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ON THE SUMS OF UNILATERALLY APPROXIMATELY CONTINUOUS AND APPROXIMATE JUMP FUNCTIONS

Abstract

In paper [4] it is proved that every jump function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two unilaterally continuous jump functions. In this article we prove that the analogous result is not true for the density topology. Moreover we show some necessary and sufficient conditions ensuring that an approximate jump function is the sum of two unilaterally approximately continuous and approximate jump functions.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$. For a Lebesgue measurable set $A \subset \mathbb{R}$ and a point x we define the right (left) density $D_+(A, x)$ ($D_-(A, x)$) of the set A at the point x as

$$\lim_{h\to 0^+} \frac{\mu(A \cap [x, x+h])}{h} \ \ (\lim_{h\to 0^+} \frac{\mu(A \cap [x-h, x])}{h} \text{ respectively}),$$

whenever these limits exist. A point x is called a right density point (a left density point) of a set A if there is a Lebesgue measurable set $B \subset A$ such that $D_+(B, x) = 1$ ($D_-(B, x) = 1$).

A function $f : A \to \mathbb{R}$ is said to be approximately continuous from the right (from the left) at a point x if there is a Lebesgue measurable set $B \subset A$ such that $x \in B$, $D_+(B, x) = 1$ ($D_-(B, x) = 1$) and the restricted function $f \restriction B$ is continuous at x. If a function f is simultaneously approximately continuous at a point x from the right and from the left,, then we will say that f is approximately continuous at x ([2]). A function f is called unilaterally approximately continuous at x if f is approximately continuous at x from the right or from the left. A function $f : \mathbb{R} \to \mathbb{R}$ is called an unilaterally approximately continuous function if it is unilaterally approximately continuous at each point $x \in \mathbb{R}$.

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For a function $f : \mathbb{R} \to \mathbb{R}$ and a point x we will say that the right approximate limit $\operatorname{ap}\lim_{t\to x^+} f(t)$ (respectively the left approximate limit $\operatorname{ap}\lim_{t\to x^-} f(t)$) is equal $a \in [-\infty, \infty]$ if there is a Lebesgue measurable set B with $D_+(B,x) = 1$ (respectively $D_-(B,x) = 1$) such that $\lim_{B\ni t\to x} f(t) = a$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an approximate jump function if for each point $x \in \mathbb{R}$ there are both finite unilateral approximate limits $\operatorname{ap}\lim_{t\to x^+} f(t)$ and $\operatorname{ap}\lim_{t\to x^-} f(t)$.

Since the operations of the calculation of unilateral approximate limits are linear, the sums and the products of approximate jump functions are also approximate jump functions.

Remark 1. *Each approximate jump function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.*

PROOF. Let η be a positive real and let $A \subset \mathbb{R}$ be a Lebesgue measurable set with $\mu(A) > 0$. Let $a \in A$ be a density point of A. Since the left approximate limit $\operatorname{ap}\lim_{t\to x^-} f(t)$ is finite, there is a Lebesgue measurable set $B \subset A$ such that $D_-(B,a) = 1$ and $\operatorname{diam}(f(B)) < \eta$, where $\operatorname{diam}(f(B))$ denotes the diameter of the set $f(B)$. So, by Davies' Lemma 2 from [3] the function f is Lebesgue measurable. □

Remark 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then there is a sequence (f_n) of approximate jump functions such that $f = \lim_{n\to\infty} f_n$.*

PROOF. Of course, there is a Baire 2 function $g : \mathbb{R} \to \mathbb{R}$ such that the set $A = \{x : f(x) \neq g(x)\}$ is of measure zero. By Preiss' theorem from [6] there is a sequence (g_n) of approximately continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ such that $g = \lim_{n\to\infty} g_n$. Consequently, the functions

$$f_n(x) = \begin{cases} g_n(x) & \text{for } x \in \mathbb{R} \setminus A \\ f(x) & \text{for } x \in A \end{cases}$$

are approximate jump functions and $f = \lim_{n\to\infty} f_n$.

Let $D_{ap}(f)$ denote the set of all points x at which f is not approximately continuous. Since for each set A with $\mu(A) = 0$ the function $f_A(x) = 1$ for $x \in A$ and $f_A(x) = 0$ for $x \in \mathbb{R} \setminus A$ is an approximate jump function and there are uncountable sets A with $\mu(A) = 0$, there are approximate jump functions (also without the Baire property) with the uncountable sets $D_{ap}(f)$.

From Belowska's construction in [1] follows that there are unilaterally approximately continuous functions for which the sets $D_{ap}(f)$ are not countable. But the following theorem is true.

Theorem 1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is an approximate jump and unilaterally approximately continuous function, then the set $D_{ap}(f)$ is countable.*

Proof. Assume by a contradiction, that the set $D_{ap}(f)$ is not countable. Let

$$A = \{x : \operatorname{ap}\lim_{t\to x^+} f(t) < \operatorname{ap}\lim_{t\to x^-} f(t)\}$$

and

$$B = \{x : \operatorname{ap}\lim_{t\to x^+} f(t) > \operatorname{ap}\lim_{t\to x^-} f(t)\}.$$

Since f is an unilaterally approximately continuous and approximate jump function, we obtain $D_{ap}(f) = A \cup B$. So, at least one of the sets A, B is uncountable. Without loss of the generality we may assume that A is not countable. For each point $x \in A$ there are rationals $a(x)$, $b(x)$ and a positive rational $r(x)$ such that:

$$\begin{gathered}\operatorname{ap}\lim_{t\to x^+} f(t) < a(x) < b(x) < \operatorname{ap}\lim_{t\to x^-} f(t)\\ \frac{\mu([x, x+h] \cap f^{-1}((-\infty, a(x))))}{h} > \frac{7}{8} \text{ for } h \in (0, r(x)],\\ \frac{\mu([x-h, x] \cap f^{-1}((b(x), \infty)))}{h} > \frac{7}{8} \text{ for } h \in (0, r(x)].\end{gathered}$$

But the set of all triplets of rationals is countable; so there are rationals a, b, r such that the set $E = \{x \in A : a(x) = a, b(x) = b, r(x) = r\}$ is uncountable. Let $u, v \in E$ be bilateral condensation points of E such that $u < v < u + r$. Then $0 < h = v - u < r$ and consequently

$$\frac{\mu([u, u+h] \cap f^{-1}((-\infty, a)))}{h} > \frac{7}{8} \text{ and} \tag{*}$$

$$\frac{\mu([v-h, v] \cap f^{-1}((b, \infty)))}{h} > \frac{7}{8}. \tag{**}$$

But $[u, u+h] = [u, v] = [v-h, v]$, so, by (*) and (**), there is a point $w \in [u, v]$ with $f(w) < a$ and $f(w) > b$. This contradiction finishes the proof. □

Conclusion 1. *If $f : \mathbb{R} \to \mathbb{R}$ is an approximate jump and unilaterally approximately continuous function, then there is a Baire 1 function $g : \mathbb{R} \to \mathbb{R}$ such that the set $\{x : g(x) \neq f(x)\}$ is countable.*

Proof. The restricted function $f\restriction(\mathbb{R} \setminus D_{ap}(f))$ is in Baire 1 class ([2]) and $\mathbb{R} \setminus D_{ap}(f)$ is an G_δ-set, so there is ([5], p. 341) a Baire 1 function $g : \mathbb{R} \to \mathbb{R}$ such that $g\restriction(\mathbb{R} \setminus D_{ap}(f)) = f\restriction(\mathbb{R} \setminus D_{ap}(f))$. This completes the proof. □

In [4] it is proved that every jump function $f : \mathbb{R} \to \mathbb{R}$ (i.e., both unilateral limits $\lim_{t\to x^+} f(t)$ and $\lim_{t\to x^-} f(t)$ exist and are finite at each point $x \in \mathbb{R}$) is the sum of two unilaterally continuous jump functions.

From the above considerations follows that this result is not true for the density topology. Of course, from Theorem 1 follows that if f is the sum of two unilaterally approximately continuous and approximate jump functions, then the set $D_{ap}(f)$ is countable. So, if f is an approximate jump function such that the set $D_{ap}(f)$ is uncountable, then it is not the sum of two unilaterally approximately continuous functions which are approximately jump functions.

Moreover, if f is the sum of two unilaterally approximately continuous and approximate jump functions, then, by Conclusion 1, there are a Baire 1 function $g : \mathbb{R} \to \mathbb{R}$ and a countable set A such that $f(x) = g(x)$ for $x \in \mathbb{R} \setminus A$.

However observe that if f is the function of Dirichlet, then the set $\{x : f(x) \neq 0\} = \mathbb{Q}$ is countable ($\mathbb{Q}$ denotes the set of all rationals), but f is not the sum of two unilaterally aproximately continuous and approximate jump functions.

Remark 3. *If $f(x) = 1$ for $\in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, then f is not the sum of two unilaterally approximately continuous and approximate jump functions.*

PROOF. Assume by a contradiction that there are unilaterally approximately continuous and approximate jump functions g, h such that $f = g + h$. Let $B = D_{ap}(g) \cup D_{ap}(h)$. By Theorem 1 the set B is countable. Moreover $Q \subset B$. Fix $a \in (0, \frac{1}{8})$. Since the functions g, h are approximately continuous at each point $x \in \mathbb{R} \setminus B$, for every point $x \in \mathbb{R} \setminus B$ there is a positive rational $r(x)$ such that for each nondegenerate interval I containing x of the length $d(I) \leq r(x)$

$$\frac{\mu(I \cap g^{-1}((g(x) - a, g(x) + a)))}{\mu(I)} > \frac{1}{2}$$

and

$$\frac{\mu(I \cap h^{-1}((h(x) - a, h(x) + a)))}{\mu(I)} > \frac{1}{2}.$$

Since the set of all rationals is countable, there is a positive rational r such that the set $E = \{x \in \mathbb{R} \setminus B : r(x) = r\}$ is of the second category. Consequently, there is an open interval J_1 such that the set $J_1 \cap E$ is dense in J_1. Let $t \in J_1 \setminus B$ be a point and let $J \subset J_1$ be an open interval of the length $d(J) < r$ containing t and such that

$$\frac{\mu(J \cap g^{-1}((g(t) - a, g(t) + a)))}{\mu(J)} > \frac{1}{2}$$

and

$$\frac{\mu(J \cap h^{-1}((h(t)-a, h(t)+a)))}{\mu(J)} > \frac{1}{2}.$$

If y is a point of the intersection $J \cap E$, then

$$\frac{\mu(J \cap g^{-1}((g(y)-a, g(y)+a)))}{\mu(J)} > \frac{1}{2}$$

and

$$\frac{\mu(J \cap h^{-1}((h(y)-a, h(y)+a)))}{\mu(J)} > \frac{1}{2}.$$

Consequently, there is a point

$$y_0 \in J \cap g^{-1}((g(y)-a, g(y)+a)) \cap g^{-1}((g(t)-a, g(t)+a)).$$

So, $|g(y)-g(t)| \leq |g(y)-g(y_0)|+|g(y_0)-g(t)| < a+a = 2a$. Similarly we can prove that $|h(y)-h(t)| < 2a$. So for arbitrary points $y_1, y_2 \in J \cap E$ we obtain $|g(y_1)-g(y_2)| \leq |g(y_1)-g(t)|+|g(t)-g(y_2)| < 2a+2a = 4a$ and analogously $|h(y_1)-h(y_2)| < 4a$.

Let $u \in Q \cap J$ be a point. Since the function g is unilaterally approximately continuous at u, there is a nondegenerate interval $J_2 \subset J$ with an endpoint $u \in J_2$ such that

$$\frac{\mu(J_2 \cap g^{-1}((g(u)-a, g(u)+a)))}{\mu(J_2)} > \frac{1}{2}.$$

There is a point $v \in J_2 \cap E$. Then

$$\frac{\mu(J_2 \cap g^{-1}((g(v)-a, g(v)+a)))}{\mu(J_2)} > \frac{1}{2}$$

and consequently, there is a point

$$w \in J_2 \cap g^{-1}((g(u)-a, g(u)+a)) \cap g^{-1}((g(v)-a, g(v)+a)).$$

So, $|g(u)-g(v)| \leq |g(u)-g(w)|+|g(w)-g(v)| < a+a = 2a$. Since the function h is unilaterally approximately continuous at u, as above we can prove that there is a point $z \in J \cap E$ such that $|h(u)-h(z)| < 2a$. Since $v, z \in E \subset \mathbb{R} \setminus Q$, we have $h(v) = -g(v)$, $g(z) = -h(z)$, $|g(v)-g(z)| < 4a$ and $|h(v)-h(z)| < 4a$. So,

$$\begin{aligned} 1 =& f(u) = g(u) + h(u) < g(v) + 2a + h(z) + 2a = g(v) - g(z) + 4a \\ \leq& 4a + 4a = 8a < 1, \end{aligned}$$

and this contradiction finishes the proof. □

We show some sufficient condition which implies that a function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two unilaterally approximately continuous and simultanuously approximate jump functions. For this, the notions of the density topology and of the approximate oscillation are necessary.

The family T_d of all Lebesgue measurable subsets $A \subset \mathbb{R}$ such that for each point $x \in A$ both unilateral densities $D_+(A, x) = D_-(A, x) = 1$ is a topology called the density topology ([2]).

For a set A denote by $\mathrm{int}_d(A)$ and respectively by $\mathrm{cl}_d(A)$ the interior and the closure of A with respect to the topology T_d. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then the approximate oscillation of f at a point x is defined as $\mathrm{ap\,osc}\, f(x) = \inf\{\mathrm{diam}(f(U)) : x \in U \in T_d\}$.

Observe that for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ we have $D_{ap}(f) = \{x : \mathrm{ap\,osc} f(x) > 0\}$.

For the approximate jump functions $f : \mathbb{R} \to \mathbb{R}$ the approximate oscillation $\mathrm{ap\,osc}\, f(x)$ of f at a point x may be defined as

$$\max(|f(x) - \operatorname*{ap\,lim}_{t\to x^+} f(t)|, |f(x) - \operatorname*{ap\,lim}_{t\to x^-} f(t)|, |\operatorname*{ap\,lim}_{t\to x^+} f(t) - \operatorname*{ap\,lim}_{t\to x^-} f(t)|).$$

We will say that a countable set $A = \{a_i : i \geq 1\}$ satisfies the condition (a) if there are pairwise disjoint sets $U_i \in T_d$ such that $a_i \in U_i$ for integer $i \geq 1$ and such that for each point $x \in \mathbb{R} \setminus \bigcup_{i\geq 1} cl_d(U_i)$ the unilateral densities $D_+(\bigcup_{i\geq 1} U_i, x) = D_-(\bigcup_{i\geq 1} U_i, x) = 0$.

Observe that isolated sets $A \subset \mathbb{R}$ and sets B whose derived sets are finite satisfy the condition (a).

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an approximate jump function such that the set $D_{ap}(f)$ is countable and for each positive real r*

$$A_r = \{x \in D_{ap}(f) : \mathrm{ap\,osc} f(x) \geq r\}$$

satisfies the condition (a). Then there are unilaterally approximately continuous and approximate jump functions g, h with $f = g + h$.

PROOF. If f is approximately continuous, then we can write $g = f$ and $h = 0$ and Theorem 2 is true in this case. So we suppose that f is not approximately continuous. There is a positive number $p \leq \frac{1}{2}$ such that $B_1 = A_p \neq \emptyset$. For $n > 1$ let $B_n = A_{\frac{p}{2^n}}$. Since $B_1 \neq \emptyset$ satisfies the condition (a), there are pairwise disjoint sets $U_{1,i} \in T_d$, $i \geq 1$ such that $U_{1,i} \cap (D_{ap}(f) \setminus B_1) = \emptyset$, the cardinality $\text{card}(B_1 \cap U_{1,i}) = 1$ and $B_1 \subset \bigcup_{i \geq 1} U_{1,i}$ and $\mathbb{R} \setminus \bigcup_{i \geq 1} cl_d(U_{1,i}) \in T_d$. Without loss of generality we may assume that $U_{1,i}$, $i \geq 1$ are F_σ-sets. For integer $i \geq 1$ denote by $u_{1,i}$ the only point belonging to $B_1 \cap U_{1,i}$.

By Zahorski's Lemma ([2, 7]) for $i \geq 1$ there are approximately continuous functions $\phi_{1,i} : \mathbb{R} \to [0,1]$ such that $\phi_{1,i}(u_{1,i}) = 1$, $0 < \phi_{1,i}(x) \leq 1$ for $x \in U_{1,i}$ and $\phi_{1,i}(\mathbb{R} \setminus U_{1,i}) = \{0\}$. Let $g_1(x) = f(x) + (f(u_{1,i}) - \text{ap}\lim_{t \to u_{1,i}^+} f(t))\phi_{1,i}(x)$ and $h_1(x) = f(x) - g_1(x)$ for $x \in U_{1,i} \cap (u_{1,i}, \infty)$ and $i \geq 1$ and $h_1(x) = (f(x) - \text{ap}\lim_{t \to u_{1,i}^-} f(t))\phi_{1,i}(x)$ and $g_1(x) = f(x) - h_1(x)$ for $x \in U_{1,i} \cap (-\infty, u_{1,i})$ and $i \geq 1$. Moreover let $g_1(x) = f(x)$ and $h_1(x) = 0$ otherwise on $\mathbb{R}$. Then $f = g_1 + h_1$, $D_{ap}(g_1) \cup D_{ap}(h_1) \subset D_{ap}(f)$ and the functions g_1, h_1 are unilaterally approximately continuous at all points $x \in B_1$ and approximate jump on $\mathbb{R}$.

In the second step we consider the set $B_2 \setminus B_1$. If $B_2 \setminus B_1 = \emptyset$, then we put $g_2 = g_1$ and $h_2 = h_1$. If $B_2 \setminus B_1 \neq \emptyset$, then we write $B_2 \setminus B_1 = \{u_{2,i} : i \geq 1\}$. Since the set B_2 satisfies the condition (a), there are pairwise disjoint F_σ-sets $U_{2,i} \in T_d$ such that for integer $i \geq 1$ we have $U_{2,i} \cap (B_1 \cup (D_{ap}(f) \setminus B_2)) = \emptyset$,

$$U_{2,i} \cap (B_2 \setminus B_1) = \{u_{2,i}\}, \;\; B_2 \setminus B_1 \subset \bigcup_{i \geq 1} U_{2,i}, \; \text{diam}(f(U_{2,i})) < \frac{1}{2},$$

and $\mathbb{R} \setminus \bigcup_{i \geq 1} cl_d(U_{2,i}) \in T_d$. By Zahorski's Lemma from [2, 7] for $i \geq 1$ there are approximately continuous functions $\phi_{2,i} : \mathbb{R} \to [0,1]$ such that $\phi_{2,i}(x) = 0$ for $x \in \mathbb{R} \setminus U_{2,i}$, $\phi_{2,i}(u_{2,i}) = 1$ and $0 < \phi_{2,i}(x) \leq 1$ for $x \in U_{2,i}$. Let $g_2(x) = g_1(x) + (g_1(u_{2,i}) - \text{ap}\lim_{t \to u_{2,i}^+} g_1(t))\phi_{2,i}(x)$ and $h_2(x) = f(x) - g_2(x)$ for $x \in U_{2,i} \cap (u_{2,i}, \infty)$ and $i \geq 1$,

$$h_2(x) = h_1(x) + (h_1(u_{2,i}) - \text{ap}\lim_{t \to u_{2,i}^-} h_1(t))\phi_{2,i}(x) \;\text{ and }\; g_2(x) = f(x) - h_2(x)$$

for $x \in U_{2,i} \cap (-\infty, u_{2,i})$ and $i \geq 1$. Moreover let $g_2(x) = g_1(x)$ and $h_2(x) = h_1(x)$ otherwise on $\mathbb{R}$. Then the functions g_2, h_2 are approximate jump functions unilaterally approximately continuous at all points $x \in B_2$ and $\max(|g_2(x) - g_1(x)|, |h_2(x) - h_1(x)|) < \frac{1}{2}$ and $g_2(x) + h_2(x) = f(x)$ for all $x \in \mathbb{R}$. Moreover $D_{ap}(g_2) \cup D_{ap}(h_2) \subset D_{ap}(f)$.

Generally in step $n > 2$ we consider the set $B_n \setminus B_{n-1}$. If $B_n \setminus B_{n-1} = \emptyset$, then we put $g_n = g_{n-1}$ and $h_n = h_{n-1}$. If $B_n \setminus B_{n-1} \neq \emptyset$ we proceed analogously as above and we construct approximate jump functions g_n and h_n unilaterally approximately continuous at all points $x \in B_n$ and such that

$$\max(|g_n(x) - g_{n-1}(x)|, |h_n(x) - h_{n-1}(x)|) < \frac{1}{2^{n-1}} \text{ for } x \in \mathbb{R},$$

$$g_n + h_n = f \text{ and } D_{ap}(g_n) \cup D_{ap}(h_n) \subset D_{ap}(f).$$

Then the sequences (g_n) and (h_n) uniformly converge and the limits $g = \lim_{n\to\infty} g_n$ and $h = \lim_{n\to\infty} h_n$ are approximate jump and unilaterally approximately continuous functions and

$$g + h = \lim_{n\to\infty} g_n + \lim_{n\to\infty} h_n = \lim_{n\to\infty}(g_n + h_n) = f. \quad \square$$

The following assertion follows from Remark 3 and Theorem 2.

Conclusion 2. *The set $\mathbb{Q}$ of all rationals does not satisfy the condition* (a)*.*

In the same way, we can prove that every set satisfying the condition (a) is nowhere dense.

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