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ON A PROBLEM OF LEIDERMAN

Abstract

In the space $B_1(0,1])$ of all Baire 1 real functions on $[0,1]$, equipped with the topology of pointwise convergence there is an uncountable discrete closed subset of Darboux functions. This affirmatively answers a question of A. Leiderman.

Let $\mathbb{R}$ be the set of all reals. Denote by $B_1([0,1])$ the space of all Baire 1 functions from $[0,1]$ to $\mathbb{R}$ equipped with the pointwise convergence topology. At the 17^{th} SUMMER CONFERENCE ON REAL FUNCTIONS THEORY in Stará Lesná, Slovakia, 2002, A. Leiderman in his lecture [1] asked if $B_1([0,1])$ contains an uncountable discrete closed subset of Darboux functions? In this article I show that the answer is affirmative. We start with the following Lemma.

Lemma 1. *If $A \subset [0,1]$ is a nonempty perfect set, then there is a Darboux Baire 1 function $f : [0,1] \to [0,1]$ such that $f(x) = 0$ for $x \in [0,1] \setminus A$ and $f(A) = [0,1]$.*

Proof. Let $h : [0,1] \to [0,1]$ be a homeomorphism from $[0,1]$ onto $[0,1]$ such that the image $h(A)$ is of positive measure ([3]). There is a nonempty F_σ-set $B \subset h(A)$ belonging to the density topology T_d ([1]). By Zahorski's lemma ([1, 4]) there is an approximately continuous function $g : [0,1] \to [0,1]$ such that $g(x) = 0$ for $x \in [0,1] \setminus A$ and $g(B) = (0,1]$. Now the function $f(x) = g(h(x))$ for $x \in [0,1]$ satisfies all the requirements. □

In the lecture [2] A. Leiderman stated that

(∗) *the uncountable family of all functions*

$$f_a(x) = \frac{1}{|x-a|} \text{ for } x \in [0,1] \setminus \{a\} \text{ and } f_a(a) = 0, \text{ where } a \in [0,1],$$

is a discrete and closed subset of $B_1([0,1])$.

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Theorem 1. *There is an uncountable closed discrete subset $E \subset B_1([0,1])$ composed of Darboux functions.*

PROOF. Let $E \subset (0,1)$ be a nonempty perfect set of measure zero and let

$$E = \{e_0, e_1, \ldots, e_\alpha, \ldots\} \text{ where } \alpha < \omega_c,$$

and ω_c denotes the first ordinal number whose cardinality is that of the continuum. Let ω_1 be the first uncountable ordinal and let $A = \{e_\alpha; \alpha < \omega_1\}$. For each point $a \in A$ there are sequences $(C_{a,n})_n$ of nonempty perfect sets of measure zero such that

(1) $C_{a,n} \cap C_{b,m} = \emptyset$ if $(a,n) \neq (b,m)$;

(2) $C_{a,2n-} \subset (0,a) \setminus A$ and $C_{a,2n} \subset (a,1) \setminus A$ for $n \geq 1$;

(3) for each $a \in A$ the sequences $(C_{a,2n-1})_n$ and $(C_{a,2n})_n$ converge in the Hausdorff metric to the set $\{a\}$.

Let

$$c_{a,n} = \max(f_a(C_{a,n})) \text{ for all pairs } (a,n) \text{ where } a \in A \text{ and } n \geq 1.$$

By Lemma 1, for $a \in A$ and for positive integers n, there are Darboux Baire 1 functions $\phi_{a,n} : [0,1] \to [0,1]$ such that

$$\phi_{a,n}(x) = 0 \text{ for } x \in [0,1] \setminus C_{a,n} \text{ and } \phi_{a,n}(C_{a,n}) = [0,1].$$

For $a \in A$ let

$$g_a(x) = \begin{cases} \max(0, f_a(x) - c_{a,n}\phi_{a,n}(x)) & \text{for } x \in C_{a,n},\ n \geq 1 \\ f_a(x) & \text{otherwise on } [0,1]. \end{cases}$$

Since for each pair $(a,n) \in A \times N$ there is a point $x \in C_{a,n}$ with $g_{a,n}(x) = 0$ and since the functions f_a are continuous at $u \neq a$, each function g_a belongs to Darboux Baire 1.

Observe that for a fixed point $a \in A$ we have

$$\inf\{g_b(a); b \neq a\} = \inf\{f_b(a); b \neq a\} = \frac{1}{\max(a, 1-a)} = r_a > 0.$$

If for $a \in A$ we put

$$U_a = \{g \in B_1([0,1]); |g(a)| < r_a\},$$

then U_a is open in the pointwise convergence topology, $g_a \in U_a$ and for each point $b \neq a$ belonging to A we have g_b is not in U_a. So the family $\{g_a; a \in A\}$ is discrete in $B_1([0,1])$.

We will prove that the family $K = \{g_a; a \in A\}$ is closed in $B_1([0,1])$. Assume, to the contrary, that there is a function $h \in B_1([0,1]) \setminus K$ which belongs to the closure (in the pointwise convergence topology T_p) $\mathrm{cl}(K)$ of the family K. Assume that there is a point $a \in A$ with $h = f_a$. Then for each $b \neq a$ belonging to A the function

$$g_b \notin W_a = \{g \in B_1([0,1]); |g(a)| < r_a\} \ni f_a.$$

Moreover, there are a pair (a,n) and a point $u \in C_{a,n}$ with $g_a(u) = 0$. Since $f_a(u) > 0$, the function

$$g_a \notin V = \{g \in B_1([01]); |g(u) - f_a(u)| < \frac{f(a)}{2}\} \ni f_a.$$

So, the equality $h = f_a$ for some $a \in A$ is not possible. Consequently, by $(*)$, there is an open family $W \in T_p$ containing h such that

$$(**) \qquad f_a \in B_1([0,1]) \setminus W \text{ for all } a \in A.$$

There are a positive real s and a point $w \in [0,1]$ such that

$$V = \{g \in B_1([0,1]); |g(w) - h(w)| < s\} \subset W.$$

If $w \in [0,1] \setminus \bigcup_{a \in A, n \geq 1} C_{a,n}$, then $f_a(w) = g_a(w)$ for all $a \in A$, and consequently $V \cap K = \emptyset$, in a contradiction with the relation $h \in \mathrm{cl}(K)$. So there is exactly one pair (a,n) with $w \in C_{a,n}$. Observe that $g_b(w) = f_b(w)$ for $a \neq b$, $b \in A$. Since $h \in cl(K)$, there is $b \neq a$ belonging to A such that $g_b \in V$. But $g_b(w) = f_b(w)$, so $f_b \in V$ and we obtain a contradiction with $(**)$. □.

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