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ON THE NON-COMPACTNESS OF MAXIMAL OPERATORS

Abstract

It is proved that if B is a convex quasi-density basis and E is a symmetric space on $\mathbb{R}^n$ with respect to Lebesgue measure, then there do not exist non-orthogonal weights w and v for which the maximal operator M_B corresponding to B acts compactly from the weight space E_w to the weight space E_v.

1 Definitions and Notation

A mapping B defined on $\mathbb{R}^n$ is said to be a *differentiation basis in* $\mathbb{R}^n$ (see, e.g., [1]) if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of open bounded sets containing the point x such that there exists a sequence $\{R_k\} \subset B(x)$ with $\operatorname{diam} R_k \to 0$ $(k \to \infty)$.

By M_B we mean the *maximal operator corresponding to the differentiation basis* B; that is,

$$M_B f(x) = \sup_{R \in B(x)} \frac{1}{|R|} \int_R |f| \quad (f \in L_{loc}(\mathbb{R}^n), \quad x \in \mathbb{R}^n).$$

The basis B is said to *differentiate the integral of the function* f if for almost every $x \in \mathbb{R}^n$ the integral mean $\frac{1}{|R|} \int_R f$ tends to $f(x)$ when $R \in B(x)$, $\operatorname{diam} R \to 0$.

The basis B is called:

a *density basis* if B differentiates the integral of the characteristic function of every measurable set,

convex if for every $x \in \mathbb{R}^n$ the collection $B(x)$ consists of convex sets,

translation invariant if $B(x) = \{x + R : R \in B(0)\}$ for any $x \in \mathbb{R}^n$,

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Busemann–Feller if for any $R \in \bigcup_{x\in\mathbb{R}^n} B(x)$ we have that $R \in B(y)$ for every $y \in R$.

We call the basis B:

quasi-density if it contains some density basis; i.e., if there exists a density basis H such that $H(x) \subset B(x)$ $(x \in \mathbb{R}^n)$,

measurable if $M_B f$ is a measurable function for any $f \in L_{loc}(\mathbb{R}^n)$.

Note that: 1) any translation invariant convex basis is a quasi-density basis (see [1, Ch. I, §3]; 2) any translation invariant basis and any Busemann-Feller basis are measurable (as is easy to verify).

Denote by $\mathbb{Q}$ the differentiation basis for which $\mathbb{Q}(x)$ $(x \in \mathbb{R}^n)$ consists of all cubic intervals containing x. Recall that $M_{\mathbb{Q}}$ is called the *Hardy–Littlewood maximal operator*.

Let (X, S, μ) be a measure space and let Δ be the class of all μ-measurable functions defined on X. The normed function space (function space for short) E is said to be *ideal* (see, e.g., [2]) if

$$x \in \Delta,\ y \in E,\ |x| \le |y|\ \mu\text{ -a.e.} \Rightarrow x \in X \text{ and } \|x\|_{_E} \le \|y\|_{_E}.$$

The function space E on (X, S, μ) is said to be *symmetric* if it is ideal and

$$x \in \Delta,\ y \in E,\ x \text{ is equimeasurable with } y \Rightarrow x \in X \text{ and } \|x\|_{_E} = \|y\|_{_E}.$$

Let E be a symmetric space on $\mathbb{R}^n$ with respect to Lebesgue measure and w be a locally integrable and non-negative function on $\mathbb{R}^n$ (i.e., w is a weight). Denote by E_w the set of all measurable functions f for which there is a function $g \in E$ such that

$$|\{|f| > t\}|_w = |\{|g| > t\}|\ (t > 0), \tag{1}$$

where $|\cdot|_w = w\,dx$ and $|\cdot| = dx$. The norm in E_w is defined as follows. For $f \in E_w$, $\|f\|_{E_w} = \|g\|_{_E}$, where g is some function from E satisfying (1). E_w is called *the space E with respect to the weight w*. Note that E_w is a symmetric space on the measure space $(\mathbb{R}^n, w\,dx)$.

For the symmetric space E on the measure space (X, S, μ) let

$$\varphi_{_E}(t) = \|\chi_A\|_{_E}\ (t \ge 0),$$

where $A \in S$ and $\mu(A) = t$. $\varphi_{_E}$ is called a *fundamental function of E*.

We call the symmetric space E *regular* if $\lim\limits_{t\to 0+} \varphi_{_E}(t) = 0$.

We call the weights w and v *non-orthogonal* if

$$|\{x \in \mathbb{R}^n:\ w(x) > 0,\ v(x) > 0\}| > 0.$$

Let w and v be non-orthogonal weights. Obviously, there is $k \in \mathbb{N}$ such that $|\{x \in \mathbb{R}^n : \frac{1}{k} \le w(x),\ v(x) \le k\}| > 0$. Due to this note we can introduce the notation

$$c_{w,v} = \sup\Big\{\frac{c_1}{c_2} : 0 < c_1 < c_2,\ |\{x \in \mathbb{R}^n : c_1 \le w(x),\ v(x) \le c_2\}| > 0\Big\}.$$

2 Results

Edmunds and Meskhi [3] proved the following theorem. *Let $1 < p < \infty$. Then there do not exist almost everywhere positive weights w and v on $\mathbb{R}^n$ for which the Hardy–Littlewood maximal operator $M_{\mathbb{Q}}$ acts compactly from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$.*

The following generalization of this result is true.

Theorem. *Let B be a convex quasi-density measurable basis, E be a symmetric space on $\mathbb{R}^n$ with respect to Lebesgue measure. Then for any non-orthogonal weights w and v on $\mathbb{R}^n$ and for any $\varepsilon \in (0,1)$ there exists a sequence of sets $\{A_k\}$ with $\{\chi_{A_k}\} \subset E_w$, $\|\chi_{A_k}\|_{E_w} > 0$ $(k \in \mathbb{N})$ such that*

$$\|(M_B f_m - M_B f_k)\chi_{A_m}\|_{E_v} > (1-\varepsilon)c_{w,v} \ \text{ when } \ m > k,$$

where $f_k = \chi_{A_k} / \|\chi_{A_k}\|_{E_w}$ $(k \in \mathbb{N})$. Furthermore, if E is a regular symmetric space, then the same conclusion is true for quasi-density measurable bases.

Corollary. *Let B be a convex quasi-density basis, E be a symmetric space on $\mathbb{R}^n$ with respect to Lebesgue measure. Then there do not exist non-orthogonal weights w and v on $\mathbb{R}^n$ for which M_B acts compactly from E_w to E_v. Furthermore, if E is a regular symmetric space, then the same conclusion is true for quasi-density bases.*

3 Auxiliary Statements

Lemma 1. *Let E be a symmetric space on $\mathbb{R}^n$ with respect to Lebesgue measure, w and v be weights on $\mathbb{R}^n$. Suppose $0 < c_1 < c_2$ and the set*

$$H = \{x \in \mathbb{R}^n : c_1 \le w(x),\ v(x) \le c_2\}$$

is of positive measure. Then for every set $A \subset H$ with $|A| > 0$ the inequalities

$$0 < \varphi_E(c_1|A|) \le \|\chi_A\|_{E_w},\ \|\chi_A\|_{E_v} \le \varphi_E(c_2|A|) \ \text{ and } \ \frac{c_1}{c_2} \le \frac{\|\chi_A\|_{E_w}}{\|\chi_A\|_{E_v}} \le \frac{c_2}{c_1}$$

hold.

Proof. As is known (see [2, Ch. II, §4]), the fundamental function of E has the properties: $\varphi_E(0)=0$, φ_E is positive and increasing on $(0,\infty)$ and $\varphi_E(t)/t$ is decreasing on $(0,\infty)$. According to the definition of a weight symmetric space we have

$$\|\chi_A\|_{E_w}=\varphi_E(|A|_w) \text{ and } \|\chi_A\|_{E_v}=\varphi_E(|A|_v).$$

Since $A\subset H$ and $|A|_w=\int\limits_A w\,dx$, $|A|_v=\int\limits_A v\,dx$, we have $c_1|A|\le|A|_w,|A|_v\le c_2|A|$. Now taking into account the properties of φ_E we easily obtain the validity of the lemma. □

Lemma 2. *Let (X,S_1,μ_1) and (X,S_2,μ_2) be measure spaces, E_1 be a function space on (X,S_1,μ_1), E_2 be an ideal function space on (X,S_2,μ_2), and $T: L^\infty(X,S_1,\mu_1)\to L^\infty(X,S_2,\mu_2)$ be a positively homogeneous operator. If there exists a sequence of sets $\{H_i\}\subset S_1\cap S_2$ with the properties:*

1) $\{\chi_{H_i}\}\subset E_1\cap E_2$,
2) $\|\chi_{H_i}\|_{E_1}>0$ $(i\in\mathbb{N})$ *and* $\|\chi_{H_i}\|_{E_1}\to 0$ $(i\to\infty)$,
3) *there is* $c>0$ *such that* $\|\chi_{H_i}\|_{E_2}\ge c\|\chi_{H_i}\|_{E_1}$ $(i\in\mathbb{N})$,
4) $T\chi_{H_i}(x)\ge 1$ *for almost every* $x\in H_i$,
5) $T\chi_{H_i}\in L^\infty(X,S_2,\mu_2)$ $(i\in\mathbb{N})$,

then for any $\varepsilon\in(0,1)$ there exists an increasing sequence of indexes $\{i(k)\}$ such that

$$\|(Tf_{i(m)}-Tf_{i(k)})\chi_{H_{i(m)}}\|_{E_2}>(1-\varepsilon)c \ \text{ when } \ m>k,$$

where $f_i=\chi_{H_i}/\|\chi_{H_i}\|_{E_1}$ $(i\in\mathbb{N})$.

Proof. Let $\alpha_i=1/\|\chi_{H_i}\|_{E_1}$ and $\beta_i=\|Tf_i\|_{L^\infty(X,S_2,\mu_2)}$ $(i\in\mathbb{N})$. Since $\alpha_i\to\infty$ $(i\to\infty)$, we can choose an increasing sequence of indexes $\{i(k)\}$ such that $\alpha_{i(m)}>\alpha_{i(k)}$ $(m>k)$ and $\varepsilon\alpha_{i(k+1)}>\beta_{i(k)}$ $(k\in\mathbb{N})$. Taking into account the condition of Lemma 1 and choosing $\{i(k)\}$, for $m>k$ we can write

$$\begin{aligned}\|(Tf_{i(m)}-Tf_{i(k)})\chi_{H_{i(m)}}\|_{E_2}&\ge\|Tf_{i(m)}\cdot\chi_{H_{i(m)}}\|_{E_2}-\|Tf_{i(k)}\cdot\chi_{H_{i(m)}}\|_{E_2}\\&\ge\|\alpha_{i(m)}\chi_{H_{i(m)}}\|_{E_2}-\|\beta_{i(k)}\chi_{H_{i(m)}}\|_{E_2}=(\alpha_{i(m)}-\beta_{i(k)})\|\chi_{H_{i(m)}}\|_{E_2}\\&\ge(1-\varepsilon)\alpha_{i(m)}c\|\chi_{H_{i(m)}}\|_{E_1}=(1-\varepsilon)c.\end{aligned}$$ □

We shall call a *strip* in $\mathbb{R}^n$ an open set bounded by two different parallel hyperplanes; i.e., a set of the form

$$\{x\in\mathbb{R}^n: a<\alpha_1x_1+\cdots+\alpha_nx_n<b\},$$

where a,b $(a<b)$, $\alpha_1,\dots,\alpha_n$ $(|\alpha_1|+\cdots+|\alpha_n|>0)$ are real numbers, and x_k $(k=1,\dots,n)$ denotes the k-th coordinate of the point $x\in\mathbb{R}^n$. The *strip width* will be called the distance between the hyperplanes that bound the strip.

Lemma 3. *Let A be a set of positive measure in $\mathbb{R}^n$. Then for any $\delta \in (0,1)$ there exists a sequence of mutually parallel strips $\{S_k\}$ such that*

1) $|S_k \cap A| > 0$ $(k \in \mathbb{N})$,
2) $\dfrac{(\text{width of } S_k)}{\operatorname{dist}(S_k, S_m)} < \delta$ $(k < m)$.

Proof. By virtue of the well-known Lebesgue theorem (see, e.g., [1]), basis $\mathbb{Q}$ differentiates the integral of every locally summable function. Thus

$$\lim_{Q \in \mathbb{Q}(x),\ |Q| \to 0} \frac{1}{|Q|} \int_Q \chi_A = \chi_A(x) \text{ a.e.} \tag{2}$$

Let us consider the point $x_0 \in A$ for which (2) is valid. Denote $N = [\frac{4}{\delta}] + 1$. Obviously, we can choose cubic intervals Q_k $(k \in \mathbb{N})$ with centers at x_0 such that

$$\frac{|Q_k \cap A|}{|Q_k|} > 1 - \frac{1}{2N} \text{ and } \ell(Q_{k+1}) = \frac{1}{2}\,\ell(Q_k) \ (k \in \mathbb{N}),$$

where $\ell(Q)$ denotes the length of the edges of the cube Q. Obviously, Q_k has the form $Q_k = I_k \times J_k$, where I_k is an interval in $\mathbb{R}$ and J_k is a cubic interval in $\mathbb{R}^{n-1}$. Let us divide I_k $(k \in \mathbb{N})$ into $2N$ equal intervals and denote by I'_k the first interval from the left. Let S_k $(k \in \mathbb{N})$ be the strip $I'_k \times \mathbb{R}^{n-1}$. It is easy to check that $\{S_k\}$ satisfies the conditions of Lemma 3. □

Lemma 4 below was proved in [4]. We present the proof here for the sake of completeness.

Lemma 4. *Let B be a convex basis and S be a strip in $\mathbb{R}^n$. Then*

$$M_B(\chi_S)(x) < \frac{2^n(\text{width of } S)}{\operatorname{dist}(x, S)} \ \textit{when} \ \operatorname{dist}(x,S) \geq (\text{width of})\ S.$$

Proof. Let $\delta = (\text{width of } S)$, $\operatorname{dist}(x,S) \geq \delta$ and $R \in B(x)$, $R \cap S \neq \varnothing$. Among the hyperplanes bounding S we denote by Γ the hyperplane which is closest to x. It is obvious that $R \cap \Gamma \neq \varnothing$. For every $y \in R \cap \Gamma$ let Δ_y be a segment connecting x and y. Let $K = \bigcup_{y \in R \cap \Gamma} \Delta_y$. Since R is convex, we have

$$K \subset R. \tag{3}$$

Let H be the homothety centered at x and with the coefficient

$$\alpha = \frac{\operatorname{dist}(x,S) + \delta}{\operatorname{dist}(x,S)}\ .$$

Let us show that

$$R \cap S \subset H(K) \setminus K. \tag{4}$$

Indeed, let z be an arbitrary point from $R \cap S$ and let y be the point at which the segment connecting x and z intersects with Γ. Since $x, z \in R$, by virtue of the convexity of R we have $y \in R$. Therefore $y \in R \cap \Gamma$. By the definitions of the set K and homothety H we easily obtain $z \in H(\Delta_y) \subset H(K)$.Since $(R \cap S) \cap K = \varnothing$. Therefore $z \notin K$. Thus, $z \in H(K) \setminus K$.(4) is proved.

Using (3), (4), the definition of H and obvious inequality $\alpha^n - 1 < \frac{2^n \delta}{\operatorname{dist}(x,S)}$ we can write

$$\frac{1}{|R|} \int_R \chi_S = \frac{|R \cap S|}{|R|} \leq \frac{|H(K) \setminus K|}{|K|} = \frac{(\alpha^n - 1)|K|}{|K|} < \frac{2^n \delta}{\operatorname{dist}(x,S)}. \qquad \square$$

4 Proof of the Theorem

Proof of the Second Part of the Theorem. Let $0 < \delta < 1 - \sqrt{1 - \varepsilon}$ and $0 < c_1 < c_2$ be such that $\frac{c_1}{c_2} > (1 - \delta) c_{w,v}$ and the set

$$H = \{x \in \mathbb{R}^n : c_1 \leq w(x),\ v(x) \leq c_2\} \tag{5}$$

is of positive measure. Let $\{H_i\}$ be a sequence of measurable sets with the properties

$$H_i \subset H,\ 0 < |H_i| < \infty\ (i \in \mathbb{N}),\ |H_i| \to 0 \ \ (i \to \infty). \tag{6}$$

Let us set $(X, S_1, \mu_1) = (\mathbb{R}^n, S, w\,dx)$, $(X, S_2, \mu_2) = (\mathbb{R}^n, S, v\,dx)$, where S is the class of all measurable sets in $\mathbb{R}^n$; $E_1 = E_w$, $E_2 = E_v$ and $T = M_B|_{L^\infty(\mathbb{R}^n, S, dx)}$.

Now let us show that $\{H_i\}$ satisfies all the conditions of Lemma 2. Every ideal space contains a characteristic function of any set with the finite measure. Therefore from (5) and (6) we have that $\{\chi_{H_i}\} \subset E_1 \cap E_2$. From Lemma 1, the regularity of the space E and (5), (6) we write

$$\begin{aligned}
&\|\chi_{H_i}\|_{E_w} \geq \varphi_E(c_1 |H_i|) > 0 \quad (i \in \mathbb{N}),\\
&\|\chi_{H_i}\|_{E_w} \leq \varphi_E(c_2 |H_i|) \to 0 \quad (i \to \infty),\\
&\frac{\|\chi_{H_i}\|_{E_v}}{\|\chi_{H_i}\|_{E_w}} \geq \frac{c_1}{c_2} > (1 - \delta) c_{w,v} \ (i \in \mathbb{N}).
\end{aligned}$$

Thus we have established that $\{H_i\}$ satisfies conditions 1)–3) of Lemma 2. (Note that in condition 3) instead of c we have $(1 - \delta) c_{w,v}$.) Condition 4) is satisfied because B is a quasi-density basis and 5) is obvious. Now by virtue of Lemma 2 and a choice of δ we conclude that the assertion is proved.

Proof of the Remaining Part of the Theorem. Due to the assertion already established it remains to prove the assertion in the case where $\lim\limits_{t\to 0+} \varphi_{_E}(t) = p > 0$. Let $0 < \delta < 1 - \sqrt{1-\varepsilon}$ and $0 < c_1 < c_2$ be such that $\frac{c_1}{c_2} > (1-\delta)c_{w,v}$ and the set

$$A = \{x \in \mathbb{R}^n : c_1 \le w(x),\ v(x) \le c_2\}$$

is of positive measure. By virtue of Lemma 3 there exists a sequence of mutually parallel strips $\{S_k\}$ such that

$$|S_k \cap A| > 0 \quad (k \in \mathbb{N}) \text{ and } \frac{(\text{width of } S_k)}{\operatorname{dist}(S_k, S_m)} < \frac{\delta}{16}\ (k < m).$$

For any $k \in \mathbb{N}$ let A_k be a set with the properties

$$A_k \subset S_k \cap A,\ |A_k| > 0 \text{ and } \varphi_{_E}(c_2|A_k|) < 2p.$$

Let $\alpha_k = 1\big/\|\chi_{A_k}\|_{_{E_w}}$ $(k \in \mathbb{N})$. Due to Lemma 1 we have

$$p \le \varphi_{_E}(c_1|A_k|) \le \|\chi_{A_k}\|_{_{E_w}} \le \varphi_{_E}(c_2|A_k|) < 2p.$$

Hence

$$\frac{1}{2p} < \alpha_k < \frac{1}{p}\ (k \in \mathbb{N}). \tag{7}$$

By virtue of Lemma 4 for $k < m$ and $x \in S_m$ we can write

$$M_B\chi_{A_k}(x) \le M_B\chi_{S_k}(x) \le \frac{4(\text{width of } S_k)}{\operatorname{dist}(S_k, S_m)} < \frac{\delta}{4}\,. \tag{8}$$

Now taking into account the condition of the Theorem, (7), (8), Lemma 1 and a choice of δ, for $k < m$ we have

$$\begin{aligned}
\|(M_Bf_m - M_Bf_k)\chi_{A_m}\|_{_{E_v}} &\ge \|M_Bf_m \cdot \chi_{A_m}\|_{_{E_v}} - \|M_Bf_k \cdot \chi_{A_m}\|_{_{E_v}} \\
&\ge \alpha_m\|\chi_{A_m}\|_{_{E_v}} - \alpha_k\|M_B\chi_{A_k} \cdot \chi_{A_m}\|_{_{E_v}} \\
&= \Big(\alpha_m - \alpha_k\,\frac{\delta}{4}\Big)\|\chi_{A_m}\|_{_{E_v}} \\
&> \alpha_m\Big(1 - \frac{\alpha_k}{\alpha_m}\,\frac{\delta}{4}\Big)\frac{c_1}{c_2}\|\chi_{A_m}\|_{_{E_w}} \\
&> (1-\delta)^2 c_{w,v}\|\alpha_m\chi_{A_m}\|_{_{E_w}} > (1-\varepsilon)c_{w,v}. \qquad \square
\end{aligned}$$

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