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ON THE POINTS OF ONE-SIDED CLIQUISHNESS

Abstract

In this paper, an attempt has been made to characterize one-sided points of cliquishness of a function in terms of generalized one-sided oscillations and study the nature of one-sided points of cliquishness of the quasi-uniform limit function of a sequence of functions.

1 Introduction.

In what follows, X is a topological space, and Y is a metric space with metric d. Through the paper $\mathbb{R}$ and $\mathbb{Q}$ are the spaces of real numbers and rational numbers, respectively, with the usual metric. Furthermore, $\mathbb{N}$ stands for the set of natural numbers, and ϕ denotes the empty set.

The notions of quasicontinuity and cliquishness of a function was introduced in [7] and [8], respectively. Recall that a function $f : X \to Y$ is said to be quasicontinuous at a point $x \in X$ if for each open neighbourhood U of x and each open neighbourhood V of $f(x)$, there is a non-empty open set $G \subseteq U$ such that $f(G) \subseteq V$ ([7]).

Again, a function $f : X \to Y$ is said to be cliquish at a point $x \in X$ ([8]) if for each $\epsilon > 0$ and each open neighbourhood U of x, there is a non-empty open set $G \subseteq U$ such that $d(f(x'), f(x'')) < \epsilon$ for all $x', x'' \in G$. A function f is said to be quasicontinuous (cliquish) if it has this property at each point.

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With $C(f)$, $Q(f)$, and $A(f)$, we define the sets of all points at which f is continuous, quasicontinuous, and cliquish, respectively. It is known that $C(f)$ is a G_δ set ([5]) and $A(f)$ is a closed set ([6]).

From the definitions, we have $C(f) \subseteq Q(f) \subseteq A(f)$.

One-sided quasicontinuity has been studied by J. Borsik in [1]. J. Ewert introduced some techniques including uniform convergency for studying one-sided quasicontinuity ([3], [4]).

A function $f : \mathbb{R} \to Y$ is said to be right-sidedly (left-sidedly) quasicontinuous at $x \in \mathbb{R}$ if for each $\delta > 0$ and each open neighbourhood V of $f(x)$, there is a non-empty open set $U \subseteq (x, x+\delta)$ (resp. $U \subseteq (x-\delta, x)$) such that $f(U) \subseteq V$. The function f is said to be right-sidedly (resp. left-sidedly) quasicontinuous if it is so at each point.

By $Q^+(f)$ and $Q^-(f)$, we denote the sets of all points at which f is right-sidedly, left-sidedly quasicontinuous, respectively. Then we have

$$Q^+(f) \cup Q^-(f) = Q(f) \text{ ([3])}.$$

Here we introduce the concept of one-sided cliquishness as follows:
A function $f : \mathbb{R} \to Y$ is said to be right-sidedly (left-sidedly) cliquish if for each $\delta > 0$ and $\epsilon > 0$, there is a non-empty open set $U \subseteq (x, x+\delta)$ (resp. $U \subseteq (x-\delta, x)$) such that $d(f(x'), f(x'')) < \epsilon$ for all $x', x'' \in U$. A function $f : \mathbb{R} \to Y$ is called right-sidedly (left-sidedly) cliquish if it has this property at each point of $\mathbb{R}$.

By $A^+(f)$ and $A^-(f)$, we denote the sets of all points at which f is right-sidedly, left-sidedly cliquish, respectively. Then, from definitions,

$$Q^+(f) \subseteq A^+(f);\ Q^-(f) \subseteq A^-(f);\ A^+(f) \cup A^-(f) = A(f).$$

The following characterization of one-sided cliquishness readily follows from the definition.

Proposition 1.1. *A function $f : \mathbb{R} \to Y$ is right-sidedly (left-sidedly) cliquish at a point $x \in \mathbb{R}$ if and only if for each $\delta > 0$ and $\epsilon > 0$, there exists at least one $y \in Y$ and a non-empty open set $U \subseteq (x, x+\delta)$ (resp. $U \subseteq (x-\delta, x)$) such that $d(f(x'), y) < \epsilon$ for all $x' \in G$.*

Theorem 1.2. *For a function $f : \mathbb{R} \to Y$, $A^+(f)$ (resp. $A^-(f)$) contains all right-sided (resp. left-sided) cluster points of $A(f)$.*

PROOF. Let x be a right-sided cluster point of $A(f)$, $\delta > 0$ and $\epsilon > 0$. Then $(x, x+\delta) \cap A(f) \neq \phi$, and suppose $x_1 \in (x, x+\delta)$ and $x_1 \in A^+(f)$. Then there is a non-empty open set $U \subseteq (x_1, x+\delta) \subseteq (x, x+\delta)$ such that $d(f(x'), f(x'')) < \epsilon$

for all $x', x'' \in U$. So $x \in A^+(f)$. Similarly, assuming $x_1 \in (x, x+\delta)$ and $x_1 \in A^-(f)$, we can show that $x \in A^+(f)$.

The other part can be proved similarly. $\square$

2 Points of One-Sided Cliquishness.

Theorem 2.1. *For a function $f : \mathbb{R} \to Y$, the sets $A^+(f) \setminus A^-(f)$ and $A^-(f) \setminus A^+(f)$ are countable.*

PROOF. Suppose that $A^+(f) \setminus A^-(f)$ is not countable. It is known that if $A \subseteq \mathbb{R}$ is not countable, then there is a point $a \in A$ such that a is a bilateral accumulation point of A. Then there is a point $a \in A^+(f) \setminus A^-(f)$ such that a is a bilateral accumulation point of $A^+(f) \setminus A^-(f)$. So by the Proposition 1.1, f is both right-sidedly and left-sidedly cliquish at a which is not true. So $A^+(f) \setminus A^-(f)$ is countable.

Similarly, we can show that $A^-(f) \setminus A^+(f)$ is countable. $\square$

Remark 2.2. For a function $f : \mathbb{R} \to Y$, the set $A(f) \setminus (A^+(f) \cap A^-(f))$ is countable.

It follows from the fact that

$$A(f) \setminus (A^+(f) \cap A^-(f)) = (A^+(f) \setminus A^-(f)) \cup (A^-(f) \setminus A^+(f)),,$$

and both the sets $A^+(f) \setminus A^-(f)$, $A^-(f) \setminus A^+(f)$ are countable.

Corollary 2.3. *A function $f : \mathbb{R} \to Y$ is cliquish if and only if $A^+(f) \cap A^-(f)$ is dense in $\mathbb{R}$.*

PROOF. Suppose $f : \mathbb{R} \to Y$ is cliquish. Then

$$\mathbb{R} \setminus (A^+(f) \cap A^-(f)) = A(f) \setminus (A^+(f) \cap A^-(f))$$

is a countable set (by the Remark 2.2), and hence $A^+(f) \cap A^-(f)$ is dense in $\mathbb{R}$.

Conversely, if $A^+(f) \cap A^-(f)$ is dense in $\mathbb{R}$, then from the the fact that $A(f)$ is closed and $A^+(f) \cap A^-(f) \subseteq A(f)$, it follows that $\mathbb{R} = A(f)$, and the proof is completed. $\square$

3 Generalized One-Sided Oscillations.

A few types of bilateral oscillations were introduced by J. Borsik in [2]. For a function $f : \mathbb{R} \to Y$, the generalized right-sidedly, left-sidedly oscillations of f at a point $x \in \mathbb{R}$ are denoted by $w^+(f, x)$ and $w^-(f, x)$, respectively, and are defined as follows:

$$w^+(f, x) = \sup_{\delta>0} \inf_{U\subseteq(x,x+\delta)} \sup_{x',x''\in U} d(f(x'), f(x'')),$$

where the infimum is taken under all non-empty open sets $U \subseteq (x, x + \delta)$, and

$$w^-(f, x) = \sup_{\delta>0} \inf_{U\subseteq(x-\delta,x)} \sup_{x',x''\in U} d(f(x'), f(x'')),$$

where the infimum is taken under all non-empty open sets $U \subseteq (x - \delta, x)$.

The sets $Q(f)$, $A(f)$ have been characterized in [6] when $Y = \mathbb{R}$. We shall now characterize the sets $A^+(f)$, $A^-(f)$ in terms of generalized, one-sided oscillations.

Theorem 3.1. *For a function* $f : \mathbb{R} \to Y$*,*
i) $A^+(f) = \{x \in \mathbb{R} : w^+(f, x) = 0\}$
ii) $A^-(f) = \{x \in \mathbb{R} : w^-(f, x) = 0\}$

Proof of *i)*. Let $x \in A^+(f)$, $\delta > 0$ and $\epsilon > 0$. Then there is a non-empty open set $U \subseteq (x, x + \delta)$ such that $d(f(x'), f(x'') < \epsilon$ for all x', $x'' \in U$. So

$$\inf_{U\subseteq(x,x+\delta)} \sup_{x',x''\in U} d(f(x'), f(x'')) = 0,$$

and consequently, $w^+(f, x) = 0$.
Conversely, suppose that $w^+(f, x) = 0$. Then for each $\delta > 0$,

$$\inf_{U\subseteq(x,x+\delta)} \sup_{x',x''\in U} d(f(x'), f(x'')) = 0.$$

So for each $\delta > 0$, $\epsilon > 0$, there is a non-empty open set $U \subseteq (x, x + \delta)$ such that $d(f(x'), f(x'')) < \epsilon$, for all x', $x'' \in U$. Hence, $x \in A^+(f)$, and the proof is completed.

The proof of ii) is analogous. □

4 Limits.

A net $\{f_j : j \in J\}$ of functions $f_j : \mathbb{R} \to Y$ is said to be quasi-uniformly convergent to a function $f : \mathbb{R} \to Y$ ([9]) if for each $x \in \mathbb{R}$, $\epsilon > 0$, there is $j_0 \in J$ such that for each $j \in J$, $j \geq j_0$, there is a neighbourhood U of x with $d(f(x'), f_j(x')) < \epsilon$ for each $x' \in U$.
For a net $\{M_j : j \in J\}$ of sets, we will use the following notations:

$$\limsup M_j = \bigcap_{j \in J} \bigcup_{i \in J, i \geq j} M_i, \ \liminf M_j = \bigcup_{j \in J} \bigcap_{i \in J, i \geq j} M_i,$$

and we will write $\lim M_j$, if $\limsup M_j = \liminf M_j$.

Now we obtain the following results.

Theorem 4.1. *Let $\{f_n\}_n$ be a sequence of functions $f_n : \mathbb{R} \to Y$ which quasi-uniformly converges to a function $f : \mathbb{R} \to Y$. Then*
i) $\limsup A^+(f_n) \subseteq A^+(f) \subseteq \liminf A^+(f_n)$
ii) $\limsup A^-(f_n) \subseteq A^-(f) \subseteq \liminf A^-(f_n)$

Proof of *i)*. Let $x_0 \in \limsup A^+(f_n)$, $\delta > 0$, and $\epsilon > 0$. From assumptions, we can choose $n \in \mathbb{N}$ and an open neighbourhood U of x_0 such that

$$x_0 \in A^+(f_n),\ d(f(x), f_n(x)) < \epsilon/2,$$

for all $x \in U$. Let us take δ_1, $0 < \delta_1 < \delta$, with $(x_0, x_0 + \delta_1) \subseteq U$. Since $x_0 \in A^+(f_n)$, by the Proposition 1.1, there is a point $y_n(x_0) \in Y$ and a non-empty open set $G \subseteq (x_0, x_0 + \delta_1)$ with

$$d(f_n(x), y_n(x_0)) < \epsilon/2,$$

for all $x \in G$. Now for $x \in G$,

$$d(f(x), y_n(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), y_n(x_0)) < \epsilon.$$

So by the Proposition 1.1, $x_0 \in A^+(f)$, and consequently $\limsup A^+(f_n) \subseteq A^+(f)$.
Again let $x_0 \in A^+(f)$, $\delta > 0$, and $\epsilon > 0$. Since $\{f_n\}_n$ converges quasi-uniformly to f, there is $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq n_0$, there is a neighbourhood U of x_0 with $d(f(x), f_n(x)) < \epsilon/2$, for all $x \in U$. For each $n \in \mathbb{N}$, $n \geq n_0$, let us take δ_n, $0 < \delta_n < \delta$ with $(x_0, x_0 + \delta_n) \subseteq U$. Since $x_0 \in A^+(f)$, by Proposition 1.1, for each $n \in \mathbb{N}$, $n \geq n_0$, there is a point $y_n(x_0) \in Y$ and a non-empty open set $G_n \subseteq (x_0, x_0 + \delta_n)$ with

$$d(f(x), y_n(x_0)) < \epsilon/2,$$

for all $x \in G_n$. Now for $x \in G_n$, $(n \geq n_0)$,

$$d(f_n(x), y_n(x_0)) \leq d(f_n(x), f(x)) + d(f(x), y_n(x_0)) < \epsilon$$

which gives $x_0 \in A^+(f_n)$, $n \geq n_0$. So $x_0 \in \bigcap_{n \geq n_0} A^+(f_n)$, and consequently $x_0 \in \liminf A^+(f_n)$. Hence, $A^+(f) \subseteq \liminf A^+(f_n)$, and the result follows.

The proof of ii) can be similarly done. $\square$

Corollary 4.2. *Let $\{f_n\}_n$ be a sequence of functions $f_n : \mathbb{R} \to Y$ which quasi-uniformly converges to a function $f : \mathbb{R} \to Y$. Then*
i) $\lim A^+(f_n) = A^+(f)$
ii) $\lim A^-(f_n) = A^-(f)$

In the sequel, we will consider $\mathbb{R} \cup \{\infty\}$ with the generalized metric d given by $d(x, y) =\mid x - y \mid$. However, we assume that

$$-\infty + \infty = \infty - \infty = 0 \text{ and } \mid \pm\infty \mid = \infty \quad ([3]).$$

The next theorem shows that one-sided points of quasi-uniform limit of a sequence of functions $\{f_n\}_n$ is translated to that of the sequences of one-sided oscillation type functions associated with the sequence of functions $\{f_n\}_n$.

Theorem 4.3. *Let $\{f_n\}_n$ be sequence of functions $f_n : \mathbb{R} \to Y$ which quasi-uniformly converges to a function $f : \mathbb{R} \to Y$. Then*
i) the sequence $\{w^+(f_n, .)\}_n$ of functions $w^+(f_n, .) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by $x \mapsto w^+(f_n, x)$ is quasi-uniformly convergent to a function $w^+(f, .) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by $x \mapsto w^+(f, x)$.
ii) the sequence $\{w^-(f_n, .)\}_n$ of functions $w^-(f_n, .) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by $x \mapsto w^-(f_n, x)$ is quasi-uniformly convergent to a function $w^-(f, .) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by $x \mapsto w^-(f, x)$.

Proof of *i)*. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq n_0$, there is a neighbourhood U_n of x_0 with $d(f_n(x), f(x)) < \epsilon/3$ for all $x \in U_n$. We fix $n \geq n_0$, $x \in U_n$, and $\delta > 0$ satisfying $(x, x + \delta) \subseteq U_n$. Assume $w^+(f, x) < \infty$. Then

$$\inf_{U \subseteq (x, x+\delta)} \sup_{x', x'' \in U} d(f(x'), f(x'')) < w^+(f, x) + \epsilon/3.$$

So there is a non-empty open set $U \subseteq (x, x + \delta)$ such that

$$d(f(x'), f(x'')) < w^+(f, x) + \epsilon/3,$$

for all x', $x'' \in U$. Now for x', $x'' \in U$,

$$\begin{aligned} d(f_n(x'), f_n(x'')) &\leq d(f_n(x'), f(x')) + d(f(x'), f(x'')) + d(f(x''), f_n(x'')) \\ &< \epsilon + w^+(f, x). \end{aligned}$$

So $w^+(f_n, x) \leq w^+(f, x) + \epsilon$. Similarly, we find $w^+(f, x) \leq w^+(f_n, x) + \epsilon$. Thus,

$$\mid w^+(f_n, x) - w^+(f, x) \mid < \epsilon.$$

If $w^+(f, x) = \infty$ for some $x \in U_n$, then $w^+(f, x) > k + \epsilon$ for each $k \in \mathbb{N}$. Now $\delta_1 > 0$ can be chosen such that $(x, x+\delta_1) \subseteq U_n$ and $\sup\limits_{x', x'' \in U} d(f(x'), f(x'')) > k$, for each $k \in \mathbb{N}$ and each non-empty open set $U \subseteq (x, x + \delta_1)$. Hence,

$$\sup_{x', x'' \in U} d(f_n(x'), f_n(x'')) > k,$$

for each $k \in \mathbb{N}$ and U as above. So $w^+(f_n, x) = \infty$, and hence

$$\mid w^+(f_n, x) - w^+(f, x) \mid < \epsilon,$$

for each $x \in U_n$. Hence, the proof is completed.

The proof of ii) is analogous. □

Remark 4.4. In theorem 4.1 and theorem 4.3, the quasi-uniform convergence cannot be replaced by the pointwise convergence which follows from the following example.

Example 4.5. Let $\{r_1, r_2, \dots\}$ be an enumeration of the set of rationals in $[0, 1]$ and $f_n : [0, 1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \dots\dots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $\{f_n\}_n$ converges pointwise to the function $f : [0, 1] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Hence, we have $A^+(f_n) = [0,1] = A^-(f_n)$ for each $n \in \mathbb{N}$, and $A^+(f) = \phi = A^-(f)$. So $\limsup A^+(f_n) = [0,1] = \liminf A^+(f_n)$ and $\limsup A^-(f_n) = [0,1] = \liminf A^-(f_n)$. Hence, $\lim A^+(f_n) \neq A^+(f)$ and $\lim A^-(f_n) \neq A^-(f)$. Furthermore,

$$w^+(f_n, x) = \sup_{\delta>0} \inf_{U\subseteq(x,x+\delta)} \sup_{x',x''\in U} |f_n(x') - f_n(x'')| = 0$$

and

$$w^-(f_n, x) = \sup_{\delta>0} \inf_{U\subseteq(x-\delta,x)} \sup_{x',x''\in U} |f_n(x') - f_n(x'')| = 0$$

for each $n \in \mathbb{N}$ and for each $x \in [0,1]$. Also,

$$w^+(f, x) = \sup_{\delta>0} \inf_{U\subseteq(x,x+\delta)} \sup_{x',x''\in U} |f(x') - f(x'')| = 1,$$

and

$$w^-(f, x) = \sup_{\delta>0} \inf_{U\subseteq(x-\delta,x)} \sup_{x',x''\in U} |f(x') - f(x'')| = 1$$

for each $x \in [0,1]$.
So $w^+(f,.)$ and $w^-(f,.)$ are not pointwise limits of the sequences $\{w^+(f_n,.)\}_n$ and $\{w^-(f_n,.)\}_n$, respectively.

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