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SOME VARIATIONS ON THE BANACH-ZARECKI THEOREM

Abstract

This note concerns some variations on the classical Theorem of Banach-Zarecki [2].

In this note, we prove a slightly stronger version of the classical Theorem of Banach-Zarecki (See [2, Theorem 4, Chapter IX, Section 4]).

Theorem (Banach and Zarecki)**.** *Let f be a continuous function on $[a, b]$ that maps sets of measure zero to sets of measure zero, and let f be a function of bounded variation on $[a, b]$. Then f is an absolutely continuous function on $[a, b]$.*

In [1], J. J. Koliha restated it in part, as follows.

Theorem (Koliha)**.** *Let f be a continuous function on $[a, b]$, let f be Lebesgue summable on $[a, b]$, and let $F'(t) = f(t)$ for all but at most countably many t in $[a, b]$. Then f is an absolutely continuous function on $[a, b]$, and*

$$\int_a^b f = F(b) - F(a)\,.$$

We will give a proof of the following theorem that subsumes these results. We will use only standard results about Lebesgue measure and integration. We do employ the result that a function of bounded variation has a finite derivative almost everywhere on (a, b), and this derivative is summable on $[a, b]$. (This can be derived from the Vitali Covering Theorem and Fatou's Lemma. Consult [2, Theorem 5, Chapter VIII, Section 2 and Theorem 6, Chapter VIII, Section 3].)

We will prove the following.

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Theorem 1. *Let f be a continuous function on $[a,b]$, and let P be a measurable subset of $[a,b]$ such that the set $f\big([a,b]\setminus P\big)$ has measure zero and f is differentiable at each point of P. Then a necessary and sufficient condition that f be absolutely continuous on $[a,b]$ is that there exists a positive summable function g on $[a,b]$ such that $g \geq f'$ on P.*

In [3, (6.9), Chapter IX], we have a version of this proof which, however, is much more complex. The essential difference between Theorem 1 and the Koliha and Banach-Zarecki Theorems is that f' is dominated from above, but not from below, in the hypotheses.

Before tackling the proof of Theorem 1, we make an easy application of an old technique that should be familiar. We provide a proof of the following lemma for the sake of completeness.

Lemma 1. *Let f and F be as in Theorem 1, and let*

$$P_+ = \big\{x \in P : f'(x) \geq 0\big\}\,.$$

Then

$$\max\big(0\,,\ f(b)-f(a)\big) \leq m\Big(f\big((a,b)\cap P_+\big)\Big)\,.$$

Proof. Without loss of generality, we assume that $f(b) > f(a)$. By hypothesis, $f\big((a,b)\setminus P\big)$ has measure 0. Hence, the set

$$\big(f(a)\,,\ f(b)\big)\setminus f\big((a,b)\setminus P\big)$$

has measure $f(b)-f(a)$.

So choose

$$y \in \big(f(a)\,,\ f(b)\big)\setminus f\big((a,b)\setminus P\big).$$

Let x_0 be the maximal element in the compact set $f^{-1}(y)\cap(a,b)$. Then $f'(x_0) < 0$ is necessarily impossible, so $f'(x_0) \geq 0$. Finally, $x_0 \in (a,b)\cap P_+$, so

$$y = f(x_0) \in f\big((a,b)\cap P_+\big)\,,$$

and it follows that

$$m\Big(f\big((a,b)\cap P_+\big)\Big) \geq f(b)-f(a)\,. \qquad \square$$

Proof of Theorem 1. Necessity is obvious so we prove sufficiency. Let there exist a summable function $g > 0$ such that $g \geq f'$ on P. Let

$$P_+ = \{x \in P : f'(x) \geq 0\}.$$

Choose indices i and j, and choose any $\epsilon > 0$. Put

$$E_i = \{x \in P_+ \ : \ i-1 \leq g < i\}$$

and

$$S_{ij} = \Big\{x \in E_i : m\big(f(I)\big) < im(I) \quad \text{for any interval } I \text{ containing } x \text{ for which } m(I) < \frac{1}{j}\Big\}.$$

Then

$$S_{i1} \subset S_{i2} \subset S_{i3} \subset \cdots \subset \quad \text{and} \quad \cup_j S_{ij} = E_i. \tag{1}$$

Let U be an open set containing S_{ij} such that $m(U \setminus S_{ij}) < \epsilon$. We assume, by deleting finitely many points from S_{ij} if necessary, that $m\big(I_k\big) < \frac{1}{j}$ for each component I_k of U. Delete any component I_k of U disjoint from S_{ij}. Then each component I_k of U contains a point of S_{ij}, and

$$\begin{aligned} m\big(f(S_{ij})\big) &\leq m\big(f(U)\big) \leq \sum_k m\big(f(I_k)\big) \\ &\leq \sum_k im(I_k) = im(U) \leq im(S_{ij}) + i\epsilon\,. \end{aligned}$$

But ϵ is independent of i, so $m\big(f(S_{ij})\big) \leq im(S_{ij})$. We let $j \to \infty$ and deduce from (1) that

$$m\big(f(E_i)\big) \leq im(E_i)\,. \tag{2}$$

From the definition of E_i, we deduce that $im(E_i) \leq \int_{E_i}(g+1)$, and from (2), that

$$m\big(f(E_i)\big) \leq \int_{E_i} (g+1)\,. \tag{3}$$

We sum on i and use the definition of E_i to obtain

$$m\big(f(P_+)\big) \leq \sum_i m\big(f(E_i)\big) \leq \sum_i \int_{E_i} (g+1) = \int_{P_+} (g+1)\,. \tag{4}$$

By Lemma 1, we have

$$m\big(f(P_+)\big) \geq \max\big(0,\ f(b) - f(a)\big)\,.$$

From (4), we obtain

$$\int_{P_+} (g+1) \geq \max\big(0,\ f(b) - f(a)\big)\,.$$

Likewise for any subinterval (u, v) of $[a, b]$, we have

$$\max\big(0,\ f(v) - f(u)\big) \leq \int_u^v (g+1) \tag{5}$$

because $g + 1 > 0$. If $\big\{(u_j, v_j)\big\}_j$ are mutually disjoint subintervals of $[a, b]$, it follows from this that

$$\sum_j \big(f(v_j) - f(u_j)\big) \leq \sum_j \int_{u_j}^{v_j} (g+1) \leq \int_a^b (g+1)\,. \tag{6}$$

From (6), we deduce that f has bounded upper variation on $[a, b]$. But f is continuous and bounded on $[a, b]$, so in fact f is of bounded variation on $[a, b]$. As mentioned in our introductory remarks, f is differentiable almost everywhere on $[a, b]$, and f' is summable on $[a, b]$. Moreover, $|f'| \geq -f'$ on P, and we repeat our arguments for $-f$ to obtain

$$\sum_j -\big(f(v_j) - f(u_j)\big) \leq \sum_j \int_{u_j}^{v_j} \big(|f'| + 1\big)\,. \tag{7}$$

From the absolute continuity of the integrals of $g + 1$ and $|f'| + 1$, it follows (using (6) and (7)) that f is absolutely continuous on $[a, b]$. □

Our first Corollary is inspired by the Banach-Zarecki Theorem.

Corollary 1. *Let f be a continuous function differentiable almost everywhere on $[a, b]$. Let f map sets of measure zero to sets of measure zero. Then a necessary and sufficient condition that f be absolutely continuous on $[a, b]$ is that there exists a positive summable function g on $[a, b]$ such that $g \geq f'$ almost everywhere on $[a, b]$.*

We leave the proof that is immediate from Theorem 1.

We can use Theorem 1 to prove another corollary inspired by the Koliha Theorem.

Corollary 2. *Let f be a continuous function differentiable everywhere except possibly at countably many points on $[a, b]$. Let g be a positive summable function on $[a, b]$ such that $g \geq f'$ almost everywhere on $[a, b]$. Then f is an absolutely continuous function on $[a, b]$.*

Let f and P be as in the proof of Theorem 1. Let E be a measurable subset of P at each point of which $f' = 0$. For any fixed positive number i, we see from (2) that $m\big(f(E)\big) \leq im(E)$. But i is arbitrary, so $m\big(f(E)\big) = 0$. We state the following.

Lemma 2. *Let f and P be as in Theorem 1, and let E be a measurable subset of P such that $f'(x) = 0$ for any $x \in E$. Then $m\big(f(E)\big) = 0$.*

We conclude with an elementary proof of the following.

Corollary 3. *Let f be continuous on $[a, b]$ and $f' \leq 0$ almost everywhere. Let f map sets of measure zero to sets of measure zero. Then f is nonincreasing on $[a, b]$.*

PROOF. Assume to the contrary, that there exists an interval (u, v) with $f(u) < f(v)$. It follows from Lemma 1 that

$$m\bigg(f\Big(\big\{x \in (u, v) \ : \ f'(x) = 0\big\}\Big)\bigg) \geq f(v) - f(u) > 0\,.$$

This conflicts with Lemma 2. □

References

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