Real Analysis Exchange
Vol. 28(2), 2002/2003, pp. 429–437

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MAXIMAL CLASSES FOR THE FAMILY OF STRONG ŚWIĄTKOWSKI FUNCTIONS

Abstract

In this paper we introduce the class of extra strong Świątkowski functions, and we characterize the following maximal classes for the families of strong Świątkowski and extra strong Świątkowski functions: the maximal additive class, the maximal multiplicative class, and the maximal class with respect to maximums.

1 Preliminaries

The letters $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the real line, the set of rationals, and the set of positive integers, respectively. The symbol $\mathrm{I}(a,b)$ denotes the open interval with endpoints a and b. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{cl} A$, $\operatorname{card} A$, and χ_A to denote the closure, the cardinality, and the characteristic function of A, respectively.

Let I be a nondegenerate interval and $f \colon I \to \mathbb{R}$. The symbols $\mathcal{C}(f)$ and $\mathcal{A}(f)$ will stand for the set of all points of continuity of f and the set of all local maximums (not necessarily strict) of f, respectively. We say that f is a *Darboux function* ($f \in \mathcal{D}$), if it maps connected sets onto connected sets. We say that f is a *strong Świątkowski function* [1] ($f \in \acute{\mathcal{S}}_s$), if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}\big(f(\alpha), f(\beta)\big)$, there is an $x_0 \in (\alpha,\beta) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. We will say that f is an *extra strong Świątkowski function* ($f \in \acute{\mathcal{S}}_{es}$), if $f\big[[a,b]\big] = f\big[[a,b] \cap \mathcal{C}(f)\big]$ for all $a, b \in I$, $a < b$. (Clearly $\acute{\mathcal{S}}_{es} \subset \acute{\mathcal{S}}_s \subset \mathcal{D}$ and both inclusions are proper.) Finally $f \in \mathcal{C}onst$ iff $f[I]$ is a singleton. Moreover, for each $x \in I$ we write $\overline{\lim}(f,x) = \overline{\lim}_{t \to x} f(x)$, and similarly we define the symbols $\overline{\lim}(f,x^-)$ and $\overline{\lim}(f,x^+)$.

Key Words: Darboux function, strong Świątkowski function, extra strong Świątkowski function.
Mathematical Reviews subject classification: Primary 26A21, 54C30. Secondary 26A15, 54C08.
Received by the editors September 27, 2002
*Supported by Bydgoszcz Academy.

If $\mathcal{L}$ and $\mathcal{L}_1$ are families of real functions, then we define:

$$\begin{aligned}\mathcal{M}_a(\mathcal{L}_1,\mathcal{L}) &= \big\{f : (\forall g \in \mathcal{L}_1)\ f+g \in \mathcal{L}\big\},\\ \mathcal{M}_m(\mathcal{L}_1,\mathcal{L}) &= \big\{f : (\forall g \in \mathcal{L}_1)\ fg \in \mathcal{L}\big\},\\ \mathcal{M}_{\max}(\mathcal{L}_1,\mathcal{L}) &= \big\{f : (\forall g \in \mathcal{L}_1)\ \max\{f,g\} \in \mathcal{L}\big\}.\end{aligned}$$

Moreover we let

$$\mathcal{M}_a(\mathcal{L}) = \mathcal{M}_a(\mathcal{L},\mathcal{L}),\ \mathcal{M}_m(\mathcal{L}) = \mathcal{M}_m(\mathcal{L},\mathcal{L}),\ \mathcal{M}_{\max}(\mathcal{L}) = \mathcal{M}_{\max}(\mathcal{L},\mathcal{L}).$$

The above classes are called the maximal additive class for $\mathcal{L}$, the maximal multiplicative class for $\mathcal{L}$, and the maximal class with respect to maximums for $\mathcal{L}$, respectively.

Remark 1.1. Clearly if $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{L}_1' \supset \mathcal{L}_1$, then $\mathcal{M}_a(\mathcal{L}_1',\mathcal{L}') \subset \mathcal{M}_a(\mathcal{L}_1,\mathcal{L})$. Similar inclusions hold for $\mathcal{M}_m$ and $\mathcal{M}_{\max}$.

2 Auxiliary Lemmas

First we will show some properties of extra strong Świątkowski functions.

Lemma 2.1. *Let $f\colon \mathbb{R}\to\mathbb{R}$ and $x_0 \in \mathbb{R}$. Assume that $f\restriction(-\infty,x_0) \in \acute{\mathcal{S}}_{es}$, $x_0 \in \mathcal{C}(f)$, and $f\restriction(x_0,\infty) \in \acute{\mathcal{S}}_{es}$. Then $f \in \acute{\mathcal{S}}_{es}$.*

Proof. Let $\alpha<\beta$. We can assume that $\alpha \le x_0 \le \beta$. Since $f\restriction(-\infty,x_0) \in \acute{\mathcal{S}}_{es}$, we have

$$\begin{aligned}f\big[[\alpha,x_0)\big] &= f\big[\textstyle\bigcup_{n\in\mathbb{N}}[\alpha,x_0-1/n]\big] = \textstyle\bigcup_{n\in\mathbb{N}} f\big[[\alpha,x_0-1/n]\big]\\ &= \textstyle\bigcup_{n\in\mathbb{N}} f\big[[\alpha,x_0-1/n]\cap\mathcal{C}(f)\big] = f\big[[\alpha,x_0)\cap\mathcal{C}(f)\big].\end{aligned}$$

Similarly, since $f\restriction(x_0,\infty) \in \acute{\mathcal{S}}_{es}$, we have $f\big[(x_0,\beta]\big] = f\big[(x_0,\beta]\cap\mathcal{C}(f)\big]$. Hence

$$\begin{aligned}f\big[[\alpha,\beta]\big] &= f\big[[\alpha,x_0)\big]\cup\big\{f(x_0)\big\}\cup f\big[(x_0,\beta]\big]\\ &= f\big[[\alpha,x_0)\cap\mathcal{C}(f)\big]\cup\big\{f(x_0)\big\}\cup f\big[(x_0,\beta]\cap\mathcal{C}(f)\big] = f\big[[\alpha,\beta]\cap\mathcal{C}(f)\big].\end{aligned}$$

Consequently, $f \in \acute{\mathcal{S}}_{es}$. □

Lemma 2.2. *Let $f\colon \mathbb{R}\to\mathbb{R}$ and $x_0 \in \mathbb{R}$. Assume that $f\restriction(-\infty,x_0] \in \mathcal{C}onst$, $f\restriction(x_0,\infty) \in \acute{\mathcal{S}}_{es}$, and $f(x_0) \in \bigcap_{\delta>0} f\big[[x_0,x_0+\delta]\cap\mathcal{C}(f)\big]$. Then, $f \in \acute{\mathcal{S}}_{es}$.*

Proof. Let $\alpha < \beta$. We may assume that $\alpha = x_0$ and $x_0 \notin \mathcal{C}(f)$. (Cf. Lemma 2.1.) Since $f\restriction(x_0,\infty) \in \acute{\mathcal{S}}_{es}$, we have $f\big[(x_0,\beta]\big] = f\big[(x_0,\beta] \cap \mathcal{C}(f)\big]$. Using the assumptions, we obtain $f(x_0) \in f\big[(x_0,\beta] \cap \mathcal{C}(f)\big]$. Hence

$$\begin{aligned} f\big[[x_0,\beta]\big] &= \big\{f(x_0)\big\} \cup f\big[(x_0,\beta]\big] = \big\{f(x_0)\big\} \cup f\big[(x_0,\beta] \cap \mathcal{C}(f)\big] \\ &= f\big[(x_0,\beta] \cap \mathcal{C}(f)\big] = f\big[[x_0,\beta] \cap \mathcal{C}(f)\big]. \end{aligned}$$

Consequently, $f \in \acute{\mathcal{S}}_{es}$. □

Lemma 2.3. *If $f, \varphi \in \acute{\mathcal{S}}_{es}$, then $f \circ \varphi \in \acute{\mathcal{S}}_{es}$.*

Proof. Let $\alpha < \beta$. We consider two cases.

Case 1. $\varphi\restriction[\alpha,\beta] \in \mathcal{C}onst$.

Then $(f \circ \varphi)\restriction[\alpha,\beta] \in \mathcal{C}onst \subset \acute{\mathcal{S}}_{es}$ and

$$(f \circ \varphi)\big[[\alpha,\beta]\big] = (f \circ \varphi)\big[[\alpha,\beta] \cap \mathcal{C}(f \circ \varphi)\big].$$

Case 2. $\varphi\restriction[\alpha,\beta] \notin \mathcal{C}onst$.

Since $\varphi \in \acute{\mathcal{S}}_{es}$, $\varphi\big[[\alpha,\beta]\big]$ is a nondegenerate interval. Since $f \in \acute{\mathcal{S}}_{es}$,

$$f\big[\varphi\big[[\alpha,\beta]\big]\big] = f\big[\varphi\big[[\alpha,\beta]\big] \cap \mathcal{C}(f)\big].$$

Hence

$$\begin{aligned} (f \circ \varphi)\big[[\alpha,\beta]\big] &= f\big[\varphi\big[[\alpha,\beta]\big]\big] = f\big[\varphi\big[[\alpha,\beta] \cap \mathcal{C}(\varphi)\big]\big] \\ &= f\big[\varphi\big[[\alpha,\beta] \cap \mathcal{C}(\varphi)\big] \cap \mathcal{C}(f)\big] \\ &\subset f\big[\varphi\big[[\alpha,\beta] \cap \mathcal{C}(f \circ \varphi)\big]\big] = (f \circ \varphi)\big[[\alpha,\beta] \cap \mathcal{C}(f \circ \varphi)\big]. \end{aligned}$$

Consequently, $f \circ \varphi \in \acute{\mathcal{S}}_{es}$. □

The next three lemmas are purely technical. The first one is probably known, but I could not find an appropriate reference.

Lemma 2.4. *If $f\colon \mathbb{R} \to \mathbb{R}$, then the set $f[\mathcal{A}(f)]$ is at most countable.*

Proof. Let $y \in f[\mathcal{A}(f)]$. Then $y = f(x)$ for some $x \in \mathcal{A}(f)$. Choose arbitrary $p_x, q_x \in \mathbb{Q}$ such that $p_x < x < q_x$ and $f(t) \le y$ for each $t \in (p_x, q_x)$. Define the function $\varphi\colon f[\mathcal{A}(f)] \to \mathbb{Q}^2$ by $\varphi(y) = (p_x, q_x)$. If $\varphi(y_1) = \varphi(y_2)$, then $(p_{x_1}, q_{x_1}) = (p_{x_2}, q_{x_2})$ for some $x_1, x_2 \in \mathcal{A}(f)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $x_2 \in (p_{x_1}, q_{x_1})$, so $y_2 \le y_1$, and similarly $y_1 \le y_2$. Whence $y_1 = y_2$. We proved that φ is an injection. Consequently, $\operatorname{card} f[\mathcal{A}(f)] \le \operatorname{card} \mathbb{Q}^2$, which completes the proof. □

Lemma 2.5. *If $f \in \acute{\mathcal{S}}_s \setminus \mathcal{C}onst$, then there are $a < b$ such that $f{\restriction}[a,b]$ is bounded and nonconstant.*

PROOF. Define $S = \{x \in \mathbb{R} : \overline{\lim}(|f|, x) = \infty\}$. Then S is closed. Let $\{I_n : n \in \mathbb{N}\}$ be a family of compact intervals such that $\mathbb{R} \setminus S = \bigcup_{n\in\mathbb{N}} I_n$. Since $f \in \acute{\mathcal{S}}_s \setminus \mathcal{C}onst$, $f\big[\mathcal{C}(f)\big]$ is a nondegenerate interval. One can easily see that $f[\mathcal{C}(f)] \subset f[\mathbb{R} \setminus S] = \bigcup_{n\in\mathbb{N}} f[I_n]$. So, there is an $n \in \mathbb{N}$ such that $f[I_n]$ is not a singleton. Then $f{\restriction}I_n \notin \mathcal{C}onst$ and $\overline{\lim}(|f|, x) < \infty$ for each $x \in I_n$. But I_n is compact; so $f{\restriction}I_n$ is bounded. $\square$

Lemma 2.6. *Let I be an open interval (maybe unbounded) and $f\colon I \to \mathbb{R}$. If $f \in \acute{\mathcal{S}}_s$ is a nonconstant upper semicontinuous function, then there is an $x_0 \notin \mathcal{A}(f)$ and a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset I$ and either*

$$f(x) < f(x_0) \text{ for each } x \in (x_0 - \delta, x_0) \tag{1}$$

or

$$f(x) < f(x_0) \text{ for each } x \in (x_0, x_0 + \delta). \tag{2}$$

PROOF. Let $a, b \in I$, $a < b$, and $f(a) \neq f(b)$. We may assume that $a, b \in \mathcal{C}(f)$ and $f(a) < f(b)$. (The case $f(a) > f(b)$ is analogous.) By Lemma 2.4, we can choose a $y \in \big(f(a), f(b)\big) \setminus f[\mathcal{A}(f)]$. Since $f(b) > y$ and $b \in \mathcal{C}(f)$, we can define

$$x_0 = \inf\{x \in [a, b) : f(x) \geq y\}.$$

Recall that $a \in \mathcal{C}(f)$ and $f(a) < y$. So, $x_0 > a$. Let $\delta = x_0 - a > 0$. Notice that if $x \in (x_0 - \delta, x_0) = (a, x_0)$, then $f(x) < y$. To complete the proof we will show that $f(x_0) = y$. Since $f \in \mathcal{D}$, we have $f(x_0) \leq \overline{\lim}(f, x_0^-) \leq y$. On the other hand, since f is upper semicontinuous and $x_0 < b$, $f(x_0) \geq \overline{\lim}(f, x_0^+) \geq y$. Hence $f(x_0) = y$. $\square$

3 Main Results

First we consider the maximal additive classes for the families of strong Świątkowski and extra strong Świątkowski functions.

Theorem 3.1. $\mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s) \subset \mathcal{C}onst$.

PROOF. Let $f \notin \mathcal{C}onst$. We will show that $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. If $f \notin \acute{\mathcal{S}}_s$, then since $\chi_\emptyset \in \acute{\mathcal{S}}_{es}$ and $f = f + \chi_\emptyset \notin \acute{\mathcal{S}}_s$, we obtain $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. So, we may assume that $f \in \acute{\mathcal{S}}_s$. By Lemma 2.5, there are $a < b$ such that $f{\restriction}[a, b] \notin \mathcal{C}onst$ and f is bounded on $[a, b]$. Then clearly $f{\restriction}(a, b) \notin \mathcal{C}onst$.

First suppose $(a,b) \subset \mathcal{C}(f)$. By Lemma 2.6, there is an $x_0 \in (a,b) \setminus \mathcal{A}(f)$ and a $\delta \in (0, x_0 - a)$ such that; e.g., condition (1) holds. (Similarly we can proceed if there is an $x_0 \in (a,b) \setminus \mathcal{A}(f)$ and a $\delta \in (0, b - x_0)$ such that condition (2) holds.) Since $x \notin \mathcal{A}(f)$, we may choose a sequence $(x_i) \subset (x_0, b)$ such that $x_i \searrow x_0$ and $f(x_i) > f(x_0)$ for each i. Define

$$g(x) = \begin{cases} 0 & \text{if } x \in (-\infty, x_0] \cup [x_1, \infty), \\ g_i(x) & \text{if } x \in [x_{i+1}, x_i],\ i \in \mathbb{N}, \end{cases}$$

where

$$g_i(x) = \max\bigl\{2\bigl(f(x_0) - f(x)\bigr), 1 - \bigl(2|x - (x_{i+1} + x_i)/2|\bigr)/(x_i - x_{i+1})\bigr\}.$$

Then x_0 is the only point of discontinuity of g, and by Lemma 2.2, $g \in \acute{\mathcal{S}}_{es}$. We will show that $f + g \notin \acute{\mathcal{S}}_s$.

Put $\alpha = x_0 - \delta/2$ and $\beta = x_1$. Notice that by (1), for each $x \in [\alpha, x_0)$ we have $(f+g)(x) = f(x) < f(x_0)$. Hence $f(x_0) \in \bigl((f+g)(\alpha), (f+g)(\beta)\bigr)$. Fix an $x \in (x_0, \beta)$. Then $x \in [x_{i+1}, x_i)$ for some $i \in \mathbb{N}$. We will show that $(f+g)(x) > f(x_0)$. We consider three cases:

- If $f(x) < f(x_0)$, then
$$(f+g)(x) = (f+g_i)(x) \geq f(x) + 2f(x_0) - 2f(x) > f(x_0).$$
- If $x = x_{i+1}$, then $(f+g)(x) = f(x) > f(x_0)$.
- If $f(x) \geq f(x_0)$ and $x \neq x_{i+1}$, then
$$(f+g)(x) \geq f(x) + 1 - \bigl(2|x - (x_{i+1} + x_i)/2|\bigr)/(x_i - x_{i+1}) > f(x) \geq f(x_0).$$

Finally observe that $x_0 \notin \mathcal{C}(f+g)$. So, $(f+g)(x) \neq f(x_0)$ for each $x \in (\alpha, \beta) \cap \mathcal{C}(f+g)$ and $f + g \notin \acute{\mathcal{S}}_s$. Now let $(a,b) \setminus \mathcal{C}(f) \neq \emptyset$. Without loss of generality we may assume that $f(x_0) < \overline{\lim}(f, x_0^+)$ for some $x_0 \in (a,b)$. Choose a $y \in \bigl(f(x_0), \overline{\lim}(f, x_0^+)\bigr)$. Since $f \in \acute{\mathcal{S}}_s$, there is a sequence $(x_i) \subset (x_0, b) \cap \mathcal{C}(f)$ such that $x_i \searrow x_0$ and $f(x_i) = y$ for each i. For each i, by [2, Lemma 4.1], there is a continuous function $g_i \colon [x_{i+1}, x_i] \to \mathbb{R}$ such that $g_i = -y$ on $\{x_{i+1}, x_i\}$ and $g_i > -f$ on (x_{i+1}, x_i). Let

$$g(x) = \begin{cases} -y & \text{if } x \in (-\infty, x_0] \cup [x_1, \infty), \\ g_i(x) & \text{if } x \in [x_{i+1}, x_i],\ i \in \mathbb{N}. \end{cases}$$

Then x_0 is the only point of discontinuity of g, and by Lemma 2.2, $g \in \acute{\mathcal{S}}_{es}$. We will show that $f + g \notin \mathcal{D} \supset \acute{\mathcal{S}}_s$.

If $x \in (x_0, x_1)$, then $x \in [x_{i+1}, x_i)$ for some $i \in \mathbb{N}$, whence

$$(f+g)(x) = (f+g_i)(x) \geq 0 > f(x_0) - y = (f+g)(x_0).$$

Consequently, $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. $\square$

Clearly $\mathcal{C}onst \subset \mathcal{M}_a(\acute{\mathcal{S}}_s) \cap \mathcal{M}_a(\acute{\mathcal{S}}_{es})$. So, using Theorem 3.1 and Remark 1.1, we obtain the following corollary.

Corollary 3.2. $\mathcal{M}_a(\acute{\mathcal{S}}_s) = \mathcal{M}_a(\acute{\mathcal{S}}_{es}) = \mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s) = \mathcal{C}onst.$ $\square$

Now we turn to the maximal multiplicative classes for the families of strong Świątkowski and extra strong Świątkowski functions.

Theorem 3.3. $\mathcal{M}_m(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s) \subset \mathcal{C}onst.$

PROOF. Let $f \in \mathcal{M}_m(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. Since $\chi_{\mathbb{R}} \in \acute{\mathcal{S}}_{es}$, we obtain $f = f \cdot \chi_{\mathbb{R}} \in \acute{\mathcal{S}}_s$. We will show that $f[\mathcal{C}(f)] \subset f[\mathcal{A}(f)] \cup \{0\}$. Let $z \in f[\mathcal{C}(f)] \setminus \{0\}$. Then $z = f(x)$ for some $x \in \mathcal{C}(f)$. Assume that $z > 0$. (The other case is similar.) There is an open interval I such that $x \in I$ and $f{\restriction}I > 0$. Let $\varphi \colon I \to \mathbb{R}$ be an increasing homeomorphism. Define $\psi = \ln \circ f \circ \varphi^{-1}$. We will show that $\psi \in \mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. Let $g \in \acute{\mathcal{S}}_{es}$, $\alpha < \beta$ and $y \in \mathrm{I}\big((\psi+g)(\alpha), (\psi+g)(\beta)\big)$. Put

$$\bar{g}(x) = \begin{cases} \exp\big(g\big(\varphi(x)\big)\big) & \text{if } x \in \big[\varphi^{-1}(\alpha), \varphi^{-1}(\beta)\big], \\ \text{constant} & \text{on } \big(-\infty, \varphi^{-1}(\alpha)\big] \text{ and } \big[\varphi^{-1}(\beta), \infty\big). \end{cases}$$

Then by Lemmas 2.3 and 2.2, $\bar{g} \in \acute{\mathcal{S}}_{es}$.

For each $x \in \big[\varphi^{-1}(\alpha), \varphi^{-1}(\beta)\big] \subset I$ we have

$$\begin{aligned}(f\bar{g})(x) &= \big((f \circ \varphi^{-1})(\bar{g} \circ \varphi^{-1})\big)\big(\varphi(x)\big) = \big((\exp \circ \psi)(\exp \circ g)\big)\big(\varphi(x)\big) \\ &= \big(\exp \circ (\psi + g)\big)\big(\varphi(x)\big).\end{aligned}$$

Whence

$$\big(\ln \circ (f\bar{g}) \circ \varphi^{-1}\big)\big(\varphi(x)\big) = \big(\psi + g\big)\big(\varphi(x)\big). \tag{3}$$

Recall that $y \in \mathrm{I}\big((\psi+g)(\alpha), (\psi+g)(\beta)\big)$. Hence

$$\begin{aligned}\exp(y) &\in \mathrm{I}\big(\big(\exp \circ (\psi+g)\big)(\alpha), \big(\exp \circ (\psi+g)\big)(\beta)\big) \\ &= \mathrm{I}\big((f\bar{g})\big(\varphi^{-1}(\alpha)\big), (f\bar{g})\big(\varphi^{-1}(\beta)\big)\big).\end{aligned}$$

By assumption, $f \in \mathcal{M}_m(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. Thus $f\bar{g} \in \acute{\mathcal{S}}_s$, and $(f\bar{g})(x_0) = \exp(y)$ for some $x_0 \in \big(\varphi^{-1}(\alpha), \varphi^{-1}(\beta)\big) \cap \mathcal{C}(f\bar{g})$. Since φ is a homeomorphism, $(f\bar{g}) \circ \varphi^{-1}$

is continuous at $\varphi(x_0)$. So by (3), $\varphi(x_0) \in (\alpha, \beta) \cap \mathcal{C}(\psi + g)$. One can easily see that $(\psi + g)(\varphi(x_0)) = y$, and consequently, $\psi + g \in \acute{\mathcal{S}}_s$.

We showed that $\psi \in \mathcal{M}_a(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. So by Theorem 3.1, $\psi \in \mathcal{C}onst$. But for each $x \in I$ we have

$$(\ln \circ f)(x) = (\ln \circ f \circ \varphi^{-1})(\varphi(x)) = \psi(\varphi(x)).$$

So $f \restriction I \in \mathcal{C}onst$ and finally $z \in \mathcal{A}(f)$. Since $f \in \acute{\mathcal{S}}_s$, the set $f[\mathcal{C}(f)] \subset f[\mathcal{A}(f)] \cup \{0\}$ is connected. By Lemma 2.4, the set $f[\mathcal{C}(f)]$ is at most countable. Whence $f[\mathcal{C}(f)] = \{y\}$ for some $y \in \mathbb{R}$. But $(\inf f[\mathbb{R}], \sup f[\mathbb{R}]) \subset f[\mathcal{C}(f)] = \{y\}$; so $f \in \mathcal{C}onst$. $\square$

Clearly $\mathcal{C}onst \subset \mathcal{M}_m(\acute{\mathcal{S}}_s) \cap \mathcal{M}_m(\acute{\mathcal{S}}_{es})$. So, using Theorem 3.3 and Remark 1.1, we obtain the following corollary.

Corollary 3.4. $\mathcal{M}_m(\acute{\mathcal{S}}_s) = \mathcal{M}_m(\acute{\mathcal{S}}_{es}) = \mathcal{M}_m(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s) = \mathcal{C}onst.$ $\square$

Finally we will characterize the maximal classes with respect to maximums for the families of strong Świątkowski and extra strong Świątkowski functions.

Theorem 3.5. *$\mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s) \subset \mathcal{C}onst$.*

Proof. Let $f \notin \mathcal{C}onst$. We will show that $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. If $f \notin \acute{\mathcal{S}}_s$, then there are $\alpha < \beta$ and $y \in \mathrm{I}(f(\alpha), f(\beta))$ such that $f(x) \neq y$ for each $x \in (\alpha, \beta) \cap \mathcal{C}(f)$. Put $g = \min\{f(\alpha), f(\beta)\}$ and $h = \max\{f, g\}$. Then clearly $g \in \acute{\mathcal{S}}_{es}$, $y \in \mathrm{I}(h(\alpha), h(\beta))$ and $h(x) \neq y$ for each $x \in (\alpha, \beta) \cap \mathcal{C}(h)$. Whence $h \notin \acute{\mathcal{S}}_s$. So, $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$, and we may assume that $f \in \acute{\mathcal{S}}_s$.

Now assume that f is upper semicontinuous. By Lemma 2.6, there are an $x_0 \notin \mathcal{A}(f)$ and a $\delta > 0$ such that; e.g., condition (1) holds. (Similarly we can proceed if there are an $x_0 \notin \mathcal{A}(f)$ and a $\delta > 0$ such that condition (2) holds.) Choose a sequence (x_i) such that $x_i \searrow x_0$ and $f(x_i) > f(x_0)$ for each i. Since $f \in \acute{\mathcal{S}}_s$, we may assume that $(x_i) \subset \mathcal{C}(f)$. For each i, since $x_i \in \mathcal{C}(f)$, there is a $\delta_i > 0$ such that $f(x) > f(x_0)$ for each $x \in (x_i - \delta_i, x_i + \delta_i)$. Without loss of generality we may assume that $x_{i+1} + \delta_{i+1} < x_i - \delta_i$ for each $i \in \mathbb{N}$. Let

$$g(x) = \begin{cases} f(x_0) - 1 & \text{if } x \in (-\infty, x_0] \cup \{x_i : i \in \mathbb{N}\} \cup [x_1, \infty), \\ f(x_0) + 1 & \text{if } x \in \bigcup_{i=1}^{\infty} [x_{i+1} + \delta_{i+1}, x_i - \delta_i], \\ \text{linear} & \text{in each interval } [x_{i+1}, x_{i+1} + \delta_{i+1}] \text{ and } [x_i - \delta_i, x_i]. \end{cases}$$

Then x_0 is the only point of discontinuity of g, and by Lemma 2.2, $g \in \acute{\mathcal{S}}_{es}$. We will show that $h = \max\{f, g\} \notin \acute{\mathcal{S}}_s$.

Put $\alpha = x_0 - \delta/2$ and $\beta = x_1 - \delta_1$. Notice that by (1), for each $x \in [\alpha, x_0)$ we have $h(x) = \max\{f(x), f(x_0) - 1\} < f(x_0)$. Hence $f(x_0) \in \big(h(\alpha), h(\beta)\big)$. Fix an $x \in (x_0, \beta)$. Then $x \in [x_{i+1}, x_i)$ for some $i \in \mathbb{N}$. We will show that $h(x) > f(x_0)$. We consider two cases.

- If $x \in [x_{i+1}, x_{i+1} + \delta_{i+1}) \cup (x_i - \delta_i, x_i)$, then $h(x) \geq f(x) > f(x_0)$.
- If $x \in [x_{i+1} + \delta_{i+1}, x_i - \delta_i]$, then $h(x) \geq g(x) > f(x_0)$.

Finally, observe that

$$\overline{\lim}(h, x_0^+) \geq \overline{\lim}(g, x_0^+) = f(x_0) + 1 > f(x_0) = h(x_0),$$

so $x_0 \notin \mathcal{C}(h)$. So, $h(x) \neq f(x_0)$ for each $x \in (\alpha, \beta) \cap \mathcal{C}(h)$ and $h \notin \acute{\mathcal{S}}_s$. Consequently, $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$, and we may assume that f is <u>not</u> upper semicontinuous.

We will assume that $f(x_0) < \overline{\lim}(f, x_0^+)$ for some $x_0 \in \mathbb{R}$. (The case $f(x_0) < \overline{\lim}(f, x_0^-)$ for some $x_0 \in \mathbb{R}$ is similar.) Let $y \in \big(f(x_0), \overline{\lim}(f, x_0^+)\big)$. There is a sequence $(x_i) \subset \mathcal{C}(f)$ such that $x_i \searrow x_0$ and $f(x_i) > y$ for each i. For each i, since $x_i \in \mathcal{C}(f)$, there is a $\delta_i > 0$ such that $f(x) > y > f(x_0)$ for each $x \in (x_i - \delta_i, x_i + \delta_i)$. Without loss we may assume that $x_{i+1} + \delta_{i+1} < x_i - \delta_i$ for each $i \in \mathbb{N}$. Let

$$g(x) = \begin{cases} f(x_0) & \text{if } x \in (-\infty, x_0] \cup \{x_i : i \in \mathbb{N}\} \cup [x_1, \infty), \\ y & \text{if } x \in \bigcup_{i=1}^{\infty} [x_{i+1} + \delta_{i+1}, x_i - \delta_i], \\ \text{linear} & \text{in each interval } [x_{i+1}, x_{i+1} + \delta_{i+1}] \text{ and } [x_i - \delta_i, x_i]. \end{cases}$$

Then x_0 is the only point of discontinuity of g, and by Lemma 2.2, $g \in \acute{\mathcal{S}}_{es}$. We will show that $h = \max\{f, g\} \notin \mathcal{D} \supset \acute{\mathcal{S}}_s$. If $x \in (x_0, x_1)$, then $x \in [x_{i+1}, x_i)$ for some $i \in \mathbb{N}$: so

- either $x \in [x_{i+1}, x_{i+1} + \delta_{i+1}) \cup (x_i - \delta_i, x_i)$ and $h(x) \geq f(x) > y$,
- or $x \in [x_{i+1} + \delta_{i+1}, x_i - \delta_i]$ and $h(x) \geq g(x) = y$.

But $y > f(x_0) = h(x_0)$. Consequently, $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s)$. □

Clearly $\mathcal{C}onst \subset \mathcal{M}_{\max}(\acute{\mathcal{S}}_s) \cap \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es})$. So, using Theorem 3.5 and Remark 1.1, we obtain the following corollary.

Corollary 3.6. $\mathcal{M}_{\max}(\acute{\mathcal{S}}_s) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}, \acute{\mathcal{S}}_s) = \mathcal{C}onst.$

4 An Excerpt from [2]

Lemma 4.1. *If* $g\colon [a,b] \to (-\infty, M)$ *is upper semicontinuous both at* a *and at* b*, then there is a continuous function* $\psi\colon [a,b] \to [\min\{g(a), g(b)\}, M]$ *such that* $\psi = g$ *on* $\{a,b\}$ *and* $\psi > g$ *on* (a,b)*.*

Proof. Let $\delta_0 = (b-a)/2$. For each $n \in \mathbb{N}$ find a $\delta_n \in (0, \delta_{n-1}/2)$ such that $g < g(a) + n^{-1}$ on $[a, a+\delta_n]$ and $g < g(b) + n^{-1}$ on $[b - \delta_n, b]$. Define

$$\psi(x) = \begin{cases} M & \text{if } x \in [a+\delta_1, b-\delta_1], \\ g(a) + n^{-1} & \text{if } x = a + \delta_{n+1},\ n \in \mathbb{N}, \\ g(b) + n^{-1} & \text{if } x = b - \delta_{n+1},\ n \in \mathbb{N}, \\ g(x) & \text{if } x \in \{a, b\}, \end{cases}$$

and let ψ be linear in each interval $[a+\delta_{n+1}, a+\delta_n]$ or $[b-\delta_n, b-\delta_{n+1}]$ $(n \in \mathbb{N})$. One can easily see that ψ has all required properties. □

References

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[2] A. Maliszewski, *Maximums of almost continuous functions*, in preparation.