, nor to $\mathcal{N}_{s-\dim}$, $\mathcal{N}_{s-\sigma}$ or $\mathcal{N}_s$. □

In [4] the following topology $\mathcal{T}_s$ was introduced: $\mathcal{T}_s = \{A \in \mathcal{L} : A \subset \Phi_s(A)\}$ for $s \in (0, 1)$. This topology is stronger than the Euclidean topology $\mathcal{O}$ and weaker than the density topology $\mathcal{T}$. It is also shown in [4] that

1) if $0 < s < t < 1$ then $\mathcal{T}_s \subsetneq \mathcal{T}_t$,
2) $\bigcup_{0<s<t} \mathcal{T}_s \subsetneq \mathcal{T}_t$, and
3) $\bigcup_{0<s<1} \mathcal{T}_s \subsetneq \mathcal{T}$.

Definition 3. For any $s \in (0, 1)$ let $\mathcal{T}_{s-\sigma} = \{A \in \mathcal{L} : A \subset \Phi_{s-\sigma}(A)\}$ and $\mathcal{T}_{s-\dim} = \{A \in \mathcal{L} : A \subset \Phi_{s-\dim}(A)\}$.

Theorem 4. *Let $0 < s < t < 1$. The families $\mathcal{T}_{s-\sigma}$ and $\mathcal{T}_{s-\dim}$ are topologies on the real line and $\mathcal{T}_s \subsetneq \mathcal{T}_{s-\sigma} \subsetneq \mathcal{T}_{s-\dim} \subsetneq \mathcal{T}_t$.*

Proof. The fact that these families are topologies is easily seen and also the inclusions are obvious. We only need to show the latter are proper. We begin by proving $\mathcal{T}_{s-\sigma} \setminus \mathcal{T}_s \neq \emptyset$.

From Theorem 5.4 of [1], it follows that there exists a compact set $F \subset [\frac{3}{4}, 1]$ such that $0 < \mathcal{H}^s < +\infty$. So, $F \in \mathcal{N}_{s-\sigma} \setminus \mathcal{N}_s$. Let $\mathcal{A} = \bigcup_{n=1}^{\infty} \frac{1}{2^{n-1}} \cdot F$ and

$B = \mathbb{R} \setminus A$. We claim that $B \in \mathcal{T}_{s-\sigma} \setminus \mathcal{T}_s$. Indeed, each point of the set B, except 0, is an inner point of this set in the Euclidean topology. It remains to show that $0 \in \Phi_{s-\sigma}(B)$. Let $\{n_k\}_{k\in\mathbb{N}}$ be an arbitrary increasing sequence of natural numbers. Then

$$\limsup_k (n_k \cdot B') \cap [-1,1] = (\bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} (n_m \cdot A)) \cap [-1,1] \subset \bigcup_{k=1}^{\infty} (n_k \cdot A) \cap [-1,1].$$

Since $F \in \mathcal{N}_{s-\sigma}$, and this σ-ideal is closed under multiplication by numbers, $A \in \mathcal{N}_{s-\sigma}$ and, of course, $\bigcup_{k=1}^{\infty} (n_k \cdot A) \cap [-1,1]$ also belongs to $\mathcal{N}_{s-\sigma}$.

Now the task is to show $0 \notin \Phi_s(B)$. For this, it is enough to find an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that for each subsequence $\{n_{k_p}\}_{p\in\mathbb{N}}$ the set $\limsup_p (n_{k_p} \cdot B') \cap [-1,1]$ is not from the σ-ideal $\mathcal{N}_s$. Let $n_k = 2^k$ for $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, $(2^k \cdot A) \cap [-1,1] = A \cap [-1,1]$ and for any subsequence $\{n_{k_p}\}_{p\in\mathbb{N}} \subset \{n_k\}_{k\in\mathbb{N}}$ we have

$$\limsup_p (n_{k_p} \cdot B') \cap [-1,1] = \limsup_p (n_{k_p} \cdot A) \cap [-1,1] = A \cap [-1,1] \notin \mathcal{N}_s,$$

since $F \subset A \cap [-1,1]$.

The same proof remains valid for the next pairs of topologies, since all of them were defined in the same way, the ideals which were used are invariant under multiplication by numbers, and inclusions between them are proper (Theorem 1). $\square$

Some properties of the topologies $\mathcal{T}_s$, $\mathcal{T}_{s-\sigma}$ and $\mathcal{T}_{s-\dim}$ for any $s \in (0,1)$ are listed below:

1. The topologies $\mathcal{T}_s, \mathcal{T}_{s-\sigma}$, $\mathcal{T}_{s-\dim}$ are stronger then the Euclidean topology, so each of them is Hausdorff.
2. Each countable set belongs to $\mathcal{N}_s$, so it is a closed set in each topology $\mathcal{T}_s, \mathcal{T}_{s-\sigma}$ and $\mathcal{T}_{s-\dim}$. Therefore, these spaces are not separable and every compact subspace of $(\mathbb{R}, \mathcal{T}_s)$, $(\mathbb{R}, \mathcal{T}_{s-\sigma})$ or $(\mathbb{R}, \mathcal{T}_{s-\dim})$ is finite.
3. None of $\mathcal{T}_s$, $\mathcal{T}_{s-\sigma}$ or $\mathcal{T}_{s-\dim}$ forms a Lindelöf space. Indeed, for every $s \in (0,1)$, there exists an uncountable set C belonging to $\mathcal{N}_s$ (It is enough to take the Cantor set with Hausdorff dimension greater then s [5, Cor. 29.23].), so that $\{(\mathbb{R} \setminus C) \cup \{x\}\}_{x\in C}$ is a $\mathcal{T}_s$-open cover of $\mathbb{R}$ without any subcover of power less than the continuum. The same example allows us to prove the property for $\mathcal{T}_{s-\sigma}$ and $\mathcal{T}_{s-\dim}$.
4. None of the spaces $(\mathbb{R}, \mathcal{T}_s)$, $(\mathbb{R}, \mathcal{T}_{s-\sigma})$, $(\mathbb{R}, \mathcal{T}_{s-\dim})$ is first countable. Indeed, take $x \in \mathbb{R}$ and let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{T}_s$-open neighborhoods of x. For each $n \in \mathbb{N}$ choose $x_n \in E_n \setminus \{x\}$ and put $E = E_1 \setminus \{x_n : n \in \mathbb{N}\}$. Then E is a $\mathcal{T}_s$-open neighborhood of x which does not include any E_n. For other topologies we can proceed similarly.

5. The family of $\mathcal{T}_s, (\mathcal{T}_{s-\sigma}$ or $\mathcal{T}_{s-\dim})$-connected sets coincides with the family of sets connected in the natural topology.

Using σ-ideals we can consider also Hashimoto type topologies [2].

Definition 4. For any $s \in (0,1)$ let

$$\begin{aligned}\mathcal{T}_s^* &= \{G \setminus N : G \in \mathcal{O} \wedge N \in \mathcal{N}_s\},\\ \mathcal{T}_{s-\sigma}^* &= \{G \setminus N : G \in \mathcal{O} \wedge N \in \mathcal{N}_{s-\sigma}\},\\ \mathcal{T}_{s-\dim}^* &= \{G \setminus N : G \in \mathcal{O} \wedge N \in \mathcal{N}_{s-\dim}\}.\end{aligned}$$

Obviously, these families are topologies and, by Theorem 2, we have the following proper inclusions

$$\mathcal{T}_s^* \subsetneq \mathcal{T}_{s-\sigma}^* \subsetneq \mathcal{T}_{s-\dim}^* \subsetneq \mathcal{T}_t^*, \tag{2}$$

for any $s, t \in (0,1)$, $s < t$.

Remark 1. Of course each Hashimoto type topology is contained in the density type topology defined by using the same σ-ideal; for example $\mathcal{T}_s^* \subset \mathcal{T}_s$, since if $A \in \mathcal{T}_s^*$, then $A = G \setminus N$, where $G \in \mathcal{O}$ and $N \in \mathcal{N}_s$, so $A \subset G \subset \phi_s(G) = \phi_s(A)$ since $\mathcal{O} \subset \mathcal{T}_s$ and $A \sim_s G$. Therefore $A \in \mathcal{T}_s$. The situation is analogous for other σ-ideals.

Theorem 4, (2) and the last remark yield the following scheme

$$\begin{array}{ccccccc}\mathcal{T}_s^* & \subsetneq & \mathcal{T}_{s-\sigma}^* & \subsetneq & \mathcal{T}_{s-\dim}^* & \subsetneq & \mathcal{T}_t^*\\ \cap & & \cap & & \cap & & \cap\\ \mathcal{T}_s & \subsetneq & \mathcal{T}_{s-\sigma} & \subsetneq & \mathcal{T}_{s-\dim} & \subsetneq & \mathcal{T}_t\end{array}$$

for any $s, t \in (0,1)$, $s < t$.

To see that the inclusions $\mathcal{T}_s^* \subset \mathcal{T}_s$, $\mathcal{T}_{s-\sigma}^* \subset \mathcal{T}_{s-\sigma}$, $\mathcal{T}_{s-\dim}^* \subset \mathcal{T}_{s-\dim}$ are proper, it is enough to show that $\mathcal{T}_s$ is not contained in $\mathcal{T}_t^*$. Let B denote an interval set $B = \bigcup_{n=1}^{\infty}[a_n, b_n]$ such that $a_{n+1} < b_{n+1} < a_n$ for any $n \in \mathbb{N}$, the sequence $\{b_n\}$ tends to zero, $\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = 0$ and $\lim_{n\to\infty} \frac{a_n - b_{n+1}}{a_n} = 1$. Then $0 \in \Phi_s(\mathbb{R} \setminus B)$ and $A = \mathbb{R} \setminus B \in \mathcal{T}_s$, but for any interval (a,b) containing zero, there exists an interval $[a_n, b_n] \subset (a,b)$ of positive Lebesgue measure, so it is does not belong to $\mathcal{N}_t$. Therefore $A \notin \mathcal{T}_t^*$. Moreover, it shows also that none of topologies $\mathcal{T}_s, \mathcal{T}_{s-\sigma}, \mathcal{T}_{s-\dim}$ and $\mathcal{T}_t$ is contained in any topology of Hashimoto type.

Some pairs of considered topologies are incomparable: $\mathcal{T}_{s-\sigma}^* \setminus \mathcal{T}_s \neq \emptyset$, $\mathcal{T}_{s-\dim}^* \setminus \mathcal{T}_{s-\sigma} \neq \emptyset$ and $\mathcal{T}_t^* \setminus \mathcal{T}_{s-\dim} \neq \emptyset$. To show, for example, that the first family is not empty we take a compact set $F \subset [\frac{3}{4}, 1]$ with $F \in \mathcal{N}_{s-\sigma} \setminus \mathcal{N}_s$. Then $A = \bigcup_{n=1}^{\infty} \frac{1}{2^{n-1}} \cdot F \in \mathcal{N}_{s-\sigma} \setminus \mathcal{N}_s$. The complement of the set A belongs to $\mathcal{T}^*_{s-\sigma}$ and $0 \notin \Phi_s(\mathbb{R} \setminus A)$. (See the proof of Theorem 2.)

Summarizing, we have the following scheme:

$$
\begin{array}{ccccccc}
\mathcal{T}^*_s & \subsetneq & \mathcal{T}^*_{s-\sigma} & \subsetneq & \mathcal{T}^*_{s-\mathrm{dim}} & \subsetneq & \mathcal{T}^*_t \\
\rotatebox{90}{$\supsetneq$} & & \rotatebox{90}{$\supsetneq$} & & \rotatebox{90}{$\supsetneq$} & & \rotatebox{90}{$\supsetneq$} \\
\mathcal{T}_s & \subsetneq & \mathcal{T}_{s-\sigma} & \subsetneq & \mathcal{T}_{s-\mathrm{dim}} & \subsetneq & \mathcal{T}_t
\end{array}
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