Real Analysis Exchange
Vol. 28(2), 2002/2003, pp. 415–428

Stewart Baldwin, Department of Mathematics, Auburn University, AL 36849-5310. e-mail: `baldwsl@mail.auburn.edu`

THE MARCZEWSKI HULL PROPERTY AND COMPLETE BOOLEAN ALGEBRAS

Abstract

Let $\mathcal{A}$ be an algebra of subsets of an underlying set A (which is not the entire power set of A in general), and let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal over A. The pair $(\mathcal{A}, \mathcal{I})$ is said to have the *hull property* iff whenever $X \subseteq A$, there is a $Y \in \mathcal{A}$ such that $X \subseteq Y$ and Y is "least" mod $\mathcal{I}$, i.e., if $Z \in \mathcal{A}$ and $X \subseteq Z$, then $Y \setminus Z \in \mathcal{I}$. It has been observed that in many cases for which $(\mathcal{A}, \mathcal{I})$ satisfies the hull property, the quotient Boolean algebra $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra. That, and the superficial similarity between the definitions themselves, along with the similar proofs that have sometimes resulted when using these properties, leads to the natural question of how the two properties "$(\mathcal{A}, \mathcal{I})$ satisfies the hull property" and "$\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra" are related to each other. Examples will be produced which show that neither of these two properties implies the other. In addition, we examine the question of what additional hypotheses would cause one of these properties to imply the other.

1 Introduction

Let $\mathcal{A}$ be an algebra of sets, and $\mathcal{I}$ an ideal. The Marczewski hull property (see [M]) and the property of having a complete quotient algebra $\mathcal{A}/\mathcal{I}$ are two similar looking properties for which there are some natural questions. Recently, an argument that had earlier been used by John Walsh (in [W]) to prove that $((s), (s)^0)$ has the hull property was modified by Jack Brown and myself (in [BB]) to get a similar proof that $(s)/(s)^0$ is a complete Boolean algebra (see the following for the definitions). This result, along with the apparent similarity in the definitions of the two properties themselves, prompted the question about what the relationship was between the hull property and complete quotient algebras.

Key Words: Complete Boolean Algebra, Marczewski Hull Property, Marczewski Measurable, Universally Measurable
Mathematical Reviews subject classification: Primary 28A05, Secondary 03G05
Received by the editors September 18, 2002

This paper is an attempt at answering some of the basic questions of the relationships between these two properties. The remainder of section 1 will give some of the basic definitions. In section 2, examples will be constructed to show that neither of the two properties implies the other. In sections 3 and 4, we examine what additional hypotheses might be required to get one of the two properties to imply the other, and section 5 summarizes the basic results and states some problems for further research.

Definition 1. Let A be a set. A collection $\mathcal{A}$ of subsets of A is called an *algebra* if it is closed under finite unions, finite intersections, and complements with respect to A. $\mathcal{A}$ is a *σ-algebra* if, in addition, it is closed under countable unions and intersections. A collection $\mathcal{I}$ of subsets of A is called an *ideal* over A iff $\mathcal{I}$ is closed under finite unions and subsets. $\mathcal{I}$ is a *σ-ideal* if $\mathcal{I}$ is also closed under countable unions. If $\mathcal{A}$ is an algebra on a set A, and $\mathcal{I} \subseteq \mathcal{A}$ define $\mathcal{I}^+ = \mathcal{A} \setminus \mathcal{I}$, and $\mathcal{I}^* = \{A \setminus X : X \in \mathcal{I}\}$. Throughout this paper, we shall assume that if an algebra is denoted by a calligraphic letter, then its underlying set will be denoted by the corresponding Roman letter. Let K be the class consisting of all pairs $(\mathcal{A}, \mathcal{I})$ such that $\mathcal{A}$ is an algebra with underlying set A and $\mathcal{I}$ is an ideal over A such that $\mathcal{I} \subseteq \mathcal{A}$. Let K_σ be the class of all $(\mathcal{A}, \mathcal{I})$ in K such that $\mathcal{A}$ is a σ algebra and $\mathcal{I}$ is a σ-ideal.

The pair $(\mathcal{A}, \mathcal{I}) \in K$ is said to satisfy the *hull property* iff for every $X \subseteq A$ there is a $Y \in A$ such that $X \subseteq Y$, and whenever $Z \in A$ is such that $X \subseteq Z$, we have $Y \setminus Z \in \mathcal{I}$ (i.e., Y is "least" mod $\mathcal{I}$). This property has turned out to be a useful property in the study of σ-algebras of separable metric spaces. (See, e.g., [W].) Observe that the hull property is trivially equivalent to what might be called the "dual hull property", i.e., if $X \subseteq A$, then there is a $Y \in \mathcal{A}$ such that $Y \subseteq X$ and Y is "greatest possible mod $\mathcal{I}$" (i.e., if $Z \in \mathcal{A}$ is such that $Z \subseteq X$, then $Z \setminus Y \in \mathcal{I}$).

Given $(\mathcal{A}, \mathcal{I}) \in K$, the quotient algebra $\mathcal{A}/\mathcal{I}$ can be formed by defining an equivalence relation $\equiv$ by $X \equiv Y$ iff $(X \setminus Y) \cup (Y \setminus X) \in \mathcal{I}$, for $X, Y \in \mathcal{A}$, and then dropping to the corresponding equivalence classes, giving us the quotient Boolean algebra $\mathcal{A}/\mathcal{I}$. Every Boolean algebra has a natural partial ordering (which corresponds to the $\subseteq$ relation for an algebra of sets). A Boolean algebra is said to be *complete* iff every subset has a least upper bound with respect to this natural ordering. Many standard references (for example, [J]) can be consulted for the basic properties of Boolean algebras. However, rather than dealing with this ordering at the level of the quotient algebra, it will be more convenient to do some of our arguments at the level of the algebra $\mathcal{A}$, thus avoiding an additional layer of notation, so we shall take several standard definitions of Boolean algebras, and pull them up to the level of the algebra $\mathcal{A}$. Thus, we define, for subsets V and W of X, the relation $V \leq_{\mathcal{I}} W$ iff

$V \setminus W \in \mathcal{I}$. If $\mathcal{I}$ is obvious from context (as will usually be the case), then we shall drop the subscript and write just $\leq$ instead. Then this relation is transitive and reflexive, but not antisymmetric in general. The terms *upper bound*, *least upper bound*, etc., are then defined in the obvious way, noting that least upper bounds (if they exist at all) may not be unique (since if U is a least upper bound of some subset of $\mathcal{A}$, then any other element of $\mathcal{A}$ which is equivalent (mod $\mathcal{I}$) to U will also be a least upper bound). However, least upper bounds (if they exist) will be unique mod $\mathcal{I}$. The Boolean algebra $\mathcal{A}/\mathcal{I}$ is said to be *complete* iff every subset of $\mathcal{A}$ has a least upper bound in this natural ordering. Observe that this is equivalent to the dual statement that every subset of $\mathcal{A}$ has a greatest lower bound.

Let $\mathcal{A}$ be an algebra of subsets of the set A, and let $\mathcal{I}$ be an ideal over A such that $\mathcal{I} \subseteq \mathcal{A}$. A subset $\mathcal{S}$ of $\mathcal{I}^+$ will be called *dense* in $\mathcal{I}^+$ iff for every $U \in \mathcal{I}^+$ there is a $V \in \mathcal{S}$ such that $V \subseteq U$. A subset $\mathcal{S}$ of $\mathcal{I}^+$ will be called *predense* in $\mathcal{I}^+$ iff for every $U \in \mathcal{I}^+$ there is a $V \in \mathcal{S}$ such that $U \cap V \in \mathcal{I}^+$. It is easy to check that $\mathcal{S}$ is predense in $\mathcal{I}^+$ iff $\mathcal{S}' = \{W \in \mathcal{I}^+ : W \subseteq V$ for some $V \in \mathcal{S}\}$ is dense in $\mathcal{I}^+$. We can also observe that, in this language, a subset $\mathcal{S}$ of $\mathcal{I}^+$ is predense in $\mathcal{I}^+$ iff A is a least upper bound of $\mathcal{S}$. If $\mathcal{C}, \mathcal{D} \subseteq \mathcal{I}^+$, then we say that $\mathcal{D}$ is *predense below* $\mathcal{C}$ iff for every $X \in \mathcal{C}$ and for every $W \subseteq X$ such that $W \in \mathcal{I}^+$, there is a $Y \in \mathcal{D}$ such that $W \cap Y \in \mathcal{I}^+$. If $\mathcal{C}, \mathcal{D} \subseteq \mathcal{A}$, we say that $\mathcal{D}$ is a *refinement* of $\mathcal{C}$ iff for every $X \in \mathcal{D}$, there is a $Y \in \mathcal{C}$ such that $X \subseteq Y$. Given $(\mathcal{A}, \mathcal{I}) \in K$ and $\mathcal{C} \subseteq \mathcal{I}^+$, $\mathcal{C}$ is called an *antichain* iff for any two distinct elements X and Y of $\mathcal{C}$, $X \cap Y \in \mathcal{I}$. We remind the reader of the well known fact that if $\mathcal{C}$ is any subset of $\mathcal{I}^+$, then there is an antichain $\mathcal{D} \subseteq \mathcal{A}$ which is a refinement of $\mathcal{C}$ and predense below $\mathcal{C}$. (Among all antichains which are refinements of $\mathcal{C}$, use Zorn's Lemma to get a maximal one with that property.)

2 Complete Boolean Algebras and the Hull Property

In this section, we show that neither the complete Boolean algebra property nor the hull property implies the other. An example which satisfies the hull property but does not give a complete quotient algebra is easy to obtain by using the trivial observation that a power set always satisfies the hull property, added to the well known fact that quotient Boolean algebras which are not complete are very common.

Theorem 1. *If A is the set of real numbers, $\mathcal{A}$ is the set of all subsets of A, and $\mathcal{I}$ is the σ-ideal of all countable subsets of A, then $(\mathcal{A}, \mathcal{I})$ satisfies the hull property, but $\mathcal{A}/\mathcal{I}$ is not a complete Boolean algebra.*

Proof. $(\mathcal{A},\mathcal{I})$ trivially satisfies the hull property. To see that $\mathcal{A}/\mathcal{I}$ is not a complete Boolean algebra, let $\mathcal{S}$ be an uncountable set of pairwise disjoint uncountable subsets of $\mathbb{R}$. To see that $\mathcal{S}$ has no least upper bound with respect to the ideal $\mathcal{I}$, suppose that U is an upper bound (mod $\mathcal{I}$) of $\mathcal{S}$ for some $U \subseteq \mathbb{R}$. Then for each $S \in \mathcal{S}$, $S \setminus U \in \mathcal{I}$, i.e., $S \setminus U$ is countable, and therefore $S \cap U$ is uncountable, so we can pick an element $a_S \in S \cap U$. Since the elements of $\mathcal{S}$ are pairwise disjoint, the a_S's are distinct. Now, let $U' = U \setminus \{a_S : S \in \mathcal{S}\}$. Then $S \setminus U' = S \setminus U \cup \{a_S\}$ is still countable for each $S \in \mathcal{S}$, so U' is also an upper bound of $\mathcal{S}$. But $U \setminus U'$ is uncountable, and therefore not in $\mathcal{I}$, so $U' \subseteq U$ shows that U is not a least uppper of $\mathcal{S}$. Thus, since U was an arbitrarily chosen upper bound of $\mathcal{S}$, $\mathcal{S}$ has no least upper bound, and $\mathcal{A}/\mathcal{I}$ is not complete. $\square$

Getting the hull property to fail while maintaining completeness of the quotient algebra is more difficult. We use a trick which will allow us to destroy the hull property without altering the quotient algebra.

Definition 2. A set $X \in \mathcal{I}^+$ will be called an *$\mathcal{I}$-atom* iff X is a minimal element of $\mathcal{I}^+$ (mod $\mathcal{I}$), i.e, for all $Y \in \mathcal{I}^+$, if $Y \leq X$, then $Y \equiv X$. The pair $(\mathcal{A},\mathcal{I}) \in K$ will be called *atomless* iff $\mathcal{I}^+$ has no $\mathcal{I}$-atoms (equivalently, $\mathcal{A}/\mathcal{I}$ is atomless in the usual definition of the term for Boolean algebras). A set $\mathcal{U} \subseteq \mathcal{A}$ is called an *ultrafilter* on $\mathcal{A}$ iff $\mathcal{U}$ is maximal with respect to having the finite intersection property. If $(\mathcal{A},\mathcal{I}) \in K$, and $\mathcal{U}$ is an ultrafilter on $\mathcal{A}$ such that $\mathcal{U} \cap \mathcal{I}$ is empty (i.e., the set corresponding to $\mathcal{U}$ is an ultrafilter on $\mathcal{A}/\mathcal{I}$ in the usual definition of ultrafilter for Boolean algebras), then $\mathcal{U}$ will be called an $(\mathcal{A},\mathcal{I})$-ultrafilter.

Theorem 2. *If $(\mathcal{A},\mathcal{I}) \in K$ is atomless, then there exists an algebra $\mathcal{B}$ and an ideal $\mathcal{J}$ such that $(\mathcal{B},\mathcal{J})$ does not satisfy the hull property, and such that $\mathcal{A}/\mathcal{I}$ and $\mathcal{B}/\mathcal{J}$ are isomorphic (as Boolean algebras). The algebra $\mathcal{B}$ can be constructed such that the underlying set B of $\mathcal{B}$ has any cardinality greater than or equal to the cardinality of A.*

Proof. Note that since $(\mathcal{A},\mathcal{I})$ is atomless, $\mathcal{A}$ is infinite, and therefore A is infinite. Let $\kappa \geq |A|$ be the cardinality of the desired new set B, and let C be a set of cardinality κ which is disjoint from A. Fix an infinite cardinal $\lambda \leq \kappa$, and let $\mathcal{L}$ be the set of all subsets of C having cardinality strictly less than λ. Let $\mathcal{L}^* = \{C \setminus S : S \in \mathcal{L}\}$. Let $B = A \cup C$, and note that B has cardinality κ. Define $\mathcal{J}$ to be all sets of the form $S \cup T$, where $S \in \mathcal{I}$ and $T \in \mathcal{L}$. To define the algebra $\mathcal{B}$, let $\mathcal{U} \subseteq \mathcal{A}$ be a $(\mathcal{A},\mathcal{I})$-ultrafilter (which exists by a standard Zorn's Lemma argument). Now, let $\mathcal{B} = \{X \cup Y : \text{either } (X \in \mathcal{U} \text{ and } Y \in \mathcal{L}^*) \text{ or } (X \in \mathcal{A} \setminus \mathcal{U} \text{ and } Y \in \mathcal{L})\}$. It remains to show that $\mathcal{B}, \mathcal{J}$ are as desired. It is easy to see that $\mathcal{J}$ is an ideal with $\mathcal{J} \subseteq \mathcal{B}$. Using the well known fact about ultrafilters that if $X \in \mathcal{A}$, then exactly one of X and $A \setminus X$ is in $\mathcal{U}$, it is easy to see that $\mathcal{B}$ is closed under finite unions and intersections. To see that $\mathcal{B}$ is closed under complements, note that for every $X \in \mathcal{A}$, $X \in \mathcal{U}$ iff $A \setminus X \notin \mathcal{U}$. To see that $\mathcal{A}/\mathcal{I}$ and $\mathcal{B}/\mathcal{J}$ are isomorphic, define $f : \mathcal{B} \to \mathcal{A}$ by $f(X) = X \cap A$ for all $X \in \mathcal{B}$. Also, $X \equiv_{\mathcal{J}} Y$ iff $f(X) \equiv_{\mathcal{I}} f(Y)$, for all $X, Y \in \mathcal{B}$, and it is therefore routine to check that f induces a Boolean algebra isomorphism between $\mathcal{A}/\mathcal{I}$ and $\mathcal{B}/\mathcal{J}$. Finally, to see that $(\mathcal{B}, \mathcal{J})$ does not satisfy the hull property, we observe that since $\mathcal{A}$ is atomless, $\mathcal{U}$ cannot have a least element (mod $\mathcal{I}$). Thus, we look at $C \subseteq B$. If Y is any element of $\mathcal{B}$ which covers C, then $Y = X \cup C$ for some $X \in \mathcal{U}$, for those are the only elements of $\mathcal{B}$ containing C. However, since $\mathcal{U}$ has no least element (mod $\mathcal{I}$), these sets can have no least element (mod $\mathcal{J}$). □

Corollary 3. *Neither of the properties "$\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra" and "$(\mathcal{A}, \mathcal{I})$ satisfies the hull property" implies the other.*

Proof. There are many atomless pairs $(\mathcal{A}, \mathcal{I})$ which have a complete quotient algebra (for example, the Lebesgue measurable sets and the Lebesgue null sets), to which the previous theorem can be applied. □

After the above proof was found, an alternate way of getting examples was noticed: If $\mathcal{B}$ is a complete atomless Boolean algebra, $\mathcal{A}$ is the usual Stone algebra of $\mathcal{B}$, and $\mathcal{I}$ is the empty ideal, then $(\mathcal{A}, \mathcal{I})$ has a complete quotient algebra (isomorphic to $\mathcal{B}$) and does not satisfy the hull property (as it is routine to show that any singleton from A cannot be covered by a least member of $\mathcal{A}$).

The above arguments show that we can get an example with any complete atomless Boolean algebra as our quotient algebra. However, note that the pair $(\mathcal{B}, \mathcal{J})$ in the above proof will not be a σ-algebra and σ-ideal pair unless $(\mathcal{A}, \mathcal{I}) \in \mathcal{K}$ is a σ-algebra and σ-ideal pair, λ is uncountable, and the ultrafilter $\mathcal{U}$ is countably complete (i.e., closed under countable intersections). The first two of these requirements present no problems, but it is not difficult to show that there exists an atomless $(\mathcal{A}, \mathcal{I}) \in \mathcal{K}$ with a complete quotient algebra and a countably complete $(\mathcal{A}, \mathcal{I})$-ultrafilter iff there exists a measurable cardinal. Thus, even though the above theorem can be used to construct many different examples of algebras with the desired property, they could not be σ-algebras without additional set theoretic hypotheses. (The Stone algebra of an atomless Boolean algebra can never be a σ-algebra, so that argument will also not work to get a σ-algebra example.) The original version of this paper provided an example of a σ-algebra, using the above theorem and assuming the existence of a measurable cardinal, but a ZFC example has now been found, as in the following results.

Definition 3. A set X of reals is said to be *Marczewski Mesaurable* (written $X \in (s)$) iff for every perfect subset P of the reals there is a perfect set $Q \subseteq P$ such that either $Q \subseteq X$ or Q is disjoint from X. A set X of reals is said to be *Marczewski Null* (written $X \in (s)^0$) iff for every perfect subset P of the reals there is a perfect set $Q \subseteq P$ such that Q is disjoint from X (equivelently, X is Marczewski Null iff it is Marczewski Mesaurable and contains no perfect subset). It is well known that the pair $((s), (s)^0)$ satisfies the hull property [W] and has a complete quotient algebra (see [BB] for a simple proof), and that every Borel set is in (s). The desired example will be a modification of this pair.

Lemma 4. *Let $\mathcal{M} \subseteq (s) \setminus (s)^0$ be a maximal antichain with respect to the pair $((s), (s)^0)$. Then there is a pairwise disjoint collection $\mathcal{M}'$ of perfect sets that is also a maximal antichain with respect to the pair $((s), (s)^0)$, and such that $\mathcal{M}'$ is a refinement of $\mathcal{M}$.*

PROOF. Given such an $\mathcal{M}$, let $(P_\alpha : \alpha < c)$ be an enumeration of all perfect subsets of the reals, where c is the cardinality of the continuum. Define sets $(Q_\alpha : \alpha < c)$ by transfinite induction on $\alpha < c$. If Q_β has been defined for all $\beta < \alpha$ and there is a perfect set Q such that $Q \cap Q_\beta$ is empty for all $\beta < \alpha$ and $Q \subseteq P_\alpha \cap X$ for some $X \in \mathcal{M}$, then we let Q_α be equal to some such Q. If no such Q exists, then we let Q_α be the empty set. We then let $\mathcal{M}' = \{Q_\alpha : \alpha < c \text{ and } Q_\alpha \text{ is perfect}\}$. $\mathcal{M}'$ is clearly a pairwise disjoint collection of perfect sets such that $\mathcal{M}'$ is a refinement of $\mathcal{M}$. Thus, we only need to show that $\mathcal{M}$ is a maximal antichain with respect to the pair $((s), (s)^0)$. Suppose that this is not the case, and let $Z \in (s) \setminus (s)^0$ be such that $Z \cap Q \in (s)^0$ for all $Q \in \mathcal{M}'$. By the maximality of $\mathcal{M}$, there must be an $X \in \mathcal{M}$ such that $Z \cap X \notin (s)^0$. If we pick such an X, then (since $Z \cap X \in (s)$) there must be a perfect set P such that $P \subseteq Z \cap X$. Then $P = P_\alpha$ for some fixed $\alpha < c$. Now, since $P \subseteq Z$, $P \cap Q_\beta \in (s)^0$ for all $\beta < c$ and therefore for all $\beta < \alpha$. Thus, $P \cap Q_\beta$ is countable for each $\beta < \alpha$, because otherwise $P \cap Q_\beta$ would be an uncoutable closed set and would therefore contain a perfect subset, contradicting that $P \cap Q_\beta \in (s)^0$. Therefore, $\bigcup_{\beta<\alpha}(P \cap Q_\beta)$ has cardinality less than the continuum, and there exists a perfect set $Q \subseteq P$ such that $Q \cap \bigcup_{\beta<\alpha}(P \cap Q_\beta)$ is empty. This Q would be a possible choice for Q_α, and therefore Q_α is not the empty set. However, this contradicts the fact that P intersects all Q_β's in only an $(s)^0$-set. $\square$

Theorem 5. *There exists a pair $(\mathcal{A}, \mathcal{I}) \in K$ such that $\mathcal{A}$ is a σ-algebra, $\mathcal{I}$ is a σ-ideal, and $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra, but $(\mathcal{A}, \mathcal{I})$ does not satisfy the hull property.*

Proof. Let $\mathcal{M}$ be a collection of c-many pairwise disjoint perfect sets such that $\mathcal{M}$ is a maximal antichain with respect to the pair $((s),(s)^0)$. Such a collection can be obtained by starting with c-many pairwise disjoint perfect sets, expanding it to a maximal antichain using a Zorn's Lemma argument, and then applying the previous lemma. For each $M \in \mathcal{M}$, let $B_M \subseteq M$ be a set which is Bernstein in M (i.e., every perfect subset of M intersects both B_M and the complement of B_M). Let $\mathcal{B} = \{B_M : M \in \mathcal{M}\}$, and let $\mathcal{I}$ be the σ-ideal of subsets of the reals generated by $(s)^0$ and $\mathcal{B}$. Let $B = \bigcup_{M\in\mathcal{M}} B_M$, and note that B is a Berstein subset of the set of all reals (a simple consequence of the maximality of $\mathcal{M}$).

Claim. $(s) \cap \mathcal{I} = (s)^0$.

Proof of Claim: The direction $(s)^0 \subseteq (s)\cap\mathcal{I}$ is clear, so assume $X \in (s)\cap\mathcal{I}$. Since $X \in \mathcal{I}$, X can be represented as the union of an element of $(s)^0$ and of subsets of countably many B_M's. In particular, $X = Y \cup Z$ for some $Y \in (s)^0$ and $Z \subseteq B$. Now, to see that $X \in (s)^0$, let P be any perfect subset of the reals. Then since $X \in (s)$, there is a perfect set $Q \subseteq P$ such that either Q misses X or Q is a subset of X. Since $Y \in (s)^0$, there is a perfect set $R \subseteq Q$ such that $R \cap Y$ is empty. Since B contains no perfect sets, we cannot have $R \subseteq X$, and therefore $R \cap X$ is empty. This completes the proof of the claim.

Now, let $\mathcal{A}$ be the σ-algebra generated by (s) and $\mathcal{I}$. Then $\mathcal{A}$ consists of all sets of the form $(X \cup Y) \setminus Z$, where $X \in (s)$ and $Y, Z \in \mathcal{I}$ (an easy consequence of the routinely checked fact that the collection of all such sets is closed under complements and countable unions). Combined with the claim, it is easily seen that $\mathcal{A}/\mathcal{I}$ and $(s)/(s)^0$ are isomorphic as Boolean algebras (and therefore complete, since $(s)/(s)^0$ is). To see that $(\mathcal{A},\mathcal{I})$ does not satisfy the hull property, we show that the set B cannot be covered by a least (mod $\mathcal{I}$) member of $\mathcal{A}$. Thus, let $C \in \mathcal{A}$ be such that $B \subseteq C$. Then $C = (X \cup Y) \setminus Z$ for some $X \in (s)$, $Y, Z \in \mathcal{I}$. Since $Y = U \cup V$ for some $U \in (s)^0$ and some $V \subseteq B$ that intersects only countably many B_M's, let $M \in \mathcal{M}$ be such that $V \cap M$ is empty. Let $D = (C \setminus M) \cup B_M$. Then clearly $B \subseteq D$. In addition, since $B_M \subseteq (C \cup U) \cap M$, and B_M is Bernstein in M, we must have that $M \setminus C \in (s)^0$ (Otherwise, $M \setminus (C \cup U)$ contains a perfect set and intersects B_M, a contradiction). Thus, it is easy to check that $C \setminus D \notin \mathcal{I}$, thereby showing that B cannot be covered by an least (mod $\mathcal{I}$) member of $\mathcal{A}$. $\square$

3 Properties Implied by the Hull Property

Definition 4. Let us say that the pair $(\mathcal{A},\mathcal{I})$ has the *density property* iff $A \setminus \bigcup \mathcal{S} \in \mathcal{I}$ whenever $\mathcal{S}$ is a predense subset of $\mathcal{I}^+$. (Intuitively, the density property says that if $\mathcal{S}$ covers so much of A that A itself is a least upper bound of $\mathcal{S}$, then the set of points not covered by $\mathcal{S}$ must be small, i.e., an element of $\mathcal{I}$.) Let $(\mathcal{A}, \mathcal{I}) \in K$, and let $\mathcal{P}$ is a collection of subsets of A. Then we say that the pair $(\mathcal{A}, \mathcal{I})$ is *Marczewski-Burstin representable* ("MB-representable") by $\mathcal{P}$ iff the following two properties hold (see [BT], [BBC1]).

1. For all $Y \subseteq A$, $Y \in \mathcal{A}$ iff for every $P \in \mathcal{P}$, there is a $Q \in \mathcal{P}$, $Q \subseteq P$, such that Q is either a subset of, or disjoint from Y.

2. For all $Y \subseteq A$, $Y \in \mathcal{I}$ iff for every $P \in \mathcal{P}$, there is a $Q \in \mathcal{P}$, $Q \subseteq P$, such that Q is disjoint from Y.

An interesting fact is that one of these two conditions is equivalent to the density property in the case where $\mathcal{P} = \mathcal{I}^+$.

Theorem 6. *The pair $(\mathcal{A}, \mathcal{I})$ has the density property iff the following equivalence is true: for all $Y \subseteq A$, $Y \in \mathcal{I}$ iff for every $P \in \mathcal{I}^+$, there is a $Q \in \mathcal{I}^+$, $Q \subseteq P$, such that Q is disjoint from Y.*

PROOF. ($\Rightarrow$) Suppose $(\mathcal{A}, \mathcal{I})$ has the density property, and let $Y \subseteq A$. If $Y \in \mathcal{I}$, then for any $P \in \mathcal{I}^+$ we have that $P \setminus Y \in \mathcal{I}^+$ is disjoint from Y. In the other direction, suppose that for every $P \in \mathcal{I}^+$, there is a $Q \in \mathcal{I}^+$ such that Q is disjoint from Y. Then it is easy to see that this implies that $\mathcal{S} = \{U \in \mathcal{I}^+ : U \cap Y \text{ is empty}\}$ is predense in $\mathcal{I}^+$. Thus, by the density property, $Y \subseteq A \setminus \bigcup \mathcal{S} \in \mathcal{I}$.

($\Leftarrow$). Suppose that $Y \in \mathcal{I}$ iff for every $P \in \mathcal{I}^+$, there is a $Q \in \mathcal{I}^+$, $Q \subseteq P$, such that Q is disjoint from Y. Let $\mathcal{S}$ be a predense subset of $\mathcal{I}^+$, and let $Y = A \setminus \bigcup \mathcal{S}$. Let $P \in \mathcal{I}^+$. Then by the definition of predense, there must be a $Q \in \mathcal{I}^+$, $Q \subseteq P$, such that $Q \subseteq W$ for some $W \in \mathcal{S}$. Clearly, Q and Y are disjoint, so since $P \in \mathcal{I}^+$ was arbitrary, $Y \in \mathcal{I}$. $\square$

Theorem 7. *If $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra, and $(\mathcal{A}, \mathcal{I})$ has the density property, then $(\mathcal{A}, \mathcal{I})$ has the hull property.*

PROOF. Let $Y \subseteq A$, and let $\mathcal{U} = \{V \in \mathcal{I}^+ : Y \subseteq V\}$. Then, since $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra, $\mathcal{U}$ has a greatest lower bound $U \in \mathcal{A}$. Let $\mathcal{S} = (\{U\} \cup \{A \setminus V : V \in \mathcal{U}\}) \cap \mathcal{I}^+$. It is easy see that $\mathcal{S}$ is predense in $\mathcal{I}^+$, for otherwise, there would be a $W \in \mathcal{I}^+$ such that $W \cap V \in \mathcal{I}$ for all $V \in \mathcal{S}$, and then $U \cup W$ would be a lower bound of $\mathcal{U}$ which would contradict the fact that U is the greatest lower bound of $\mathcal{U}$. Thus, since $\mathcal{S}$ is predense, $Z = A \setminus \bigcup \mathcal{S} \in \mathcal{I}$. Then $U \cup Z$ is equivalent to U (mod $\mathcal{I}$), and is therefore also a greatest lower bound of $\mathcal{U}$. But $Y \subseteq U \cup Z$, so $U \cup Z$ is the desired set which witnesses the hull property. $\square$

Theorem 8. *If $(\mathcal{A}, \mathcal{I})$ satisfies the hull property, then $(\mathcal{A}, \mathcal{I})$ is MB-representable by $\mathcal{I}^+$.*

PROOF. Assume the hypothesis of the theorem, and let $Y \subseteq A$. We need to show that $Y \in \mathcal{A}$ iff for every $P \in \mathcal{I}^+$ there is a $Q \in \mathcal{I}^+$, $Q \subseteq P$, such that Q is either contained in or disjoint from Y. If $Y \in \mathcal{A}$, and $P \in \mathcal{I}^+$, then since $P = (P \cap Y) \cup (P \setminus Y)$, at least one of $P \cap Y$ and $P \setminus Y$ must be in $\mathcal{I}^+$. In the other direction, suppose that $Y \notin \mathcal{A}$. By the hull property, there is a $U \in \mathcal{A}$ such that $Y \subseteq U$, and U is smallest possible (mod $\mathcal{I}$). By the dual hull property, there is a $L \in \mathcal{A}$ such that $L \subseteq Y$ and L is largest possible (mod $\mathcal{I}$). Now, since $Y \notin \mathcal{A}$, and $L \subseteq Y \subseteq U$, neither $U \setminus Y$ nor $Y \setminus L$ can be elements of $\mathcal{A}$, so neither are elements of $\mathcal{I}$, and thus $U \setminus L \notin \mathcal{I}$. Let $P = U \setminus L$, and since $P \in \mathcal{A}$, $P \in \mathcal{I}^+$. Let Q be any element of $\mathcal{I}^+$ which is a subset of P. Then Q cannot be a subset of $U \setminus Y$, because we would then have $Y \subseteq U \setminus Q$, contradicting leastness of U. For a similar reason, Q cannot be a subset of $Y \setminus L$. Thus, Q cannot be either contained in or disjoint from Y, and we have therefore finished proving part (1) of Definition 2. Since $Y \in \mathcal{I}$ iff $Y \in \mathcal{A}$ and Y contains no member of $\mathcal{I}^+$, part (2) of definition follows directly from what we have already proven. □

Theorem 9. *There exists a pair* $(\mathcal{A}, \mathcal{I}) \in K$ *having the density property, but for which* $(\mathcal{A}, \mathcal{I})$ *is not MB-representable by* $\mathcal{I}^+$. *Examples can be found in which* A *has cardinality continuum,* $\mathcal{A}$ *is a* σ*-algebra, and* $\mathcal{I}$ *is a* σ*-ideal.*

PROOF. Let κ be any uncountable regular cardinal, and for each $\alpha < \kappa$, let $(\mathcal{A}_\alpha, \mathcal{I}_\alpha) \in K$ be (σ-algebra, σ-ideal) pairs satisfying the hull property such that the sets A_α are pairwise disjoint, and for each $\alpha < \kappa$, there is an $Z_\alpha \in \mathcal{A}_\alpha$ such that both Z_α and $A_\alpha \setminus Z_\alpha$ are members of $\mathcal{I}_\alpha^+$. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$, and define $\mathcal{I} = \{X \subseteq A\colon \text{For all } \alpha < \kappa,\ X \cap A_\alpha \in \mathcal{I}_\alpha\}$, and $\mathcal{A} = \{X \subseteq A\colon$ For all $\alpha < \kappa$, $X \cap A_\alpha \in \mathcal{A}_\alpha$, and for all but countably many $\alpha < \kappa$, either $X \cap A_\alpha \in \mathcal{I}_\alpha$ or $A_\alpha \setminus X \in \mathcal{I}_\alpha$. Then it is easy to verify that $(\mathcal{A}, \mathcal{I})$ satisfies (2) of Definition 4, but that $Z = \bigcup_{\alpha<\kappa} Z_\alpha$ is a counterexample for (1) of Definition 4. □

4 Properties Weaker Than the Complete Boolean Algebra Property

Definition 5. Let us say that the pair $(\mathcal{A}, \mathcal{I})$ has the *splitting property* iff whenever $\mathcal{C}$ and $\mathcal{D}$ are antichains of $(\mathcal{A}, \mathcal{I})$ such that $C \cap D \in \mathcal{I}$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$, then there are sets $E_C, C \in \mathcal{C}$ and $F_D, D \in \mathcal{D}$ such that $E_C, F_D, \in \mathcal{I}$ and $C \setminus E_C$ and $D \setminus F_D$ are disjoint for all $C \in \mathcal{C}$, $D \in \mathcal{D}$. We define $(\mathcal{A}, \mathcal{I})$ to have the *weak splitting property* iff whenever $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{I}^+$ are such that $X_1 \cap X_2 \in \mathcal{I}$ for all $X_i \in \mathcal{C}_i$ $(i = 1, 2)$, there are $\mathcal{D}_i$ such that $\mathcal{D}_i$ is predense below $\mathcal{C}_i$ $(i = 1, 2)$, and such that $X_1 \cap X_2$ is empty for all $X_i \in \mathcal{D}_i$ $(i = 1, 2)$. Since the relations "refinement of" and "predense below" are easily seen to be transitive relations on subsets of $\mathcal{A}$, it is easy to check that the splitting property implies the weak splitting property.

Proposition 10. *If $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra, then $(\mathcal{A},\mathcal{I})$ has the splitting property.*

PROOF. Let S and T be least upper bounds for $\mathcal{C}$ and $\mathcal{D}$ respectively. Then $S \cap T \in \mathcal{I}$, so by shrinking one of S or T by a member of $\mathcal{I}$, if necessary, we may assume without loss of generality that $S \cap T$ is empty. Then $E_C = C \setminus (C \cap S)$ and $F_D = D \setminus (D \cap T)$ are as desired. (Note that it was not required in this proof that $\mathcal{C}$ and $\mathcal{D}$ be antichains.) $\square$

The desired result is to prove that the hull property plus the weak splitting property implies a complete quotient algebra. In fact, we get a better result than this.

Theorem 11. *If $(\mathcal{A},\mathcal{I})$ satisfies the weak splitting property, and $(\mathcal{A},\mathcal{I})$ is MB-representable by $\mathcal{I}^+$, then $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra.*

PROOF. Assume the hypotheses of the theorem, and let $\mathcal{C} \subseteq \mathcal{A}$. We need to show that $\mathcal{C}$ has a least upper bound in $\mathcal{A}$ (mod $\mathcal{I}$). Let $\mathcal{C}_1 = \mathcal{C}$ and define $\mathcal{C}_2$ to consist of all $X \in \mathcal{I}^+$ such that $X \cap Y \in \mathcal{I}$ for all $Y \in \mathcal{C}_1$. Then, let $\mathcal{D}_1, \mathcal{D}_2$ be the sets guaranteed by the weak splitting property. Finally, let $L = \bigcup \mathcal{D}_1$, and we want to show that L is the desired least upper bound. We first need to show that $L \in \mathcal{A}$, and we use the MB-representation to do this. Thus, let $P \in \mathcal{I}^+$. Since $\mathcal{C}_1 \cup \mathcal{C}_2$ is predense by definition of $\mathcal{C}_2$, so is $\mathcal{D}_1 \cup \mathcal{D}_2$. Thus, there is an $i \in \{1,2\}$ and a $Q \in \mathcal{I}^+$ with $Q \subseteq P$ such that $Q \subseteq D$ for some $D \in \mathcal{D}_i$. Then Q is either contained in or disjoint from L, depending on whether i is 1 or 2. Since P was arbitrary, this proves that $L \in \mathcal{A}$. Since L contains every member of $\mathcal{D}_1$, L is an upper bound of $\mathcal{D}_1$, and therefore of $\mathcal{C}_1$ (since $\mathcal{C}_1$ is predense below $\mathcal{D}_1$). Thus, L is an upper bound of $\mathcal{C}$. Since L misses every member of $\mathcal{D}_2$, L must be a least upper bound, for the same reason. $\square$

Corollary 12. *If $(\mathcal{A},\mathcal{I})$ satisfies the weak splitting property, and $(\mathcal{A},\mathcal{I})$ is MB-representable by $\mathcal{I}^+$, then $(\mathcal{A},\mathcal{I})$ satisfies the hull property.*

PROOF. By the previous theorem, $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra. Thus, by results of the previous section, $(\mathcal{A},\mathcal{I})$ has the density property, and it therefore also has the hull property. $\square$

Definition 6. Let us say that the pair $(\mathcal{A},\mathcal{I})$ is *Borel dense* iff

1. $\mathcal{A}$ is a σ-algebra of subsets of some complete, separable metric space X, and $\mathcal{I} \subseteq \mathcal{A}$ is a σ-ideal over X.

2. $\mathcal{A}$ contains the set $\mathcal{B}$ of all Borel subsets of X.
3. $\mathcal{I}^+ \cap \mathcal{B}$ is a dense subset of $\mathcal{I}^+$.

Proposition 13. *If $(\mathcal{A}, \mathcal{I}) \in K_\sigma$ is Borel dense, and the continuum hypothesis holds, then $(\mathcal{A}, \mathcal{I})$ has the splitting property.*

Proof. If $\mathcal{C}$ is an antichain in $\mathcal{I}^+$, then by Borel denseness, we can pick a Borel set $B_C \in \mathcal{I}^+$ such that $B_C \subseteq C$ for each $C \in \mathcal{C}$. These sets B_C must be distinct (since otherwise $\mathcal{C}$ would not have been an antichain). Thus, since there are no more than ω_1 Borel sets (assuming CH), no antichain can have size more than ω_1. Thus, since we have a σ-algebra and a σ-ideal, it is easy to see that any antichain can be replaced by a pairwise disjoint antichain in which the members are equivalent (mod $\mathcal{I}$) to the original antichain. This is clearly stronger than the splitting property. $\square$

Since many pairs $(\mathcal{A}, \mathcal{I})$ which are of interest are Borel-dense, the above theorem gives a convenient way of proving the splitting property when CH holds. This will give us some examples (under CH) of pairs satisfying the splitting property that do not have a complete quotient algebra. It is not clear if the CH hypothesis can be eliminated. However, if "splitting" is replaced by "weak splitting", it will be shown that the same examples need no additional set theoretic hypotheses. The examples given here will use known algebras and ideals from the literature. For convenience, the "definitions" given here will use known equivalences which are somewhat shorter to state. See [BC] for an extensive survey of these ideas.

Definition 7. A subset X of the set of real numbers is said to be *universally measurable* (written $X \in \mathcal{U}$) iff for every homeomorphism h of the reals, $h(A)$ is Lebesgue measurable, and X has *universal measure zero* (written $x \in \mathcal{U}_0$) iff $h(A)$ has Lebesgue measure zero for every such h. X is said to have the *Baire property* (written $X \in \mathcal{B}_w$) iff it the the symmetric difference of an open set and a first category set. X has the *Baire property in the restricted sense* (written $X \in \mathcal{B}_r$) iff $X \cap P$ has the Baire property (in the relative topology for P) for every perfect subset P of the set of real numbers, and X is *always of first category* (written $x \in$AFC) iff $X \cap P$ is of first category in P for every perfect set P.

The pairs of current interest are $(\mathcal{U}, \mathcal{U}_0)$ and ($\mathcal{B}_r$,AFC), and some well known facts that will be used below are that $\mathcal{U}, \mathcal{B}_r \subseteq (s)$, $\mathcal{U}_0$,AFC$\subseteq (s)^0$, and that $(\mathcal{U}_0)^+$ and AFC$^+$ are dense subsets of $((s)^0)^+$. We give the proof for the pair $(\mathcal{U}, \mathcal{U}_0)$, with the proof for the pair ($\mathcal{B}_r$,AFC) (and for the pair (Borel, Countable)) being very similar.

Theorem 14. *$\mathcal{U}/\mathcal{U}_0$ is not a complete Boolean algebra.*

PROOF. By [MP], there is a Vitali set V which is Marczewski measurable. Let $\mathcal{S} = \{X \in \mathcal{U} : X \subseteq V\}$. We shall show that $\mathcal{S}$ does not have any least upper bound (mod $\mathcal{U}_0$) in $\mathcal{U}$. Suppose otherwise, and let $W \in \mathcal{U}$ be a least upper bound of $\mathcal{S}$. Then, since $\mathcal{U} \subseteq (s)$, $\mathcal{U}_0 \subseteq (s)^0$, and $\mathcal{U}_0^+$ is a dense subset of $((s)^0)^+$, W is also a least upper bound of S in (s). In addition, denseness of $\mathcal{U}_0^+$ in $((s)^0)^+$ implies that V is also a least upper bound of $\mathcal{S}$ in (s). Thus, V and W must be equivalent mod $(s)^0$, and there are sets $A, B \subseteq (s)^0$ with $W = (V \cup A) \setminus B$ (and, without loss of generality, A disjoint from V and $B \subseteq V$).

Let $Z = \bigcup_{q\in\mathbf{Q}}((A\cup B) + q)$, where $\mathbf{Q}$ is the set of rational numbers, and $(A\cup B) + q$ is the translation of $A\cup B$ by q. Clearly, $(A\cup B) + q \in (s)^0$ for every real number q, and therefore $Z \in (s)^0$, since $(s)^0$ is a σ-ideal. Let $Y = \bigcup_{q,r\in\mathbf{Q},q\neq r}((W + q) \cap (W + r))$. Since W is Lebesgue measurable, $W + q$ is Lebesgue measurable for all real numbers q, and therefore Y is Lebesgue measurable. Let $X = W\setminus Y$. Then X is Lebesgue measurable, and satisfies the property that $(X + q) \cap (X + r)$ is empty for any distinct rational numbers q and r, i.e., X is a subset of some Vitali set. Let $T = \mathbb{R}\setminus\bigcup_{q\in\mathbf{Q}}(X + q)$. Then T is clearly Lebesgue measurable, since X is. Our goal is to show that T is Lebesgue null, and for that we need the following claim.

Claim. $T \subseteq Z$.

PROOF OF CLAIM: Let $x \in T$. Then by the definition of a Vitali set, $x\in V + q$ for some rational number q. There are now two cases:

Case 1: $x\notin W + q$. Then $x - q\in V\setminus W\subseteq B$, and therefore $x\in B + q\subseteq Z$.

Case 2: $x\in W+q$. Then since $x\notin X+q$, $x\in Y$. Thus, $x\in(W+p)\cap(W+r)$ for some distinct rational numbers p and r. On the other hand, $x\notin(V+p)\cap(V+r)$ (by the definition of a Vitali set), and we must therefore have either $x \in A+q$ or $x \in A + r$, i.e., $x \in Z$. This finishes the proof of the claim.

Now, $T \subseteq Z$ implies that $T \in (s)^0$, and T must therefore have Lebesgue measure zero, since every set of positive Lebesgue measure contains a perfect subset (and a set containing a perfect set cannot be in $(s)^0$). Thus, X can be extended to a Vitali set V' by adding elements of T. But then V' would be Lebesgue measurable (a union of X and a Lebesgue null set), contradicting that no Vitali set can be Lebesgue measurable. □

Theorem 15. *$(\mathcal{U},\mathcal{U}_0)$ satisfies the weak splitting property.*

PROOF. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{U}_0^+$ be as in the definition of weak splitting property, and let $\mathcal{M}$ be a collection of perfect sets such that for every $P \in \mathcal{M}$, either $P \subseteq C$ for some $C \in \mathcal{C}_1 \cup \mathcal{C}_2$ or $P \cap C \in \mathcal{U}_0$ for every $C \in \mathcal{C}_1 \cup \mathcal{C}_2$. Since $\mathcal{U}_0^+$ is dense in $((s)^0)^+$, it is easy to check that $\mathcal{M}$ is a maximal antichain in $((s)^0)^+ = (s)\backslash(s)^0$, and therefore by Lemma 4 there is a pairwise disjoint maximal antichain $\mathcal{M}'$ refining $\mathcal{M}$. If we let $\mathcal{D}_i = \{D \in \mathcal{M}' : D \subseteq C$ for some $C \in \mathcal{C}_i\}$ $(i = 1, 2)$, then it is easy to check that $\mathcal{D}_1$ and $\mathcal{D}_2$ are as desired. $\square$

Corollary 16. *The weak splitting property does not imply a complete quotient algebra. Under CH, the splitting property does not imply a complete quotient algebra.*

5 Summary and Questions

The main properties can be given in the following two diagrams.

$(\mathcal{A}, \mathcal{I})$ satisfies the Marczewski hull property.
$\Downarrow$ ($\Uparrow$?)
$(\mathcal{A}, \mathcal{I})$ is MB-representable by $\mathcal{I}^+$.
$\Downarrow$ ($\not\Uparrow$)
$(\mathcal{A}, \mathcal{I})$ satisfies the density property.

$\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra.
$\Downarrow$ ($\not\Uparrow$ if CH holds)
$(\mathcal{A}, \mathcal{I})$ satisfies the splitting property.
$\Downarrow$ ($\Uparrow$?)
$(\mathcal{A}, \mathcal{I})$ satisfies the weak splitting property.

In addition, combining the results, we know that if $(\mathcal{A}, \mathcal{I})$ satisfies the weak splitting property, and $(\mathcal{A}, \mathcal{I})$ is MB-representable by $\mathcal{I}^+$, then all of the above properties hold.

The following two questions suggested by the above figures, which were listed as open problems in the original version of this paper, have recently been solved in the affirmative by M. Balcerzak, A. Bartoszewicz, and K. Ciesielski. These results will appear in [BBC2], which will also give a ZFC example of an algebra with the splitting property but a non-complete quotient algebra.

Question 1. Does there exist a pair $(\mathcal{A}, \mathcal{I})$ such that $(\mathcal{A}, \mathcal{I})$ is MB-representable by $\mathcal{I}^+$, but $(\mathcal{A}, \mathcal{I})$ does not satisfy the hull property.

Another interesting open question which is not evident in the above figures is:

Question 2. Are there examples of pairs $(\mathcal{A}, \mathcal{I})$ which satisfy the weak splitting property and the density property, but for which $(\mathcal{A}, \mathcal{I})$ is not MB-representable by $\mathcal{I}^+$?

References

[BB] S. Baldwin, J. B. Brown, *A Simple Proof that* $(s)/(s^0)$ *is a complete Boolean Algebra,* Real Analysis Exchange, **24** (1998/9), 855–8.

[BBC1] M. Balcerzak, A. Bartoszewicz, K. Ciesielski, *On Marczewski-Burstin representations of certain algebras of sets,* Real Analysis Exchange, **26** (2) (2000–2001), 581–591.

[BBC2] M. Balcerzak, A. Bartoszewicz, K. Ciesielski, *Algebras with inner MB-represention,* preprint.

[BC] J. B. Brown, G. V. Cox, *Classical Theory of Totally Imperfect Spaces,* Real Analysis Exchange, **7** (1981-2), 185–232.

[BT] J. B. Brown, H. Elalaoui-Talibi, *Marczewski-Burstin-like characterizations of* σ*-algebras, ideals, and measurable functions,* Colloquium Mathematicum, **82** (1999), 277–286.

[J] T. Jech, *Set Theory*, Academic Press, New York, San Francisco, London, 1978.

[M] E. Szprilrajn (Marczewski), *Sur une classe de fonction de M. Sierpinski et la classe correspondante d'ensembles,* Fundamenta Mathematica, **24** (1935), 17–34.

[MP] A. W. Miller, S. G. Posvassilev, *Vitali sets and Hamel bases that are Marczewski measurable,* Fundamenta Mathematicae, **166** (2000), 269–279.

[S] M. H. Stone, *The theory of representations for Boolean algebras,* Transactions of the American Mathematical Society, **40** (1936), 37–111.

[W] J. T. Walsh, *Marczewski Sets, Measure, and the Baire Property, II,* Proceedings of the American Mathematical Society, **106** (1989), 1027–1030.