RECOLLECTIONS OF WORKING WITH CAS GOFFMAN

By Andrew M. Bruckner

1 Purdue 1965.

In the early 1960s I was trying to determine conditions under which a weak local superadditivity condition implies convexity of a function defined on an interval. I was able to obtain the desired result if the function f was assumed to be differentiable with f' continuous *a.e.* The condition failed, however if one assumed only that f' exists *a.e.* and is continuous wherever it exists. The Cantor function provided a counter example. I wondered whether differentiability alone would suffice, but did not know enough about derivatives to be able to answer that question. So I decided to study derivatives.

I searched the literature and came across a number of fascinating papers by Z. Zahorski, Solomon Marcus, C. Neugebauer, and of course Cas Goffman. I studied these and many other papers on the subject and decided to spend my sabbatical leave in 1965–1966 at Purdue University, to see what I could learn from Goffman, Neugebauer and their group at Purdue. By the time I arrived at Purdue, Neugebauer had switched his emphasis to Fourier Analysis, but Goffman was still working with questions centered on derivatives and related subjects. We quickly identified a problem to work on. (I'll describe it a bit later.) It was a pleasure to work with Cas in a relaxed, informal way.

I learned from Cas that it was possible for one to focus total attention on more than one thing at a time. One day we were working in my apartment at the same time the Purdue–Illinois football game was on television. I described a possible approach using certain sets E_{rsn} to prove a theorem we wanted.

> "Do you think that can work, Cas?"

Silence.
Purdue had just failed to gain two yards for a touchdown. I repeated my question –

> "Can this work?"

> "How can it work?", asked Cas. "Illinois was watching for the running play. Griese should have passed."

I focused on the next play and saw Purdue fail in another running play.

"I guess you're right – that can't work.", I commiserated.

"Of course it can", said Cas. "You just have to be careful to define the sets E_{rsn} properly. Those sets should go with $\frac{\epsilon}{3n}$, not $\frac{\epsilon}{3r}$".

Purdue was not able to score, but we were able to prove the theorem.

The problems that concerned us involved analogues, when sets of density zero are neglected, of several theorems on the boundary behavior of functions defined on an open half plane. To motivate the circle of ideas, let F be a continuous function defined on the half plane H above the line $L = \{(x, y) : x = y\}$.

Let θ be a direction, and consider $S = \{p \in L :$ the total cluster set of F at p is the same as the cluster set of F at p in the direction $\theta\}$

Theorem. [3] S is a residual subset of L.

As an application, consider the formula

$$F(x, y) = \frac{f(y) - f(x)}{y - x}$$

where $f : R \to R$ is continuous. Then F is continuous on H. Let $p \in L$. Then

$$\limsup_{y \to p} F(p, y) = D^+ f(p)$$

while

$$\limsup_{x \to p} F(x, p) = D^- f(p),$$

that is, the vertical and horizontal upper limits of F as (x, y) approaches (p, p) are the upper Dini derivates. By the theorem, these two upper Dini derivates are equal except for a first category subset of the line. In fact, they are equal (residually) to the unrestricted upper limit

$$\limsup_{y \to p, x \to p, y \neq x} F(x, y) = \frac{f(y) - f(x)}{y - x}.$$

The same is true for other methods of approach — for example, approaching (p, p) in the direction normal to L we obtain the upper symmetric derivate.

When one ignores sets of density zero, one cannot obtain a full analogue of this theorem. A full analogue would involve replacing full cluster sets or directional cluster sets with essential cluster sets or essential directional cluster sets, leading to upper and lower approximate limits. But Cas came up with a continuous F whose perpendicular upper approximate limit equals one on a residual subset of the bounding line L, while it is zero at every point of L in every other direction.

If one replaces approaches along line segments with approaches in "sectors" — pie shaped regions with vertices on L — the analogue of the Theorem holds:

Theorem. Let F be a measurable function on a half plane bounded by the line L, and let σ be a fixed sector. Then the unrestricted essential cluster set of F

at $p \in \mathrm{L}$ equals the essential cluster set of F at p restricted to the sector σ with vertex p except for those p in a set of first category.

The sector σ can be "arbitrarily thin", that is, the vertex of the sector can have an arbitrarily small angle. But as Cas's example shows, this angle can't be zero, in which case the sector would degenerate to a half line.

When we finished the paper, Cas mentioned he was happy to see he could still work. He had thought he might never write another paper. He was 52 years old at the time and had written 52 papers. Over the next 30 years or so, he wrote over 52 more papers.

2 Santa Barbara 1970s.

Cas visited Santa Barbara a couple of times in the 1970s where we worked on problems involving homeomorphic changes of variables (or changes of scale). The idea behind such problems is to create desirable properties for functions via homeomorphic changes of scale. For example, we proved [4] that to each function f that is continuous and of bounded variation on $[0,1]$ corresponds a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has a bounded derivative on $[0,1]$. The condition is obviously necessary as well, since neither continuity nor bounded variation of a function can be created or destroyed by a homeomorphic change of scale.

If one has finitely many such functions $f_1, \ldots, f_n$, one can choose a single homeomorphism h such that each of the functions $f_1 \circ h, f_2 \circ h, \ldots, f_n \circ h$ has a bounded derivative. Thus, a rectifiable curve in n-space given parametrically by $x_i = f_i(t), i = 1, \ldots, n$ allows a homeomorphic reparametrization of the parameter interval $[0,1]$ such that the coordinate functions

$$y_i(t) = (f_i \circ h)(t), i = 1, \ldots, n, \quad t \in [0,1]$$

are differentiable with bounded derivatives.

We also obtained necessary and sufficient conditions for a function f to be transformed into one with a continuous derivative. A few years later Laczkovich and Preiss [8] obtained conditions under which a function f can be transformed into a function in $\mathcal{C}^n$ via a homeomorphic change of scale.

Cas has done a great deal of work on problems involving homeomorphic changes of variables. Much of this work is developed in the beautiful book *"Homeomorphisms in Analysis"* [6].

During a second visit, Cas suggested we try to improve a result of W. Gorman's [7].Cas noted that every measurable function f is equivalent (that is, equal *a.e.*) to a Baire 2 function, but not necessarily to a Baire 1 function. Gorman had shown that a measurable f with finite range can be transformed into a function that is equal *a.e.* to a Baire 1 function, and had given an example of a measurable function with countable range that cannot be transformed this way. Cas and I worked on this problem, and with Roy Davies obtained the result that every absolutely measurable f can be transformed in this way [1]. (A function

f on an interval I is absolutely measurable if $f \circ h$ is Lebesgue measurable for every homeomorphism h of I onto itself. This is equivalent to saying that f is measurable with respect to every Lebesgue-Stieltjes measure derived from a strictly increasing continuous distribution function).

These two visits of Cas also illustrated the difference between Cas's logical thinking and his wife Eve's straightforward thought processes. On the first visit Cas learned quickly how to get from his bedroom in our house to the kitchen, some 75 feet away, and how to get from the kitchen to the bathroom, which is directly across the hall from the bedroom. After a week Cas reasoned logically that he did not have to go to the kitchen in order to get to the bathroom. On the second visit Eve visited also. She "reasoned" immediately that the direct route would work.

On another visit, we thought it would be useful to write an expository paper dealing with the concept of approximate differentiation. This is a useful concept because the approximate derivative (or approximate differential in the case of several variables) can often serve as an excellent substitute for the ordinary derivative (or total differential) when the latter does not exist. We agreed that I would write a first draft of a section about "everywhere" theory of the approximate derivative, emphasizing how well approximate derivatives mimic the behavior of ordinary derivatives. Cas would write a first draft of the "almost everywhere" theory of approximate differentiation, including such topics as the theorems of Stepanoff and of Whitney, as well as applications to topics such as surface area. Cas described what he thought should be included.

> "Great," I said. "Let's do it."

Silence.

> "I can't," he said. "I can't write it. I know what I want to say, but I can't write it."

I didn't understand.

> "What do you want to say?", I asked

He started to tell me. What followed was a lecture — it sounded like a well prepared, elegant lecture by a master, complete with many details. Basically he dictated a full section to me. The next day, another flawless lecture on surface area. He essentially dictated a major portion of our article [5]. This included an outline in some detail about how approximate differentiation figures in the proof that the area formula works for Sobolev mappings. The dictation included precise statements of deep theorems of Cesari and of Calderon and Serrin.

3 Special Year.

During the Academic year 1983–1984 UCSB hosted a number of visitors, for several months each, to participate in a "Special Year in Real Analysis". Cas and Eve visited for about six months and Cas interacted with all who were involved in the Special Year. We wrote our last joint paper towards the end of their stay in Santa Barbara.

Cas had mentioned a problem dealing with the perimeter of a set in $\mathbb{R}^n$.He explained a theorem of Volpert, which improved a theorem of Federer, by replacing a so called "reduced boundary" with the essential boundary. Cas said that on the basis of work with Roy Davies and discussion with J. M. Marstrand, he conjectured that for a compact totally disconnected set of positive measure in $\mathbb{R}^n$, the essential boundary can be of Hausdorff dimension $n-1$, but cannot have σ- finite $(n-1)$ Hausdorff measure. He explained some more, but I didn't really understand what he was talking about.

It was then that I discovered another quality Cas had — the ability to simplify a problem one's co-author doesn't understand so that the co-author can contribute something useful. (A good quality when advising graduate students!) Cas suggested, if memory serves, we try to show that certain sets in the plane have the property that for almost every $y \in \mathbb{R}$, the set

$$\{x : \underline{d}(E,(x,y)) = 0 \text{ and } \overline{d}(E,(x,y)) = 1\} \cap E^y$$

is residual in the ordinary boundary of E^y. Here $\underline{d}$ and $\overline{d}$ denote the lower and upper densities, and $E^y = \{x : (x,y) \in E\}$.

This question I *did* understand. We solved it, and eventually Cas and Roy Davies were able to expand the result for $\mathbb{R}^n$ and finish off the problem — Cas's conjecture was correct [2].

The preceding paragraphs give no more than a glimpse or two into Cas's personality and abilities. Those of us who knew him recognize his warmth, his sense of humor and his strong concern for justice. He was a strong defender of the underdog and often jumped to the defense of mathematicians he felt were not treated fairly.

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