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Ondřej Zindulka,*Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic.
e-mail: `zindulka@mat.fsv.cvut.cz`

SMALL OPAQUE SETS

Abstract

A set in a separable metric space is called *Borel–opaque* if it meets every Borel set of positive topological dimension. We show that if there is a set of reals with cardinality of the continuum and universal measure zero, then each separable space contains a Borel–opaque set that is of universal measure zero. Similar results hold for opaque sets that are perfectly meager, λ-sets, λ'-sets etc., and can be extended to some nonseparable spaces. On the other hand, we show that a σ-set is zero–dimensional. Using opacity we also construct universal measure zero sets of positive Hausdorff dimension.

1 Introduction

There are deep space nebulae whose density is less than the most perfect vacuum ever achieved in a laboratory and yet they are visible and look opaque. So they are simultaneously material and phantom. This is a motivation for what we call a small opaque set.

As explained below, opacity is related to dimension. Based on a result of Hilgers [8] (see also [11, §24a,VIa–1b]), Mazurkiewicz and Szpilrajn [12] proved the following result on the existence of a big set that is at the same time small.

Theorem 1.1 (Mazurkiewicz and Szpilrajn). (i) *If there is a universal measure zero set of reals with cardinality of the continuum, then for each $n \in \omega$ there is a universal measure zero set $Y \subseteq \mathbb{R}^{n+1}$ of dimension n.*

(ii) *If there is a λ-set of reals with cardinality of the continuum, then for each $n \in \omega$ there is a λ-set $Y \subseteq \mathbb{R}^{n+1}$ of dimension n.*

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In [22, Theorem 3.4], a similar set was constructed using Martin's Axiom.

Theorem 1.2 (Assume Martin's Axiom)**.** *Each analytic metrizable space X contains a universal measure zero set Y such that* $\dim Y = \dim X - 1$*. Moreover, the set meets every Borel subset of positive dimension.*

The present paper merges the two methods (that indeed are to some extent different) in order to get small opaque sets (and in particular small sets of positive topological dimension). We consider the following kinds of small sets: universal measure zero, perfectly and universally meager, σ-sets, λ and λ'-sets, $(s)_0$-sets and sets that do not admit a proper Borel–based *ccc* σ-ideal. The results also answer four out of five questions posed in the last paragraph of [22].

In Section 2 the notion of opacity is introduced. The construction of opaque sets is done in Section 3, the main result is Theorem 3.6. As a by–product of the construction, we also draw a result on the existence of zero–dimensional subspaces of full cardinality. In Section 4 we apply the construction to various small sets obtaining improvements and variations of Theorems 1.1 and 1.2.

It turns out that while the existence of universal measure zero sets of positive *topological* dimension is independent of ZFC, the Zermelo–Fraenkel set theory including the Axiom of Choice, universal measure zero sets of positive *Hausdorff* dimension always exist. We prove that in Section 5.

Throughout the paper we use the following notation. ω and ω_1 denote the first infinite and uncountable cardinal, respectively, and $\mathfrak{c}$ denotes the cardinal of continuum. $|A|$ denotes the cardinality of a set A. The Continuum Hypothesis is abbreviated by CH and its negation by $\neg$ CH. We shall also make use of the following well–known cardinals: $\operatorname{non}\mathbb{L}$, the minimal cardinality of a subset of $\mathbb{R}$ that is not Lebesgue null; $\operatorname{non}\mathbb{K}$, the minimal cardinality of a subset of $\mathbb{R}$ that is not meager; $\mathfrak{b}$, the minimal cardinality of a subset of ω^ω that is unbounded modulo finite sets.

2 Opaque Sets

If X is a topological space, then $\dim X$ denotes the covering dimension of X. Recall that X has *strongly infinite dimension* if it cannot be covered by countably many zero–dimensional sets.

Definition 2.1. Let X be a metric space and $\mathcal{C}$ a family of subsets of X. A set $Y \subseteq X$ is called *$\mathcal{C}$-opaque* (or *opaque w.r.t. $\mathcal{C}$*) if

$$C \cap Y \neq \emptyset \quad \text{for each } C \in \mathcal{C} \text{ such that } \dim C > 0.$$

If $\dim X = 0$, then the empty set is $\mathcal{C}$-opaque, and any dense set of reals is $\mathcal{C}$-opaque on $\mathbb{R}$ for any family $\mathcal{C}$. Opacity becomes interesting if $\dim X > 1$, see Proposition 2.2 *infra*.

If the family $\mathcal{C}$ satisfies a mild additional property, then one gets a little more. Let us call $\mathcal{C}$ *closed–complete* (*Borel–complete*) if $C \cap F \in \mathcal{C}$ for each $C \in \mathcal{C}$ and $F \subseteq X$ closed (Borel), respectively.

The following proposition relates opacity to the two theorems mentioned in the introduction.

Proposition 2.2. *Let Y be a $\mathcal{C}$-opaque set.*

(i) *If $\mathcal{C}$ is closed–complete, then* $\dim(C \cap Y) \geqslant \dim C - 1$ *for each* $C \in \mathcal{C}$. *In particular,* $\dim(C \cap Y) = \infty$ *whenever* $\dim C = \infty$. *In particular, if* $X \in \mathcal{C}$ *and* $\dim X > 1$, *then* $\dim Y > 0$.

(ii) *If $\mathcal{C}$ is Borel–complete, then* $\dim(C \cap Y)$ *is strongly infinite for each* $C \in \mathcal{C}$ *whose dimension is strongly infinite.*

PROOF. (i) Assume that there is $C \in \mathcal{C}$ such that $\dim(C \cap Y) < \dim C - 1$. By the *Enlargement Theorem* [3, 7.4.17] there is a G_δ-set $G \supseteq C \cap Y$ such that $\dim G = \dim(C \cap Y)$. By the *Addition Theorem* [3, Theorem 7.3.10],

$$\dim C \leqslant \dim G + \dim(C \setminus G) + 1 < \dim C - 1 + \dim(C \setminus G) + 1.$$

It follows that $\dim(C \setminus G) > 0$. The set $C \setminus G$ is an F_σ-subset of C. Therefore, by the *Countable Sum Theorem* [3, Theorem 7.2.1], there is a closed set F such that $C \cap F \subseteq C \setminus G$ and $\dim(C \cap F) > 0$. As $\mathcal{C}$ is closed–complete, it follows that $C \cap F \cap Y \neq \emptyset$. On the other hand,

$$C \cap F \cap Y \subseteq (C \setminus G) \cap Y = (C \cap Y) \setminus G = \emptyset.$$

We arrived to a contradiction. (ii) is proved in a similar manner: Assuming the dimension of C is strongly infinite and dimension of $C \cap Y$ is not, there is a $G_{\delta\sigma}$-set $G \supseteq C \cap Y$ such that the dimension of $C \setminus G$ is strongly infinite. By assumption, $C \setminus G \in \mathcal{C}$, hence it should meet Y. $\square$

To illustrate the notion of opacity, we provide some examples. As we will construct opaque sets from sets of reals, we consider only families of cardinality $\mathfrak{c}$ or less. (The argument will become clear below.)

Example 2.3 (Borel opacity)**.** We shall consider mainly $\mathcal{C}$–opaque sets for $\mathcal{C}$ the family of all separable Borel subsets of the underlying metrizable space X. This family is obviously Borel–complete.

If $|X| \leqslant \mathfrak{c}$, then $|\mathcal{C}| \leqslant \mathfrak{c}$. Indeed, there are only $|X|^\omega \leqslant \mathfrak{c}^\omega = \mathfrak{c}$ many countable subsets of X and therefore there are at most $\mathfrak{c}$ many closed separable subspaces. If $B \subseteq X$ is Borel and separable, then it is a Borel subset of its closure, which is separable. As a separable space has at most $\mathfrak{c}$ many Borel sets, we are done.

It follows from Proposition 2.2 that if $Y \subseteq X$ is $\mathcal{C}$-opaque, then $\dim Y \geqslant \dim Z - 1$ for each separable subset $Z \subseteq X$. In particular, if X is separable, then $\mathcal{C}$ is the family of all Borel sets and $|\mathcal{C}| \leqslant \mathfrak{c}$ and $\dim Y \geqslant \dim X - 1$. In this case, we call Y *Borel–opaque*.

As a variation, if X is analytic, one can take for $\mathcal{C}$ all analytic or projective sets.

Example 2.4 (Mapping onto)**.** Let X be a Euclidean space, or, more generally, a topological vector space. Let $\mathcal{L}$ be a family of linear operators (continuity is not required) on X such that $|\mathcal{L}| \leqslant \mathfrak{c}$ and $|\mathrm{rng}(\Lambda)| \leqslant \mathfrak{c}$ for each $\Lambda \in \mathcal{L}$.

For each $\Lambda \in \mathcal{L}$ choose a separable linear subspace of $F_\Lambda \subseteq \mathrm{null}(\Lambda)$ such that $\dim F_\Lambda = \dim \mathrm{null}(\Lambda)$ and for each $y \in \mathrm{rng}(\Lambda)$ let F_Λ^y be the unique affine subspace of $\Lambda^{-1}(y)$ that is a shift of F_Λ. Put

$$\mathcal{C} = \{B \subseteq F_\Lambda^y : B \text{ Borel}, \Lambda \in \mathcal{L},\ y \in \mathrm{rng}(\Lambda)\}.$$

Then $\mathcal{C}$ is obviously Borel–complete and $|\mathcal{C}| \leqslant \mathfrak{c}$. If Y is a $\mathcal{C}$-opaque set, then $\Lambda(Y) = \mathrm{rng}\,\Lambda$ whenever $\Lambda \in \mathcal{L}$ is not a linear isomorphism of X and $\mathrm{rng}\,\Lambda$. Indeed, if $y \in \mathrm{rng}\,\Lambda$, then $F_\Lambda^y \subseteq \Lambda^{-1}(y)$ is an affine linear subspace of X and therefore contains a homeomorphic copy C of the real line. As $\dim C = 1$ and $C \in \mathcal{C}$, there is $x \in C \cap Y$, whence $\Lambda x = y$.

It follows from the definition of $\mathcal{C}$ and Proposition 2.2 that $\dim Y \geqslant \dim \mathrm{null}(\Lambda) - 1$ for each $\Lambda \in \mathcal{L}$.

A particular instance of this situation is $|X| \leqslant \mathfrak{c}$ and $\mathcal{L}$ is the set of all projections of X on proper linear subspaces of X. Then an opaque set projects onto each such proper subspace and its dimension is strongly infinite if $\dim X = \infty$ and $\dim X - 1$ or more otherwise.

Another instance is X a Banach space and $\mathcal{L} \subseteq X^*$ a subset of its dual, $|\mathcal{L}| \leqslant \mathfrak{c}$. Then each $x^* \in \mathcal{L}$ takes an opaque set onto $\mathbb{R}$.

Example 2.5 (Arcs)**.** Let X be a metrizable space such that $|X| \leqslant \mathfrak{c}$ and $\mathcal{C}$ the family of all arcs in X (i.e. images of one–to–one continuous mappings $\phi : [0,1] \to X$) and their Borel subsets. Then $|\mathcal{C}| \leqslant \mathfrak{c}$. If $Y \subseteq X$ a $\mathcal{C}$-opaque set, then any path between any two distinct points in X passes through Y. Also, $\dim Y \geqslant \dim K - 1$ for each arc K in X.

Example 2.6 (Visibility)**.** Let X be a topological vector space, $|X| \leqslant \mathfrak{c}$, $x \in X$ and $\mathcal{C} = \{\{x + \lambda y : \lambda > 0\} : y \in X\}$ be the set of all rays emanating from x. A $\mathcal{C}$-opaque set is visible from the point x in every direction.

3 Construction of Opaque Sets

In order to construct opaque sets, we employ the following theorem that follows easily from [22, Theorem 2.1] and its proof.

Theorem 3.1. *For each metric space X there is a sequence $\langle h_m : m \in \omega\rangle$ of Lipschitz functions $h_m : X \to [0,1]$ such that*

$$\dim\bigl(X \setminus \bigcup_{m\in\omega} h_m^{-1}(r)\bigr) = 0 \text{ for each } r \in (0,1).$$

PROOF. We recall the construction from the proof of [22, Theorem 2.1]. Let $\langle \mathcal{B}_i : i \in \omega\rangle$ be a sequence of discrete open families such that $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$ is a base of X. For each $B \in \mathcal{B}$ let $\phi_B : X \to [0,1]$ be a continuous function such that $B = \phi_B^{-1}(0,1]$. As $\mathcal{B}_i$'s are discrete, for each $i \in \omega$ the formula $f_i = \sum_{B\in\mathcal{B}_i} \phi_B$ defines a continuous function $f_i : X \to [0,1]$. For each $i, j \in \omega$ and $x \in X$ set $g_{ij}(x) = \min\bigl(1, j f_i(x)\bigr)$. The proof of [22, Theorem 2.1] shows that

$$\dim\bigl(X \setminus \bigcup_{i,j\in\omega} g_{ij}^{-1}(r)\bigr) = 0 \text{ for each } r \in (0,1). \tag{3.1}$$

Let d be the metric on X. To have g_{ij} Lipschitz, set $\phi_B(x) = \min\bigl(1, \underline{d}(x, X \setminus B)\bigr)$ (where $\underline{d}$ is the lower distance). We still have $B = \phi_B^{-1}(0,1]$, so (3.1) remains valid. As the family $\mathcal{B}_i$ is disjoint for each $i \in \omega$, routine application of triangle inequality to the definitions of ϕ_B and f_i gives $f_i(x) - f_i(y) \leqslant d(x,y)$. Hence f_i is 1-Lipschitz and g_{ij} is j-Lipschitz for each $i, j \in \omega$. Rename g_{ij}'s to h_m's. □

Throughout this section we adopt the following notation. X is a metrizable space, $\langle h_m : m \in \omega\rangle$ denotes the sequence from the preceding theorem and for $r \in (0,1)$,

$$F_r = \bigcup_{m\in\omega} h_m^{-1}(r), \quad G_r = X \setminus \bigcup_{m\in\omega} h_m^{-1}(r). \tag{3.2}$$

Claim 3.2. (i) $\dim G_r = 0$ *for each* $r \in (0,1)$,

(ii) $\bigcup_{r\in E} G_r = X$ *for each uncountable set* $E \subseteq (0,1)$.

PROOF. (i) is just a restatement of Theorem 3.1. For (ii), we have to show that $\bigcap_{r\in E} F_r = \emptyset$. So assume that there is $x \in X$ such that $x \in F_r$ for each $r \in E$. Then for each $r \in E$ there is $m(r) \in \omega$ such that $h_{m(r)}(x) = r$. As E is uncountable, there must be $m \in \omega$ and distinct $r, s \in (0,1)$ such that $m = m(r) = m(s)$. Therefore $h_m(x) = r$ and $h_m(x) = s$, a contradiction. □

We now step aside to draw two corollaries on zero–dimensional subspaces.

Theorem 3.3. *Let X be a metrizable space.*

(i) *X is a union of ω_1 many G_δ zero–dimensional subspaces.*

(ii) *For any cardinal $\kappa \leqslant \mathfrak{c}$ of uncountable cofinality, X is a union of an increasing sequence $\langle H_\alpha : \alpha < \kappa\rangle$ of zero–dimensional subspaces.*

(iii) *If $\mathrm{cf}|X| > \omega_1$ or $\mathrm{cf}|X| < \mathfrak{c}$, then X contains a zero–dimensional subspace of cardinality $|X|$.*

(iv) *If no zero–dimensional subspace of X has cardinality $|X|$, then $\mathrm{cf}|X| = \omega_1$ and CH holds.*

PROOF. (i) is an obvious consequence of Claim 3.2.

(ii) Let κ be a cardinal, $\omega_1 \leqslant \kappa \leqslant \mathfrak{c}$. Take a set $E = \{r_\alpha : \alpha < \kappa\} \subseteq (0,1)$ and consider the sets G_{r_α}, $\alpha < \kappa$, defined by (3.2). For each $\alpha < \kappa$ put

$$H_\alpha = \bigcap_{\alpha\leqslant\beta<\kappa} G_{r_\beta}. \tag{3.3}$$

Using $\mathrm{cf}\,\kappa \geqslant \omega_1$ it is routine to deduce from 3.2(ii) that $\bigcup_{\alpha<\kappa} H_\alpha = X$.

(iii) If $\mathrm{cf}|X| > \omega_1$, let $\kappa = \omega_1$. If $\mathrm{cf}|X| < \mathfrak{c}$, let $\kappa > \mathrm{cf}|X|$ be any regular cardinal such that $\kappa \leqslant \mathfrak{c}$. Consider the family (3.3). In either case there is $\alpha < \kappa$ such that $|H_\alpha| = |X|$. (iv) is a restatement of (iii). □

Hurewicz [9] constructed under CH an uncountable separable metric space X whose each finitely–dimensional subspace is countable. This together with the previous theorem yields the following consequence.

Corollary 3.4. *Each of the following is equivalent to $\neg$ CH.*

(i) *Every metrizable space X has a zero–dimensional subspace of cardinality $|X|$.*

(ii) *Every uncountable separable metrizable space has an uncountable zero–dimensional subspace.*

We now get back to the construction of opaque sets.

Definition 3.5. Let X and Y be topological spaces and $\mathcal{I}$ and $\mathcal{J}$ families of subsets of X and Y, respectively ($\mathcal{I}$ and $\mathcal{J}$ are usually ideals). We write $\mathcal{I} \leqslant \mathcal{J}$ if the following holds: *If $A \subseteq X$ and $f : X \to Y$ is a continuous mapping such that $f \restriction A : A \to Y$ is one–to–one and $f(A) \in \mathcal{J}$, then $A \in \mathcal{I}$.*

Theorem 3.6. *Let X be a metrizable space, $\mathcal{C}$ a family of subsets of X and $\mathcal{I}$ a σ-ideal on X. Let $E \subseteq \mathbb{R}$ be such that $|E| = |\mathcal{C}|$ and let $\mathcal{J}$ be a σ-ideal on $\mathbb{R}$. If $\mathcal{I} \leqslant \mathcal{J}$ and $E \in \mathcal{J}$, then X contains a $\mathcal{C}$-opaque set $Y \in \mathcal{I}$.*

Proof. We may assume that $E \subseteq (0,1)$. Arrange the family $\mathcal{C}$ in a sequence $\langle C_r : r \in E\rangle$. Consider the sequence $\langle h_m : m \in \omega\rangle$ from Theorem 3.1. For each $m \in \omega$ put

$$E_m = \{r \in E : C_r \cap h_m^{-1}(r) \neq \emptyset\}.$$

For $r \in E_m$ pick a point $y(m,r) \in C_r \cap h_m^{-1}(r)$. Finally put

$$Y_m = \{y(m,r) : r \in E_m\}, \quad Y = \bigcup_{m\in\omega} Y_m.$$

We assert that $Y \in \mathcal{I}$. Indeed, note that, for each $h_m \restriction Y_m : Y_m \to (0,1)$ is one–to–one and that $Y_m \subseteq h_m^{-1} E_m$. As $E \in \mathcal{J}$ and $\mathcal{I} \leqslant \mathcal{J}$, we have $Y_m \in \mathcal{I}$. As $\mathcal{I}$ is σ-complete, we are done.

We show that Y is $\mathcal{C}$-opaque. Let $r \in E$ and assume that $C_r \cap Y = \emptyset$. Then $C_r \cap Y_m = \emptyset$ for each $m \in \omega$, which in turn implies $C_r \cap h_m^{-1}(r) = \emptyset$. It follows (cf. (3.2)) that $C_r \cap F_r = \emptyset$, i.e. $C_r \subseteq G_r$. By Claim 3.2(i), $\dim C_r = 0$. The proof is complete. $\square$

4 Small Opaque Sets

We now apply Theorem 3.6 to investigate which of the following sets can be opaque and/or have positive dimension. The map shows inclusions in the realm of separable metric spaces. General references are [13, 14, 2]. In [2] questions related to dimension are discussed.

$$\boxed{\begin{array}{ccccccc}
 & & & & & & \text{strong measure zero} \\
 & & & & & & \downarrow \\
 & & \sigma & & \text{universally small} & \to & \text{universal measure zero} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\lambda' & \to & \lambda & \to & \text{universally meager} & \to & (s)_0
\end{array}}$$

Strong Measure Zero Sets

A set E in a metric space X is a *strong measure zero set* if, given any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals E can be covered by a sequence $\langle E_n : n \in \omega \rangle$ of sets, each E_n of diameter at most ε_n.

Fact 4.1 ([12], [22, 5.1])**.** *Every strong measure zero set has dimension zero.*

σ-Sets

A set E in a separable metric space X is a *σ-set* if every F_σ-subset of E is G_δ in E. E is of *bounded Borel rank* if there is $\alpha < \omega_1$ such that each Borel set in E belongs to Σ^0_α. Obviously, every σ-set is of bounded Borel rank.

Fact 4.2. *Each set of bounded Borel rank (in particular each σ-set) has dimension zero.*

PROOF. This is an easy consequence of a deep result of Irek Recław, see [14, Theorem 17]: *A set of bounded Borel rank does not map continuously onto* $[0,1]$. Let E be a set of bounded Borel rank and d a metric on E. Assume $\dim E > 0$. According to the definition of the small inductive dimension, there is a point $x \in E$ and $\varepsilon > 0$ such that for each $r \leqslant \varepsilon$ the sphere $\{y \in E : d(x,y) = r\}$ is nonempty. It follows that the continuous mapping $\phi : X \to [0,\varepsilon]$ defined by $\phi(y) = \min(\varepsilon, d(x,y))$ is onto. Apply Recław's theorem. $\square$

Universal Measure Zero Sets

A set E in a metric space X has *universal measure zero* if there is no nontrivial finite diffused Borel measure in E. It is well–known that for any two metric spaces, the underlying σ-ideals of universal measure zero sets $\mathcal{I}$ and $\mathcal{J}$ satisfy $\mathcal{I} \leqslant \mathcal{J}$. Thus the next assertion follows from Example 2.3 and Theorem 3.6.

Theorem 4.3. *If there is a set $E \subseteq \mathbb{R}$ of universal measure zero such that $|E| = \mathfrak{c}$, then each metrizable space such that $|X| \leqslant \mathfrak{c}$ contains a universal measure zero set Y that is opaque w.r.t. separable Borel sets.*

Grzegorek [5] shows that there is a universal measure zero set $E \subseteq \mathbb{R}$ of cardinality $\operatorname{non} \mathbb{L}$. Thus we have

Corollary 4.4. *The following are equivalent and implied by* $\operatorname{non} \mathbb{L} = \mathfrak{c}$*:*

(i) *There is a universal measure zero set $E \subseteq \mathbb{R}$ such that $|E| = \mathfrak{c}$.*

(ii) *There is a separable universal measure zero set X of positive dimension.*

(iii) *Each separable metric space X contains a universal measure zero set Y such that* $\dim Y \geqslant \dim X - 1$*. If X has strongly infinite dimension, then so has Y.*

(iv) *Each metric space of cardinality $\leqslant \mathfrak{c}$ contains a universal measure zero set that is opaque w.r.t. separable Borel sets.*

Using Grzegorek's theorem and the idea of Example 2.4 yields the following absolute result. Provide the interval $[0, \pi)$ with Lebesgue measure. Given $\theta \in [0, \pi)$, denote by $\mathrm{proj}_\theta : \mathbb{R}^2 \to L_\theta$ the orthogonal projection on the unique line L_θ through the origin that makes angle θ with the x-axis. Each L_θ hosts a linear Lebesgue measure. Let's say that a set has full outer measure if its complement has inner measure zero.

Theorem 4.5. *There is a universal measure zero set $Y \subseteq \mathbb{R}^2$ such that*

$$\bigl\{\theta \in [0, \pi) : \mathrm{proj}_\theta Y \text{ has full outer Lebesgue measure}\bigr\}$$

has full outer measure.

PROOF. Let $E \subseteq (0, 1)$ have universal measure zero and $|E| = \mathrm{non}\,\mathbb{L}$. Let $A \subseteq [0, \pi)$ and $B \subseteq \mathbb{R}$ have full outer Lebesgue measure and $|A| = |B| = \mathrm{non}\,\mathbb{L}$. Provide each L_θ with an isometric copy B_θ of B. Consider the family $\mathcal{C} = \bigl\{\mathrm{proj}_\theta^{-1} y : y \in B_\theta,\ \theta \in A\bigr\}$. Then $|\mathcal{C}| = \mathrm{non}\,\mathbb{L} = |E|$. Apply Theorem 3.6 to get a $\mathcal{C}$-opaque set $Y \subseteq \mathbb{R}^2$ that has universal measure zero. As each element of $\mathcal{C}$ is a line, it has dimension 1 and thus is met by Y. It follows that Y is the required set. $\square$

Corollary 4.6. *There is a universal measure zero set $Y \subseteq \mathbb{R}^2$ such that*

$$\dim_H \bigl\{\theta \in [0, \pi) : \dim_H \mathrm{proj}_\theta Y = 1\bigr\} = 1.$$

These theorems easily generalize to higher dimensions.

Perfectly Meager and Universally Meager Sets

A set is *perfect* if it has no isolated points. A set E in a metric space X is *perfectly meager* if every perfect subset of E is meager in itself.

Recently Piotr Zakrzewski [20] proved that Grzegorek's [6, 7] absolutely first category sets and $\overline{AFC}$ sets coincide and called them universally meager sets. By the definition, E is *universally meager* if, for every perfect Polish space Z, a subspace $Y \subseteq Z$ and a Borel one–to–one mapping $f : Y \to E$, Y is meager in Z. Obviously, every universally meager set is perfectly meager. It follows at once from results of Zakrzewski [20, Theorem 2.2], Recław [15] and Bartoszyński [1] that whether every perfectly meager set is universally meager is independent of ZFC.

Zakrzewski gives eight equivalent conditions that characterize universally meager sets. Here is one more.

Proposition 4.7. *X is universally meager if and only if for each one–to–one continuous mapping $f : Y \to X$ from a separable metric space, Y is perfectly meager.*

PROOF. We prove the if part. Let Z be a perfect Polish space, $Y \subseteq Z$ and $f : Y \to X$ a one–to–one Borel mapping. Let $\{B_n : n \in \omega\}$ be a base for X. The sets $f^{-1}B_n$ are Borel. Hence they have the Baire property. Therefore there are meager sets I_n, $n \in \omega$, such that $U_n = f^{-1}B_n \triangle I_n$ is open in Y for each $n \in \omega$. Put $I = \bigcup_{n\in\omega} I_n$, $U = Y \setminus I$ and $\phi = f \restriction U$. Obviously $\phi^{-1}B_n = U \cap U_n$ for each $n \in \omega$; so $\phi : U \to X$ is continuous. Therefore U is perfectly meager and thus meager in Z. As I is meager, we are done. $\square$

Obviously, for any two metric spaces X and Y the underlying σ-ideals of universally meager sets $\mathcal{I}$ and $\mathcal{J}$ satisfy $\mathcal{I} \leqslant \mathcal{J}$. Grzegorek [5] shows that there is a meager set in $\mathbb{R}$ of cardinality $\operatorname{non}\mathbb{K}$. Thus we have the counterpart to Theorem 4.3 and Corollary 4.4.

Theorem 4.8. *The following are equivalent and implied by* $\operatorname{non}\mathbb{K} = \mathfrak{c}$*:*

(i) *There is a universally meager set $E \subseteq \mathbb{R}$ such that $|E| = \mathfrak{c}$.*

(ii) *Each separable metric space contains a universally meager Borel–opaque set.*

(iii) *There is a universally meager set X of positive dimension.*

There is also a category couterpart of Theorem 4.5.

Theorem 4.9. *There is a universally meager set $Y \subseteq \mathbb{R}^2$ such that the set*

$$\big\{\theta \in [0,\pi) : \operatorname{proj}_\theta Y \text{ is not meager}\big\}$$

is not meager.

λ-Sets

A set E in a metric space X is a λ-*set* if every countable subset of X is G_δ. Being a λ-set is not a σ-additive property. On the other hand, if X, Y are metric spaces and $\mathcal{I}$, $\mathcal{J}$ the underlying families of λ-sets, then $\mathcal{I} \leqslant \mathcal{J}$ ([13, Lemma 9.3.1(b)]). Also, there is a λ-set of cardinality $\mathfrak{b}$ ([17], see also [14, Theorem 21]). Thus Theorem 1.1 and Theorem 3.6 yield

Theorem 4.10. *The following are equivalent and implied by* $\mathfrak{b} = \mathfrak{c}$*:*

(i) *There is a* λ*-set* $E \subseteq \mathbb{R}$ *such that* $|E| = \mathfrak{c}$.

(ii) *Each metrizable space of cardinality* $\leqslant \mathfrak{c}$ *contains a union of countably many* λ*-sets that is opaque w.r.t. separable Borel sets.*

(iii) *There is a separable* λ*-set of positive dimension.*

Note that as every λ-set is perfectly meager ([13, Theorem 5.2]), it follows from Proposition 4.7 that every λ-set is universally meager.

λ'-Sets

A set E in a metric space X is a λ'*-set* $(\mathrm{rel}\, X)$ if $E \cup D$ is a λ-set for each countable set $D \subseteq X$. So E is a λ-set iff it is a λ'-set $(\mathrm{rel}\, E)$. In particular, every λ'-set is a λ-set. Unlike other small sets, being a λ'-set is not an intrinsic property.

Being a λ'-set is a σ-additive property ([18], see [13, Theorem 7.1]). If X, Y are metric spaces and $\mathcal{I}$, $\mathcal{J}$ the underlying σ-ideals of λ'-sets, then $\mathcal{I} \leqslant \mathcal{J}$ ([19], see [13, Lemma 9.3.1(c)]; the proof therein is incorrect, yet the lemma holds). Thus Theorem 3.6 yields the following theorem.

Theorem 4.11. *The following are equivalent:*

(i) *There is a* λ'*-set* $(\mathrm{rel}\, \mathbb{R})$ *of cardinality* $\mathfrak{c}$.

(ii) *There is a compact metric space* X *that contains a* λ'*-set* $(\mathrm{rel}\, X)$ *of positive dimension.*

(iii) *Each separable metric space* X *contains a* λ'*-set* $Y (\mathrm{rel}\, X)$ *such that* $\dim Y \geqslant \dim X - 1$. *If* X *has strongly infinite dimension, then so has* Y.

(iv) *Each metric space* X *of cardinality* $\leqslant \mathfrak{c}$ *contains a* λ'*-set* $(\mathrm{rel}\, X)$ *that is opaque w.r.t. separable Borel sets.*

Note that while it is known that CH implies (i), the discussion around Theorems 21–23 in [14] reveals that $\mathfrak{b} = \mathfrak{c}$ is not enough: In the Laver model all λ'-sets in $\mathbb{R}$ have cardinality at most ω_1 while $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$.

Universally Small Sets

Following [21], call a set E in a separable metric space X *universally small* if it belongs to every Borel–based *ccc* σ-ideal on X. Equivalently, if there are no nontrivial Borel–based *ccc* σ-ideals on E. Being universally small is obviously a σ-additive property that is preserved by one–to one preimages. It is known that there is a universally small set of cardinality ω_1, see [13, Theorem 5.3]. Thus Theorem 3.6 yields

Theorem 4.12. *The following are equivalent and implied by* CH*:*

(i) *There is a universally small set in $\mathbb{R}$ of cardinality $\mathfrak{c}$.*

(ii) *There is universally small set of positive dimension.*

(iii) *Each separable metrizable space contains a Borel–opaque universally small set.*

$(s)_0$-Sets

A set E in a separable metric space X is an $(s)_0$*-set* $(\operatorname{rel} X)$ if for every perfect compact set $P \subseteq X$ there is a perfect set $Q \subseteq P \setminus E$. This property obviously depends on the space X and is not therefore intrinsic. To overcome this trouble we define E to be an *absolutely* $(s)_0$*-set* if it is an $(s)_0$-set $(\operatorname{rel} X)$ for each separable metric space $X \supseteq E$. The following follows easily from the Perfect Set Theorem.

Proposition 4.13. *If there is an analytic space $X \supseteq E$ such that E is an $(s)_0$-set* $(\operatorname{rel} X)$*, then E is absolutely $(s)_0$.*

It is known that there is an $(s)_0$-set $(\operatorname{rel} \mathbb{R})$ of cardinality $\mathfrak{c}$. By the above proposition, it is absolutely $(s)_0$. As being absolutely $(s)_0$ is a σ-additive property that is preserved by one–to–one preimages (see [2]), Theorem 3.6 yields the following fact.

Theorem 4.14. *Each separable metrizable space contains a Borel–opaque absolutely $(s)_0$-set. Thus there are absolutely $(s)_0$-sets of positive dimension.*

5 Universal Measure Zero Sets with Positive Hausdorff Dimension

In this section we apply Theorem 3.6 to Hausdorff measures. The goal is to get, without any extra axioms, universal measure zero sets with positive Hausdorff dimension. Recall that, given $s \geqslant 0$, the s-dimensional Hausdorff measure on a separable metric space X with metric d is defined as

$$\mathcal{H}^s(E) = \sup_{\delta>0} \inf \sum_n (dE_n)^s, \quad E \subseteq X,$$

where dE_n stands for the diameter of E_n and the infima are taken over all finite or countable covers $\{E_n\}$ of E by sets of diameter at most δ. We refer to [4] or [16] for properties of $\mathcal{H}^s$. The *Hausdorff dimension* of a set $E \subseteq X$ is defined by $\dim_H E = \sup\{s > 0 : \mathcal{H}^s(E) > 0\}$.

Lemma 5.1. *Let X be a metric space. If $\dim X \geqslant n \in \omega$, then there is a countable family $\mathcal{F}$ of Lipschitz mappings $f : X \to [0,1]^n$ such that for each $r \in (0,1)^n$ there is $f \in \mathcal{F}$ with $\dim f^{-1}(r) \geqslant \dim X - n$.*

Proof. We use the notation of (3.2). Consider the functions h_m from Theorem 3.1. For each $\iota \in \omega^n$ define a function $f_\iota : X \to [0,1]^n$ by

$$f_\iota(x) = \langle h_{\iota(j)}(x) : j < n\rangle.$$

As all h_m's are Lipschitz, so is f_ι. Let $\mathcal{F} = \{f_\iota : \iota \in \omega^n\}$. We assert that $\mathcal{F}$ is the required family. It is obviously countable. Let $r \in (0,1)^n$. By Claim 3.2(i), $\dim G_{r(j)} = 0$ for all $j < n$. The *Decomposition* and *Addition Theorems* [3, Theorems 7.7.9 and 7.3.10] thus yield $\dim \bigcup_{j=0}^{n-1} G_{r(j)} \leqslant n-1$ and

$$\dim \bigcap_{j=0}^{n-1} F_{r(j)} \geqslant \dim X - \dim \bigcup_{j=0}^{n-1} G_{r(j)} - 1 \geqslant \dim X - n.$$

On the other hand

$$\bigcap_{j=0}^{n-1} F_{r(j)} = \bigcap_{j=0}^{n-1} \bigcup_{m\in\omega} h_m^{-1}\bigl(r(j)\bigr) = \bigcup_{\iota\in\omega^n} \bigcap_{j=0}^{n-1} h_{\iota(j)}^{-1}\bigl(r(j)\bigr) = \bigcup_{\iota\in\omega^n} f_\iota^{-1}(r).$$

As $f_\iota^{-1}(r)$ is closed for each $\iota \in \omega^n$, by the *Countable Sum Theorem* [3, Theorem 7.2.1] there is $\iota \in \omega^n$ such that $\dim f_\iota^{-1}(r) = \dim \bigcap_{j=0}^{n-1} F_{r(j)} \geqslant \dim X - n$, as required. $\square$

In connection with the following theorem we mention the classical result of [10]: If X is a separable metric space, then $\dim_H X \geqslant \dim X$.

Theorem 5.2. *Let X be a separable metric space. If $\dim X > n \in \omega$, then there is a set $Y \subseteq X$ of universal measure zero such that $\mathcal{H}^n(Y) = \infty$.*

Proof. By Lemma 5.1 there is a countable family $\mathcal{F}$ of Lipschitz mappings $f : X \to [0,1]^n$ such that for each $r \in (0,1)^n$ there is $f \in \mathcal{F}$ with $\dim f^{-1}(r) > 0$. Choose $B \subseteq (0,1)^n$ such that $|B| = \operatorname{non} \mathbb{L}$ and $\mathcal{H}^n(B) > 0$. For each $f \in \mathcal{F}$ let $B(f) = \{r \in B : \dim f^{-1}(r) > 0\}$. Then $\bigcup_{f\in\mathcal{F}} B(f) = B$. Therefore there is $g \in \mathcal{F}$ such that $\mathcal{H}^n\bigl(B(g)\bigr) > 0$. Put $\mathcal{C} = \{g^{-1}(r) : r \in B(g)\}$. By the above metioned Grzegorek's theorem there is a set $E \subseteq \mathbb{R}$ of universal measure zero such that $|E| = \operatorname{non} \mathbb{L}$. Obviously $|E| = |\mathcal{C}|$. Apply Theorem 3.6 to get a $\mathcal{C}$-opaque set $Y \subseteq X$ that has universal measure zero.

We prove that Y is the required set. As each $C \in \mathcal{C}$ has positive dimension, it is met by Y. Therefore g maps Y onto $B(g)$. In particular, $\mathcal{H}^n\big(g(Y)\big) \geqslant \mathcal{H}^n\big(B(g)\big) > 0$. As g is Lipschitz, [4, Lemma 6.1] yields $\mathcal{H}^n\big(g(Y)\big) \leqslant L^n \mathcal{H}^n(Y)$, where L is the Lipschitz constant of g. It follows that $\mathcal{H}^n(Y) > 0$.

If $\mathcal{H}^n(Y) < \infty$, then its restriction to Y would be a finite Borel measure in Y witnessing to Y not having universal measure zero. Thus $\mathcal{H}^n(Y) = \infty$. □

Corollary 5.3. *Each separable metric space X contains a set Y of universal measure zero such that*

(i) *if $\dim X < \infty$, then $\mathcal{H}^{\dim X - 1}(Y) = \infty$ and thus $\dim_H Y \geqslant \dim X - 1$,*

(ii) *if $\dim X = \infty$, then $\mathcal{H}^s(Y) = \infty$ for all $s > 0$ and thus $\dim_H Y = \infty$.*

PROOF. (i) is obvious. To prove (ii), construct for each $n \in \omega$ a universal measure zero set $Y_n \subseteq X$ such that $\mathcal{H}^n(Y_n) > 0$ and put $Y = \bigcup_{n\in\omega} Y_n$. □

Corollary 5.4. *For each $n \in \omega$ there is a set $Y \subseteq \mathbb{R}^{n+1}$ of universal measure zero such that $\mathcal{H}^n(Y) = \infty$.*

Thus we have "real" examples of sets that have positive Hausdorff dimension and universal measure zero. I learned from David Fremlin that he has this result for $n = 1$ (published on the web).

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