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WEIGHTED ORLICZ-TYPE INTEGRAL INEQUALITIES FOR THE HARDY OPERATOR

Dedicated to the memory of Casper Goffman

Abstract

We study integral inequalities for the Hardy operator Hf of the form $\int_0^\infty \Phi[Hf^p]\,d\mu \leq c_0 \int_0^\infty \Phi[c_1 f^p]\,d\mu$, where Φ is convex, μ is a measure on $\mathbb{R}_+$, $1 \leq p < \infty$, and f is non-increasing. The results we obtain are extensions of the classical B_p- weight theory [1, 5].

1 Introduction.

Let for $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

be the Hardy operator. In this paper we will examine Orlicz - type inequalities

$$\int_0^\infty \Phi[Hf(x)^p]\,d\mu \leq c_0 \int_0^\infty \Phi[c_1 f(x)^p]\,d\mu, \tag{1}$$

where $1 \leq p < \infty$ and where μ is a Borel measure on $\mathbb{R}_+$ finite on compact sets. We also restrict ourselves to the important special case of $f \in \mathcal{D}$, where $\mathcal{D}$ is the collection of all $f : \mathbb{R}_+ \to \mathbb{R}_+$ non-increasing.

If in (1), $\Phi(u) = u$, then the study of (1) reduces to the classical B_p-theory: (1) holds with $d\mu = w(x)dx$ if and only if $w \in B_p$, that is

$$\int_r^\infty (r/x)^p w(x)\,dx \leq c\int_0^r w(x)\,dx, \tag{2}$$

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where c is independent of $0 < r < \infty$ [1, 5]. The reason why (1) is important for $f \in \mathcal{D}$ is that many operators Tf satisfy $(Tf)^*(t) \le cH(f^*)(t)$, where $g^*(t) = \inf\{y : |\{x : |g(x)| > y\}| \le t\}$, the non-increasing rearrangement of g on $\mathbb{R}_+$ [2]. Further, the B_p-condition (2) also arises in the study of the question when Lorentz spaces are Banach spaces [3].

A natural conjecture for (1) to hold for $f \in \mathcal{D}$ is

$$\int_r^\infty \Phi[(r/x)^p]\, d\mu \le c \int_0^r d\mu.$$

The restriction on $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is that Φ is convex and $\Phi(0) = 0$. However, we shall see that an additional hypothesis on Φ' is needed. This led us to the notion of an index k of Φ. The results that we obtain are then generalizations of the classical B_p-case and reduce to it when $k = 1$. This will be taken up in the first 6 sections, and additional background and examples will be discussed in section 7.

We will use standard notation. An exception is $\chi_r(x) = \chi_{[0,r]}(x), \chi^r(x) = \chi_{[r,\infty)}(x)$. The letter c stands for a constant that may change from line to line but is always independent of $f \in \mathcal{D}$.

2 Weighted Inequalities.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex, $\Phi(0) = 0$, and there exist constants $0 < \gamma < \infty, 0 < a \le 1$ such that $\Phi'(u) \ge \gamma u^{k-1}, \quad 0 \le u \le a$ for some $k \ge 1$. We call k an index of Φ. By rescaling - $\Phi(u) \to \Phi(au)/\Phi(a)$ -we may assume that $\Phi(1) = 1$ and $a = 1$. We list now some properties we shall use frequently.

(i) $\Phi(u) \le \Phi(1)u, 0 \le u \le 1$ from which $\Phi'(1) \ge \Phi(1)$.

(ii) For $c \ge 1$ we have $\Phi(cu) \ge c\Phi(u)$. This follows from

$$\Phi(cu) = \int_0^{cu} \Phi'(t)\, dt = c\int_0^u \Phi'(c\tau)\, d\tau \ge c\int_0^u \Phi'(\tau)\, d\tau = c\Phi(u),$$

since Φ' is non-decreasing.

(iii) $\Phi(cu) \le c\Phi(u)$ if $0 \le c \le 1$.

(iv) If $0 \le a \le b < \infty$, then $\Phi(b) - \Phi(a) \ge \Phi(b - a)$. This can be seen by writing

$$\Phi(b) - \Phi(a) = \int_a^b \Phi'(t)\, dt = \int_0^{b-a} \Phi'(\tau + a)\, d\tau \ge \int_0^{b-a} \Phi'(\tau)\, d\tau = \Phi(b-a).$$

We consider the following classes of Borel measures μ on $\mathbb{R}_+$ finite on compact sets:

$$B_{\Phi,p} = \left\{\mu : \int_r^\infty \Phi[(r/x)^p]\, d\mu \le c\int_0^r d\mu\right\},$$

where the constant c is independent of $0 < r < \infty$. The other class is

$$T_{\Phi,p} = \left\{ \mu : \int_0^\infty \Phi[Hf^p]\, d\mu \le c_0 \int_0^\infty \Phi[c_1 f^p]\, d\mu, f \in \mathcal{D} \right\},$$

where the constants c_0, c_1 are independent of f.

Theorem 1. *If Φ is as above with index $k \ge 1$, then*

$$B_{\Phi,p} \subset T_{\Phi,kp} \subset B_{\Phi,kp}.$$

Remark. *If $k = 1$ we have an equivalence of the above classes, and this happens if $\Phi(u) = u$ the classical B_p-case, or $\Phi(u) = ue^u, \Phi(u) = e^u - 1$.*

Proof. The last implication follows by taking $f = \chi_r(x)$. Then $\Phi[c_1 f^{kp}(x)] = \Phi(c_1)\chi_r(x)$ and since $Hf(x)^{kp} = \chi_r(x) + (r/x)^{kp}\chi^r(x)$, $\mu \in B_{\Phi,kp}$.

As noted above, we may assume that $\Phi(1) = 1$ and $a = 1$. Let now $r = r(y)$ be in $\mathcal{D}$ and let $\rho(y) = r[\Phi^{-1}(y)]$. Then $\rho(y) \in \mathcal{D}$. Since $\mu \in B_{\Phi,p}$ we get

$$L \equiv \int_0^\infty \int_{\rho(y)}^\infty \Phi[(\rho(y)/x)^p]\, d\mu\, dy \le c \int_0^\infty \int_0^{\rho(y)} d\mu\, dy \equiv R.$$

We interchange the order of integration and see that

$$R = c \int_0^\infty \int_0^{\rho^{-1}(x)} dy\, d\mu(x) = c \int_0^\infty \rho^{-1}(x)\, d\mu(x).$$

The left integral L becomes

$$L = \int_0^\infty \int_{\rho^{-1}(x)}^\infty \Phi[(\rho(y)/x)^p]\, dy\, d\mu(x).$$

By either using integration by parts or comparing areas under the curve $t = \Phi[(\rho(y)/x)^p]$, the inner integral equals

$$\begin{aligned} I(x) &= \int_0^1 \rho^{-1}[x\Phi^{-1}(t)^{1/p}]\, dt - \Phi(1)\rho^{-1}(x) \\ &= \int_0^1 \rho^{-1}(xu)\Phi'(u^p)pu^{p-1}du - \Phi(1)\rho^{-1}(x) \\ &= \int_0^1 \Phi[r^{-1}(xu)]\Phi'(u^p)pu^{p-1}du - \rho^{-1}(x), \end{aligned}$$

since $\Phi(1) = 1$. Since the measure $d\nu = \Phi'(u^p)pu^{p-1}du$ has the property $\nu([0,1]) = \Phi(1) = 1$, we can use Jensen's inequality and get

$$I(x) \geq \Phi\left\{\int_0^1 r^{-1}(xu)\Phi'(u^p)pu^{p-1}du\right\} - \rho^{-1}(x).$$

From the assumption that $\Phi'(u) \geq \gamma u^{k-1}, 0 \leq u \leq 1$, we see that

$$I(x) \geq \Phi\left\{\int_0^1 r^{-1}(xu)\gamma u^{pk-1}du\right\} - \rho^{-1}(x).$$

We choose now $r^{-1}(t) = Hf(t)^{pk-1}f(t)$. Then

$$p\gamma r^{-1}(xu)u^{pk-1} = \frac{p\gamma}{pk}\frac{1}{x^{pk}}\frac{d}{du}\left(\int_0^{xu} f(t)\,dt\right)^{pk}.$$

Therefore

$$I(x) \geq \Phi[\gamma_0 Hf(x)^{pk}] - \rho^{-1}(x),$$

where $\gamma_0 = \gamma/k$.

Since $\rho^{-1}(x) = \Phi r^{-1}(x) = \Phi[Hf(x)^{pk-1}f(x)]$, we get from the $B_{\Phi,p}$-condition

$$\int_0^\infty \Phi[\gamma_0 Hf(x)^{pk}]\,d\mu \leq c_* \int_0^\infty \Phi[Hf(x)^{pk-1}f(x)]\,d\mu.$$

We may assume that $p_* \equiv pk > 1$. Young's inequality gives us

$$Hf^{p_*-1}f = \frac{Hf^{p_*-1}}{N}Nf \leq \frac{Hf^{p_*}}{p'_* N^{p'_*}} + \frac{N^{p_*}f^{p_*}}{p_*} = \frac{\gamma_0 Hf^{p_*}}{\gamma_0 p'_* N^{p'_*}} + \frac{N^{p_*}f^{p_*}\alpha}{p_*\alpha},$$

where α is chosen so that

$$\frac{1}{\gamma_0 p'_* N^{p'_*}} + \frac{1}{p_*\alpha} = 1.$$

Since Φ is convex we get

$$\int_0^\infty \Phi[\gamma_0 Hf(x)^{p_*}]\,d\mu \leq$$

$$c_*\left\{\frac{1}{\gamma_0 p'_* N^{p'_*}}\int_0^\infty \Phi[\gamma_0 Hf(x)^{p_*}]\,d\mu + \frac{1}{p_*\alpha}\int_0^\infty \Phi[\alpha N^{p_*}f(x)^{p_*}]\,d\mu\right\}.$$

We choose now N so large that $c_*/(\gamma_0 p'_* N^{p'_*}) < 1$. Then

$$\int_0^\infty \Phi[\gamma_0 Hf(x)^{p_*}]\,d\mu \leq c_0 \int_0^\infty \Phi[cf(x)^{p_*}]\,d\mu.$$

Finally the substitution $\gamma_0^{1/p_*}f \to f$ shows that $\mu \in T_{\Phi,kp}$. □

Corollary. *Assume* $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is convex with index* $k \geq 1$. *If* $\Phi_k(u) = \Phi(u^{1/k})$ *is also convex, then*

$$B_{\Phi_k,p} = T_{\Phi_k,p}.$$

PROOF. This follows from Theorem 1 since the index of Φ_k is 1. □

There is a converse to the first inclusion of Theorem 1 which we state next.

Theorem 2. *Assume* $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is convex,* $\Phi(0) = 0$*, and* Φ *does not have a finite index. Let* $k > 1$. *Then there exists* $\mu \in B_{\Phi,1}$ *such that* $\mu \notin T_{\Phi,k}$.

PROOF. We assume that for $u \geq 1$ the function Φ is linear, i.e.,$\Phi(u) = \Phi'(1)(u-1) + \Phi(1)$. There exists $\alpha_n \to \infty$ such that $\Phi'(1/2^n) \leq (1/2^n)^{\alpha_n - 1}$. Let $s > k-1$.

We claim that $x^s dx \in B_{\Phi,1}$.

$$\begin{aligned}\int_r^\infty \Phi(r/x) x^s\,dx &= r^{s+1}\int_0^1 \Phi(t)\frac{dt}{t^{s+2}} = r^{s+1}\sum_{n\geq 0}\int_{1/2^{n+1}}^{1/2^n}\Phi(t)\frac{dt}{t^{s+2}}\\ &\leq cr^{s+1}\sum \frac{2^{(n+1)(s+1)}}{2^{n\alpha_n}} = c\int_0^r x^s\,dx,\end{aligned}$$

where the $\leq$ follows since Φ' is non-decreasing. We show now that $x^s dx \notin T_{\Phi,k}$. Let $f_m = m\chi_{1/m}, m \geq 1$. Then

$$Hf_m(x) = m\chi_{1/m}(x) + (1/x)\chi^{1/m}(x).$$

Hence

$$\begin{aligned}\int_0^\infty \Phi[Hf_m(x)^k]x^s\,dx &\geq \int_{1/m}^\infty \Phi(1/x^k)x^s\,dx = \frac{1}{k}\int_0^{m^k}\Phi(t)\frac{dt}{t^{1+1/k+s/k}}\\ &\geq \frac{1}{k}\int_0^1 \Phi(t)\frac{dt}{t^{1+1/k+s/k}} > 0.\end{aligned}$$

However, since Φ is linear for $t \geq 1$,

$$\int_0^{1/m}\Phi(c_1 m^k)x^s\,dx = c\Phi(c_1m^k)/m^{s+1} \to 0,$$

as $m \to \infty$, since $s > k-1$. □

We give now an application of Theorem 1 which gives an extension of the integral inequalities of the classical B_p-case.

Theorem 3. *Let $w(x) \in B_p$ for some $p > 1$ and let $0 < q < \infty$. Then there exists $0 < a < 1$ such that*

$$\int_0^\infty Hf(x)^p \log^q(1/Hf(x))w(x)\,dx \le c_0 \int_0^\infty (c_1 f(x))^p \log^q(1/c_1 f(x))w(x)\,dx,$$

for all $f \in \mathcal{D}$ with $f(0+) \le a$.

PROOF. Since $w \in B_p$ we know that $w \in B_{p'}$ for some $1 < p' < p$ [5]. Choose now $1 < s < \infty$ such that $s^2 p' = p$. Since $w \in B_{sp'}$ we have for the $(j+1)$-st iterated Hardy operator $H_{j+1}f$ the inequality

$$\int_0^\infty H_{j+1}f(x)^{sp'} w(x)\,dx \le c_j \int_0^\infty f(x)^{sp'} w(x)\,dx, f \in \mathcal{D}.$$

If we let $f = \chi_r$ and note that

$$H_{j+1}f(x) = \chi_r(x) + \frac{r}{x}\phi_j(x/r)\chi^r(x),$$

where $\phi_j(y) = \sum_0^j \frac{\log^i y}{i!}$, then

$$\int_r^\infty (r/x)^{sp'} \log^j(x/r)w(x)\,dx \le c_j \int_0^r w(x)\,dx.$$

If now $j < q \le j+1$, the exponent j above can be replaced by q.

The function $g(u) = u^s \log^q(1/u)$ is convex on some interval $[0, a]$, with $0 < a < 1$. We define

$$\Phi(u) = \begin{cases} g(u), 0 \le u \le a \\ g'(a)(u-a) + g(a). \end{cases}$$

Then $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is convex with index s. We claim now that

$$L \equiv \int_r^\infty \Phi[(r/x)^{p'}]w(x)\,dx \le c \int_0^r w(x)\,dx.$$

We break up $L = \int_r^{r/a} + \int_{r/a}^\infty = I_1 + I_2$. The integral

$$I_2 = \int_{r/a}^\infty c(r/x)^{sp'} \log^q(x/r)w(x)\,dx \le c \int_0^r w(x)\,dx$$

and since for $r \leq x \leq r/a$, $\Phi[(r/x)^{p'} \leq c(r/x)^{p'}$ the integral $I_1 \leq c \int_0^r w(x)\,dx$.

Since Φ has index s, by Theorem 1

$$\int_0^\infty \Phi[Hf^{sp'}]w \leq c_0 \int_0^\infty \Phi(c_1 f^{sp'})w.$$

If $f \in \mathcal{D}$ and $f(0+) \leq a$ we get our conclusion since $s^2p' = p$. □

Remark. *If $w \in B_p, 1 \leq p < \infty$, and $q > 0$, then*

$$\int_0^\infty \frac{Hf^p}{\log^q(1/Hf)} w \leq c_0 \int_0^\infty \frac{c_1 f^p}{\log^q(1/c_1 f)} w,$$

for all $f \in \mathcal{D}$ and $f(0+) \leq a$. The proof is the same as before depending upon the convexity of $g(u) = u/\log^q(1/u), 0 \leq u \leq 1/e$.

3 Iterated Hardy Operator.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex with index k and $\Phi(0) = 0$. We will examine the following classes of Borel measures μ finite on compact sets. Below $p \geq 1$.

$$T_{j,\Phi,p} = \left\{ \mu : \int_0^\infty \Phi[H_{j+1} f^p]\,d\mu \leq c_0 \int_0^\infty \Phi[c_1 H_j f^p]\,d\mu, f \in \mathcal{D} \right\},$$

where $H_j f$ is the $j-$times iterated Hardy operator.

$$W_{j,p} = \left\{ \mu : \int_N^\infty \left(\frac{\log^{j-1} x}{x} \right)^p d\mu = \infty \right\}$$

for every $N < \infty$.

Theorem 4.

$$B_{\Phi,p} \cup W_{j,k^2p} \subset T_{j,\Phi,kp} \subset B_{\Phi,kp} \cup W_{j,kp}.$$

PROOF. Since Φ has index k, $\Phi'(u) \geq \gamma u^{k-1}, 0 \leq u \leq a$, and hence $\Phi(1)u \geq \Phi(u) \geq \frac{\gamma}{k} u^k, 0 \leq u \leq a$.

If $\mu \in B_{\Phi,p}$, then Theorem 1 shows that $\mu \in T_{j,\Phi,kp}$ since $H_j f \in \mathcal{D}$ if $f \in \mathcal{D}$. Assume now that $\mu \in W_{j,k^2p}$. Since $H_j f(x) \geq c(\log^{j-1} x)/x, x \geq 1$, we see that

$$\begin{aligned} \int_0^\infty \Phi[c_1 H_j f^{kp}]\,d\mu &\geq \int_N^\infty \Phi[c_1 c^{kp} \left(\frac{\log^{j-1} x}{x} \right)^{kp}]\,d\mu \\ &\geq \int_N^\infty c_1^k c^{k^2 p} \left(\frac{\log^{j-1} x}{x} \right)^{k^2 p} d\mu = \infty, \end{aligned}$$

if N is chosen so large that $c_1 c^{kp}((\log^{j-1} x)/x)^{kp} \le a, x \ge N$.

For the second implication we need to show that

$$T_{j,\Phi,p} \subset B_{\Phi,p} \cup W_{j,p}.$$

Let $\mu \in T_{j,\Phi,p}$, and choose $k_* \ge 1$ such that

$$\phi_j(y)^p - c_1 c_0 \phi_{j-1}(y)^p \ge 1, y \ge k_*,$$

where as before $\phi_j(y) = \sum_0^j \frac{\log^i y}{i!}$ and c_0, c_1 are the constants of μ in $T_{j,\Phi,p}$.

Let $f = k_* \chi_r$. Then

$$H_j f(x) = k_* \chi_r(x) + \frac{k_* r}{x} \phi_{j-1}(x/r) \chi^r(x).$$

Suppose there exists $0 < r_0 < \infty$ such that

$$\int_{r_0}^{\infty} \Phi[c_1 (k_* r_0/x)^p \phi_{j-1}(x/r_0)^p] \, d\mu = \infty.$$

Since μ is finite on compact sets, for every $N < \infty$

$$\int_{N}^{\infty} \Phi[c_1 (k_* r_0/x)^p \phi_{j-1}(x/r_0)^p] \, d\mu = \infty.$$

Since $\Phi(1)u \ge \Phi(u), 0 \le u \le 1$, we get

$$\int_N^{\infty} \frac{1}{x^p} \phi_{j-1}(x/r_0)^p \, d\mu = \infty,$$

if N is chosen so large that the integrand is ≤ 1. Since $\log^{j-1} x$ is the dominant term in $\phi_{j-1}(x)$ and since $\log(x/r_0) \le c \log x$ we get

$$\int_N^{\infty} \left(\frac{\log^{j-1} x}{x} \right)^p \, d\mu = \infty$$

and hence $\mu \in W_{j,p}$.

Hence we may assume that

$$\int_r^{\infty} \Phi[c_1 (k_* r/x)^p \phi_{j-1}(x/r)^p] \, d\mu < \infty,$$

for every $0 < r < \infty$. Since $\mu \in T_{j,\Phi,p}$ and $f = k_*\chi_r$, we see that

$$\Phi(k_*^p)\int_0^r d\mu + \int_r^\infty \Phi[(k_*r/x)^p\phi_j(x/r)^p]\,d\mu \leq$$
$$c_0\left\{\int_0^r \Phi(c_1k_*^p)\,d\mu + \int_r^\infty \Phi[c_1(k_*r/x)^p\phi_{j-1}(x/r)^p]\,d\mu\right\}.$$

Since the integrals involved are finite

$$\int_r^\infty \{\Phi[k_*^p(r/x)^p\phi_j(x/r)^p] - c_0\Phi[c_1k_*^p(r/x)^p\phi_{j-1}(x/r)^p]\}\;d\mu \leq c\int_0^r\; d\mu.$$

Denote by L_r the left side of the above inequality. Since we may take $c_0 \geq 1$, and since $\Phi(c_0u) \geq c_0\Phi(u)$, the expression L_r decreases if we put c_0 inside Φ. For $b > a \geq 0$, $\Phi(b) - \Phi(a) \geq \Phi(b-a)$ and thus

$$L_r \geq \int_r^\infty \Phi[(k_*r/x)^p\{\phi_j(x/r)^p - c_0c_1\phi_{j-1}(x/r)^p]\,d\mu.$$

By the choice of k_*

$$L_r \geq \int_r^\infty \Phi[(k_*r/x)^p]\,d\mu \geq \int_{k_*r}^\infty \Phi[(k_*r/x)^p]\,d\mu.$$

Hence

$$\int_{k_*r}^\infty \Phi[(k_*r/x)^p]\,d\mu \leq c\int_0^{k_*r}\; d\mu,$$

and $\mu \in B_{\Phi,p}$. □

4 From $p \to p - \epsilon$.

In this section we will examine the analogue to the well-known and important property: $w \in B_p$ implies $w \in B_{p-\epsilon}$ for some $\epsilon > 0$. This is no longer the case in our more general setting. An example will be given in section 7. However, a slightly stronger hypothesis will give us this implication. Below $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, $\Phi(0) = 0$, and $\Phi(u) = \Phi'(1)(u-1) + \Phi(1), u \geq 1$. Then $\Phi(c^j) \leq \Phi'(1)c^j, c \geq 1$. This follows from: since $\Phi'(1) \geq \Phi(1)$, $\Phi(u) \leq \Phi'(1)u, u \geq 1$.

Theorem 5. . *Let $\mu \in T_{\Phi,p}$ for some $1 \leq p < \infty$. Then there exists $\epsilon > 0$ such that $\mu \in B_{\Phi,p-\epsilon}$.*

PROOF. Since $\mu \in T_{\Phi,p}$ we get by a repeated application of the integral inequality

$$\int_0^\infty \Phi[H_j f^p]\, d\mu \le c_0^j \int_o^\infty \Phi[c_1^j f^p]\, d\mu, f \in \mathcal{D}.$$

Let now $f = \chi_r$. Then

$$H_j f(x)^p = \chi_r(x) + (r/x)^p \phi_{j-1}(x/r)^p \chi_r(x),$$

where $\phi_k(y) = \sum_0^k \frac{\log^i y}{i!}$. Thus

$$\int_r^\infty \Phi[(r/x)^p \phi_{j-1}(x/r)^p]\, d\mu \le c_0^j \int_0^r \Phi(c_1^j)\, d\mu \le c^j \int_0^r \, d\mu.$$

Since $\phi_{j-1}(x/r) \ge 1, x \ge r$, $\phi_{j-1}(x/r)^p \ge \phi_{j-1}(x/r) \ge (\log^{j-1}(x/r))/(j-1)!$ and thus the left side L_r above is

$$L_r \ge \int_r^\infty \Phi[(r/x)^p \frac{\log^{j-1}(x/r)}{(j-1)!}]\, d\mu.$$

Let now $s > c$. Then

$$\int_r^\infty \sum_1^\infty \frac{1}{s^{j-1}} \Phi[(r/x)^p \frac{\log^{j-1}(x/r)}{(j-1)!}]\, d\mu \le C \int_0^r \, d\mu.$$

Let $S = \sum_{j\ge 1}(1/s^{j-1}) = s/(s-1)$. Since Φ is convex, we get with $u_{j-1} = (r/x)^p \frac{\log^{j-1}(x/r)}{(j-1)!}$,

$$\sum_{j\ge 1} \frac{1}{s^{j-1}} \Phi(u_{j-1}) = \frac{1}{S} \sum_{j\ge 1} \frac{S}{s^{j-1}} \Phi(u_{j-1}) \ge \frac{1}{S} \Phi[\sum S \frac{u_{j-1}}{s_{j-1}}] = \frac{1}{S} \Phi[S(r/x)^{p-1/s}].$$

Since $S \ge 1$, $\Phi(Su) \ge S\Phi(u)$ and thus

$$L_r \ge \int_r^\infty \Phi[(r/x)^{p-1/s}]\, d\mu.$$

This shows that $\mu \in B_{\Phi,p-1/s}$. □

5 Concave Functions and Reverse Inequalities.

We wish to examine reverse inequalities of the form

$$\int_0^\infty \Psi(f^p)\,d\mu \le c_0 \int_0^\infty \Psi[c_1 Hf^p]\,d\mu,\, f \in \mathcal{D},$$

where $0 < c_0 < 1$ is given. The functions $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ that suggest themselves are concave and non-decreasing. Analogous to the convex case, we assume that $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is concave, $\Psi(0) = 0$, and there exist $0 < \gamma < \infty, 0 < a \le 1$ such that

$$\Psi'(u) \le \gamma u^{s-1}, 0 < u \le a,$$

for some $0 < s \le 1$. If we vary s slightly we may assume that $\gamma = s$. By rescaling - $\Psi(u) \to \Psi(au)/\Psi(a)$- we may assume in addition that $\Psi(1) = 1, a = 1$. For $0 < p < \infty$ and $0 < c_0 < 1$ we introduce the following classes of measures μ on $\mathbb{R}_+$ finite on compact sets:

$$C_{\Psi,p} = \left\{\mu : \int_0^r d\mu \le \frac{c_0}{1-c_0} \int_r^\infty \Psi[(r/x)^p]\,d\mu\right\}$$

and

$$S_{\Psi,p} = \left\{\mu : \int_0^\infty \Psi(f^p)\,d\mu \le c_0 \int_0^\infty \Psi[Hf^p]\,d\mu, f \in \mathcal{D}\right\}.$$

Theorem 6.

$$C_{\Psi,p} \subset S_{\Psi,sp} \subset C_{\Psi,sp}.$$

Proof. For the first inclusion let $r = r(y) \in \mathcal{D}$ and $\rho(y) = r\Psi^{-1}(y)$. Then

$$\int_0^\infty \int_0^{\rho(y)} d\mu\,dy \le C_0 \int_0^\infty \int_{\rho(y)}^\infty \Psi[(\rho(y)/x)^p]\,d\mu.$$

We interchange the order of integration and then the left side becomes

$$L = \int_0^\infty \rho^{-1}(x)\,d\mu,$$

and the right side is

$$R = C_0 \int_0^\infty \int_{\rho(y)}^\infty \Psi[(\rho(y)/x)^p]\,dy\,d\mu$$

By integration by parts or comparing areas under the curve $t = \Psi[(\rho(y)/x)^p]$ the inner integral $I(x)$ of R is - recall that $\Psi(1) = 1$ -

$$I(x) = \int_0^1 \rho^{-1}[x\Psi^{-1}(t)^{1/p}]\,dt - \rho^{-1}(x).$$

The substitution $t = \Psi(u^p)$ gives

$$I(x) = \int_0^1 \rho^{-1}(xu)\Psi'(u^p)pu^{p-1}du - \rho^{-1}(x).$$

Since the measure $d\nu = \Psi(u^p)pu^{p-1}du$ satisfies $\nu([0,1]) = 1$ and since $\rho^{-1}(xu) = \Psi r^{-1}(xu)$, Jensen's inequality gives

$$I(x) \leq \Psi\left\{\int_0^1 r^{-1}(xu)\Psi'(u^p)pu^{p-1}du\right\} - \rho^{-1}(x).$$

By hypothesis, $\Psi'(u^p) \leq su^{p(s-1)}, 0 < u \leq 1$ and thus

$$I(x) \leq \Psi\left\{\int_0^1 r^{-1}(xu)psu^{ps-1}du\right\} - \rho^{-1}(x).$$

We choose now $r^{-1}(t) = Hf(t)^{ps-1}f(t)$ and then we see that

$$r^{-1}(xu)psu^{ps-1} = \frac{d}{du}\left(\int_0^{xu} f(t)\,dt\right)^{ps}\frac{1}{x^{ps}}.$$

Thus

$$I(x) \leq \Psi[Hf(x)^{ps}] - \rho^{-1}(x).$$

The $C_{\Psi,p}-$ condition implies that

$$\int_0^\infty \rho^{-1}(x)\,d\mu \leq C_0 \int_0^\infty \left\{\Psi[Hf(x)^{ps}] - \rho^{-1}(x)\right\}\,d\mu.$$

Since $\rho^{-1}(x) = \Psi r^{-1}(x) = \Psi[Hf(x)^{ps-1}f(x)] \geq \Psi[f(x)^{ps}]$ we get

$$(1 + C_0)\int_0^\infty \Psi[f(x)^{ps}]\,d\mu \leq C_0 \int_0^\infty \Psi[c_1 Hf(x)^{ps}]\,d\mu.$$

This gives us the $S_{\Psi,p}-$condition .

To show that $S_{\Psi,p} \subset C_{\Psi,p}$ let $f = \chi_r$. Since $\Psi(1) = 1$ we get

$$\int_0^r d\mu \leq c_0\left(\int_0^r d\mu + \int_r^\infty \Psi[(r/x)^p]\,d\mu\right).$$

This is the $C_{\Psi,p}-$condition. □

6 Changing μ, Φ.

In this section we examine when the following inequalities hold:

$$\int_0^\infty \Phi[Hf(x)^q]\,d\mu \le c_0 \int_0^\infty \Phi[c_1 f(x)^q]d\nu, f \in \mathcal{D}, \tag{3}$$

and

$$\int_0^\infty \Phi[Hf(x)^q]\,d\mu \le c' \int_0^\infty \Psi[c'' f(x)^q]\,d\mu, f \in \mathcal{D}. \tag{4}$$

For (3) we need the simple fact that the following statements below are equivalent:

$$\int_0^\infty g\,d\mu \le c \int_0^\infty g d\nu, g \in \mathcal{D}. \tag{5}$$

$$\int_0^r d\mu \le c \int_0^r d\nu, 0 \le r < \infty. \tag{6}$$

PROOF. The substitution $g = \chi_r$ in (5) proves (6), and for the implication (6) $\to$ (5) simply note that for $g \in \mathcal{D}$

$$\int_0^\infty g\,d\mu = \int_0^\infty \mu\{g > t\}\,dt \le c \int_0^\infty \nu\{g > t\}\,dt = c \int_0^\infty g d\nu. \qquad \square$$

Theorem 7. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex, $\Phi(0) = 0$, and index $k \ge 1$. If $\mu \in B_{\Phi,p}$ or $\nu \in B_{\Phi,p}$, then the following statements are equivalent:*

$$\int_0^\infty \Phi[Hf(x)^{kp}]\,d\mu \le c_0 \int_0^\infty \Phi[c_1 f(x)^{kp}]d\nu, f \in \mathcal{D} \tag{7}$$

$$\int_0^r d\mu \le c \int_0^r d\nu, 0 \le r < \infty. \tag{6}$$

PROOF. Let $f = \chi_r$ to obtain (7)$\to$(6). For the reverse implication we have two cases.

CASE 1. $\mu \in B_{\Phi,p}$.

By Theorem 1 for $f \in \mathcal{D}$

$$\int_0^\infty \Phi[Hf^{kp}]\,d\mu \le c_0 \int_0^\infty \Phi[c_1 f^{kp}]\,d\mu,$$

and (5) completes the proof.

CASE 2. $\nu \in B_{\Phi,p}$.

Since for $f \in \mathcal{D}$ we have $Hf \in \mathcal{D}$ we get from (5) and Theorem 1

$$\int_0^\infty \Phi[Hf^{kp}]\,d\mu \le c \int_0^\infty \Phi[Hf^{kp}]d\nu \le c' \int_0^\infty \Phi[c'' f^{kp}]d\nu. \qquad \square$$

Theorem 8. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex, $\Phi(0) = 0$, with index $k \geq 1$, and let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the following statements are equivalent for $1 \leq p < \infty$ and $\mu \in B_{\Phi,p}$:*

$$\int_0^\infty \Phi[Hf(x)^{kp}]\,d\mu \leq c_0 \int_0^\infty \Psi[c_1 f(x)^{kp}]\,d\mu, f \in \mathcal{D}. \tag{8}$$

$$\Phi(u) \leq c'\Psi(c''u), 0 \leq u < \infty. \tag{9}$$

PROOF. (8)→(9). Fix $0 \leq u < \infty$ and let $f = u^{1/kp}\chi_r$. Then by (8)

$$\Phi(u)\int_0^r d\mu \leq c_0\Psi(c_1 u)\int_0^r d\mu$$

and (9) follows.

(9)→(8). By Theorem 1 we know that

$$\int_0^\infty \Phi[Hf^{kp}]\,d\mu \leq c_0 \int_0^\infty \Phi[c_1 f^{kp}]\,d\mu$$

and (9) gives us $\Phi[c_1 f(x)^{kp}] \leq c'\Psi[c'' f(x)^{kp}]$. □

7 Remarks and Examples.

This section is subdivided into subsections numbered 2 through 6 corresponding to the main sections 2 through 6 and contains comments, remarks, and examples illustrating the results.

2.

(i) We allow measures μ even singular with respect to dx. This is different from the weighted version of the Hardy-Littlewood maximal operator: The inequality

$$\int_{\mathbb{R}^n} Mf^p\,d\mu \leq c\int_{\mathbb{R}^n} |f|^p\,d\mu, 1 < p < \infty,$$

holds iff $d\mu = w(x)\,dx$ and $w \in A_p$ ([4, p. 255]).

(ii) As an example let $\mu = \delta_0$, the Dirac-delta at $x = 0$. Then μ is in any $B_{\Phi,p}$ (the left side is 0) and we have the obvious $\Phi[Hf(0+)] \leq c_0\Phi[c_1 f(0+)]$.

(iii) As for the Corollary, there exist convex functions Φ whose only index is any $k > 1$ such that $\Phi_k(u) = \Phi(u^{1/k})$ is not convex. Let

$$\Phi(u) = \begin{cases} \frac{u}{\log(1/u)}, & 0 \leq u \leq 1/e \\ \Phi'(1/e)(u - 1/e) + 1/e, & u > 1/e. \end{cases}$$

Then Φ is convex with index any $k > 1$, but not $k = 1$. However

$$\Phi_k(u) = \frac{ku^{1/k}}{\log(1/u)}, 0 \le u \le 1/e,$$

is no longer convex, since $\Phi_k'(u) \to \infty$ as $u \to 0$.

3.

Theorem 4 seems to be new even in the classical B_p-case. If $k = 1$ is an index of Φ we get a characterization of $T_{j,\Phi,p}$.

4.

It is well-known that the property $p \to p - \epsilon$ is connected with the behavior of the iterated Hardy operator. This is the approach for B_p in [5] and for the maximal operator in [7].

In our general setting, $\mu \in B_{\Phi,p}$ may not imply $\mu \in B_{\Phi,p-\epsilon}$ for any $\epsilon > 0$. As an example let

$$\Phi(u) = \begin{cases} \frac{u}{\log^2(2/u)}, & 0 \le u \le 2/e \\ \Phi'(2/e)(u - 2/e) + \Phi(2/e), & u > 2/e. \end{cases}$$

Then any $k > 1$ is an index for Φ. Let $d\mu = xdx$. We claim that $\mu \in B_{\Phi,2}$. Write

$$\int_r^\infty \Phi[(r/x)^2]x\,dx = \int_r^{er/2} + \int_{er/2}^\infty = I_1 + I_2.$$

Then $I_1 = \displaystyle\int_r^{er/2} (m[(r/x) - a] + b)\,x\,dx \le cr^2$ and

$$I_2 = c\int_{er/2}^\infty (r/x)^2 \frac{x}{\log^2(cx/r)}\,dx \le cr^2.$$

Next, $\mu \notin B_{\Phi,2-\epsilon}$. The integral I_2 above is now

$$I_2 = \int_{er/2}^\infty (r/x)^{2-\epsilon} \frac{x}{\log^2(cx/r)}\,dx = r^{2-\epsilon}\int_{er/2}^\infty \frac{x^\epsilon\,dx}{x\log^2(cx/r)} = \infty.$$

5.

A reverse inequality in the classical B_p-case can be found in [5].

6.

Integral inequalities of the form $\int_{\mathbb{R}^n} \Phi[Tf]\,dx \le c_0 \int_{\mathbb{R}^n} \Psi[c_2 f]\,dx$ have been studied for various operators T [6].

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