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BLOCH AND GAP SUBHARMONIC FUNCTIONS

Abstract

For subharmonic functions $u \geq 0$ in the unit ball B_N of $\mathbb{R}^N$, the paper characterizes this kind of growth: $\sup_{x\in B_N}(1-|x|^2)^\alpha u(x) < +\infty$ (given $\alpha > 0$), through criteria involving such integrals as $\int u(x)\,dx$ or $\int u(x)(1-|x|^2)^{\alpha-N}\,dx$ over balls centered at points $a \in B_N$. Given $p \in \mathbb{R}$ and ω some non–negative function, this article compares subharmonic functions with the previous kind of growth to subharmonic functions satisfying: $\sup_{a\in B_N}\int_{B_N} u(x)(1-|x|^2)^p\omega(|\varphi_a(x)|)\,dx < +\infty$, where φ_a are Möbius transformations. The paper also studies subharmonic functions which are sums of lacunary series and their links with both previous kinds of subharmonic functions.

1 Introduction.

Throughout the paper, $N \geq 2$ denotes a fixed integer and $|\,.\,|$ the Euclidean norm in $\mathbb{R}^N$.

Definition 1. Given $\alpha > 0$, let $\mathcal{B}_\alpha$ denote the set of all positive subharmonic functions u in $B_N = \{x \in \mathbb{R}^N : |x| < 1\}$ such that

$$G_\alpha(u) := \sup_{x\in B_N}(1-|x|^2)^\alpha u(x) < +\infty. \tag{1}$$

Remark 1. When $N = 2$, the holomorphic functions f in the unit disk of $\mathbb{C}$ such that $u = |f'|$ satisfies (1) form the so–called α-Bloch space (see [8, page 10]).

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Definition 2. For any $a \in B_N$ and any $R \in [0,1[$, let $B(a,R) = \{x \in B_N : |x-a| < R\}$, with $\operatorname{Vol} B(a,R)$ the volume of this ball. In particular $\operatorname{Vol} B(a,R) = V_N R^N$ where $V_N = \frac{2\pi^{N/2}}{N\cdot\Gamma(N/2)}$ is the volume of B_N, see [2, p.29]. We note $R_a = R\dfrac{1-|a|^2}{1+R|a|}$.

Theorem 1 establishes the following characterization of $\mathcal{B}_\alpha$.

$$u \in \mathcal{B}_\alpha \Longleftrightarrow \sup_{a\in B_N} \left(\frac{1}{\operatorname{Vol} B(a,R_a)}\right)^{1-\frac{\alpha}{N}} \int_{B(a,R_a)} u(x)\,dx < +\infty \tag{2}$$

whatever $R \in]0,1[$. In Theorem 2 and Proposition 1, we observe that only implication $\Longleftarrow$ still holds when the ball $B(a,R_a)$ is replaced by an ellipsoid $E(a,R) = \{x \in B_N : |\varphi_a(x)| < R\}$, the transformation φ_a being defined by:

$$\varphi_a(x) = \frac{a - P_a(x) - \sqrt{1-|a|^2}\,Q_a(x)}{1-\langle x,a\rangle} \ \forall x \in B_N,$$

where $\langle x,a\rangle = \sum_{j=1}^N x_j a_j$ for $x = (x_1, x_2, \ldots, x_N)$, $a = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$, $P_a(x) = \frac{\langle x,a\rangle}{|a|^2} a$ and $Q_a(x) = x - P_a(x)$, with $P_a(x) = 0$ if $a = 0$. This points out a significant difference with the α–Bloch space of holomorphic functions in the unit disk of $\mathbb{C}$. This space is characterized ([9]) by a property similar to (2), with $B(a,R_a)$ replaced by $E(a,R)$ which happens to be an Euclidean disk when $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in the unit disk of $\mathbb{C}$. Another difference with the case of $\mathbb{C}$ is outlined in Section 2. Our set $\mathcal{B}_\alpha$ is not invariant under the map φ_a ($a \in B_N$, $a \neq 0$). From now on, φ_a will always denote this more general automorphism defined above. This transformation φ_a is an automorphism of the unit ball of $\mathbb{C}^N$ (cf. [1, p.115] or [5, pp.25–30]). In this paper, we work on the unit ball of $\mathbb{R}^N$, but many interesting properties of φ_a carry over to the real case.

Theorem 3 sets forth another characterization of $\mathcal{B}_\alpha$; namely for all $R \in]0,1[$,

$$u \in \mathcal{B}_\alpha \Longleftrightarrow \sup_{a\in B_N} \int_{B(a,R_a)} u(x)(1-|x|^2)^{\alpha-N}\,dx < +\infty.$$

Definition 3. Given $p \in \mathbb{R}$ and $\omega : [0,1[\to [0,+\infty[$ a measurable function, let $\mathcal{SH}(p,\omega)$ denote the set of all non–negative subharmonic functions u in B_N which satisfy

$$S_{p,\omega}(u) := \sup_{a\in B_N} \int_{B_N} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\,dx < +\infty.$$

When the function ω is decreasing and such that

$$\Omega := \int_0^1 \frac{\omega(r) r^{N-1}}{(1-r^2)^{\frac{N+1}{2}}}\, dr < +\infty, \tag{3}$$

Theorems 4 and 5 prove that $\mathcal{B}_\alpha \subset \mathcal{SH}(p,\omega) \subset \mathcal{B}_\gamma$ for $0 < \alpha < p + \frac{N+1}{2} < p + N \leq \gamma$. Propositions 4 and 5 provide counterexamples to show that the converse inclusions do not hold.

Section 5 studies the case of gap subharmonic functions of the form $u(x) = \sum_{k=1}^{+\infty} c_k |x|^{2^k}$. Theorems 6, 7 and Propositions 7, 8 give several criteria for such functions to belong to $\mathcal{B}_\alpha$ or $\mathcal{SH}(p,\omega)$.

Technical lemmas 1–7 are postponed to the appendix (see Section 6).

Let us end Section 1 with some remarks about the significance of p and ω in Definition 3. For holomorphic functions f in the unit disk of $\mathbb{C}$, let $S_{p,\omega}(|f'|^q)$ be defined, for any $q > 0$, as in Definition 3, with φ_a replaced by map $z \mapsto \frac{a-z}{1-\bar{a}z}$ where a and z belong to the unit disk of $\mathbb{C}$, identified with B_2. If $\omega(r) = \log\frac{1}{r}$ and $p = 0$, then $S_{p,\omega}(|f'|^2) < +\infty$ means that f belongs to the space $BMOA$. If $\omega(r) = \left(\log\frac{1}{r}\right)^s$ with $s > 1$, $p > -2$ and $q > 0$, then $S_{p,\omega}(|f'|^q) < +\infty$ means that f belongs to the $\frac{p+2}{q}$–Bloch space. If $\omega \equiv 1$ and $p = 1$, then $S_{p,\omega}(|f'|^2) < +\infty$ means that f belongs to the Hardy space H^2. If $\omega \equiv 1$ and $p \geq 1$, then $S_{p,\omega}(|f'|^p) < +\infty$ means that f belongs to the Bergman space L^p_a. If $\omega \equiv 1$ and $p > -1$, then $S_{p,\omega}(|f'|^2) < +\infty$ means that f belongs to the Dirichlet space D_p. If $\omega \equiv 1$ and $p > -1$, then $S_{p,\omega}(|f'|^{p+2}) < +\infty$ means that f belongs to the $(p+2)$–Besov space. More details and references about these spaces may be found in [8].

2 The Set $\mathcal{B}_\alpha$ Is Not Möbius–Invariant.

Given $a \in B_N$, if $u \in \mathcal{B}_\alpha$ is such that $u \circ \varphi_a$ remains subharmonic in B_N, then $u \circ \varphi_a \in \mathcal{B}_\alpha$.

This assertion follows from Lemma 2 (see Section 6).

Let $x \in B_N$ and $y = \varphi_a(x) = \varphi_a^{-1}(x)$. Then $1 - \langle x, a\rangle \geq 1 - |a| > 0$. Hence

$$\begin{aligned}(1-|x|^2)^\alpha u(\varphi_a(x)) = (1-|\varphi_a(y)|^2)^\alpha u(y) &= \frac{(1-|y|^2)^\alpha (1-|a|^2)^\alpha u(y)}{(1-\langle y,a\rangle)^{2\alpha}} \\ &\leq \frac{(1-|y|^2)^\alpha u(y)(1-|a|^2)^\alpha}{(1-|a|)^{2\alpha}} \leq \left(\frac{1+|a|}{1-|a|}\right)^\alpha G_\alpha(u) < +\infty.\end{aligned}$$

Remark 2. For $u \in \mathcal{B}_\alpha$, the function $u \circ \varphi_a$ is not necessarily subharmonic in B_N.

Example. Given $a \in B_N$, $a \neq 0$, the function u defined by $u(x) = 1 + \langle x, a\rangle$ $\forall x \in B_N$ belongs to $\mathcal{B}_\alpha$ (for any $\alpha > 0$) but $u \circ \varphi_a$ is not subharmonic. This function u is subharmonic and even harmonic in $\mathbb{R}^N$ since its Laplacian is identically zero. Moreover $u(x) \geq 0$ $\forall x \in B_N$ since $|\langle x, a\rangle| \leq |x|.|a| < 1$ $\forall x \in B_N$ $\forall a \in B_N$. As u is bounded on B_N, (1) obviously holds. Now

$$v(x) := u(\varphi_a(x)) = 1 + \langle \varphi_a(x), a\rangle = 1 + \frac{|a|^2 - \langle x, a\rangle}{1 - \langle x, a\rangle} =$$

$$= 1 + \frac{|a|^2 - 1 + 1 - \langle x, a\rangle}{1 - \langle x, a\rangle} = 2 - \frac{1 - |a|^2}{1 - \langle x, a\rangle}.$$

For any $j \in \{1, 2, ..., N\}$, we have:

$$\frac{\partial v}{\partial x_j}(x) = -(1-|a|^2)\frac{a_j}{(1 - \langle x, a\rangle)^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x_j^2}(x) = -(1-|a|^2)\frac{2a_j^2}{(1 - \langle x, a\rangle)^3}$$

thus $\Delta v(x) = -\dfrac{2(1 - |a|^2)|a|^2}{(1 - \langle x, a\rangle)^3} < 0$ $\forall x \in B_N$.

3 Averaging Over Balls and Ellipsoids

Theorem 1. *Given $\alpha > 0$ and $R \in]0, 1[$, a subharmonic function $u \geq 0$ belongs to $\mathcal{B}_\alpha$ if and only if*

$$M_{\alpha,R}(u) := \sup_{a \in B_N} \frac{1}{[\operatorname{Vol} B(a, R_a)]^{1-\frac{\alpha}{N}}} \int_{B(a,R_a)} u(x)\, dx < +\infty.$$

Moreover $\left(\frac{1}{1+R}\right)^\alpha G_\alpha(u) \leq \left(\frac{1}{R\sqrt[N]{V_N}}\right)^\alpha M_{\alpha,R}(u) \leq \left(\frac{1+R}{1-R}\right)^\alpha G_\alpha(u)$.

Proof. $\Longleftarrow$ Let $a \in B_N$. The subharmonicity of u yields

$$u(a) \leq \frac{1}{\operatorname{Vol} B(a, R_a)} \int_{B(a,R_a)} u(x)\, dx.$$

Now $1 - |a|^2 = \dfrac{1 + R|a|}{R} R_a \leq \dfrac{1 + R}{R} \left(\dfrac{\operatorname{Vol} B(a, R_a)}{V_N}\right)^{1/N}$. Hence

$$u(a)(1 - |a|^2)^\alpha \leq \left(\frac{1 + R}{R\sqrt[N]{V_N}}\right)^\alpha \frac{1}{[\operatorname{Vol} B(a, R_a)]^{1-\frac{\alpha}{N}}} \int_{B(a,R_a)} u(x)\, dx.$$

$\Longrightarrow$ Since $u \in \mathcal{B}_\alpha$, we have $u(x) \le \frac{G_\alpha(u)}{(1-|x|^2)^\alpha}$ $\forall x \in B_N$. Let $a \in B_N$. By Lemma 1, $\frac{1}{1-|x|^2} \le \frac{1+R}{1-R}\frac{1}{1-|a|^2}$ $\forall x \in B(a, R_a)$. Thus

$$\int_{B(a,R_a)} u(x)\,dx \le G_\alpha(u)\left(\frac{1+R}{1-R}\right)^\alpha \frac{1}{(1-|a|^2)^\alpha}\cdot \operatorname{Vol} B(a,R_a);$$

so that

$$\begin{aligned}\frac{1}{[\operatorname{Vol} B(a,R_a)]^{1-\frac{\alpha}{N}}}\int_{B(a,R_a)} u(x)\,dx &\le G_\alpha(u)\left(\frac{1+R}{1-R}\right)^\alpha \frac{[\operatorname{Vol} B(a,R_a)]^{\alpha/N}}{(1-|a|^2)^\alpha}\\ &= G_\alpha(u)\left(\frac{1+R}{1-R}\right)^\alpha \frac{1}{(1-|a|^2)^\alpha} V_N^{\alpha/N} R^\alpha \frac{(1-|a|^2)^\alpha}{(1+R|a|)^\alpha}\\ &\le G_\alpha(u)\left(\frac{1+R}{1-R} R \sqrt[N]{V_N}\right)^\alpha . \qquad \square\end{aligned}$$

Corollary 1. *Let $\alpha > 0$ and $u \in \mathcal{B}_\alpha$. Then $M_{\alpha,R}(u) < +\infty$ $\forall R \in]0,1[$. If there exist constants $C > 0$ and $\varepsilon > 0$ such that $M_{\alpha,R}(u) \le CR^{\alpha+\varepsilon}$ $\forall R \in]0,1[$, then u is the function identically zero in B_N.*

Proof. If $G_\alpha(u) \ne 0$, Theorem 1 implies $M_{\alpha,R}(u) \sim R^\alpha V_N^{\alpha/N} G_\alpha(u)$ as $R \to 0^+$, which is a contradiction. $\square$

Theorem 2. *Let $\alpha > 0$, $R \in]0,1[$ and u a non–negative subharmonic function in B_N. If*

$$L_{\alpha,R}(u) := \sup_{a\in B_N} \frac{1}{[\operatorname{Vol} E(a,R)]^{2\frac{N-\alpha}{N+1}}}\int_{E(a,R)} u(x)\,dx < +\infty,$$

then $u \in \mathcal{B}_\alpha$, with $G_\alpha(u) \le m_\alpha(R)\left[V_N R^N\right]^{1-2\frac{\alpha+1}{N+1}} L_{\alpha,R}(u)$ where $m_\alpha(R) = \frac{(1-R^2)^\alpha}{(1-R)^N}$ if $0 < \alpha \le N$ and $m_\alpha(R) = \left(\frac{2}{2\alpha-N}\right)^{2\alpha-N}(\alpha-N)^{\alpha-N}\alpha^\alpha$ if $\alpha > N$.

Remark 3. When $\alpha > N$ and $0 < R \le \frac{N}{2\alpha-N}$, the above upper bound of $G_\alpha(u)$ still holds with $m_\alpha(R) = \frac{(1-R^2)^\alpha}{(1-R)^N}$ and is even sharper.

Proof. Let $a \in B_N$. Since $u \ge 0$, Lemma 3 (Section 6) and the subharmonicity of u lead to

$$\begin{aligned}\int_{E(a,R)} u(x)\,dx \ge \int_{B(a,R_a)} u(x)\,dx &\ge u(a)\operatorname{Vol} B(a,R_a)\\ &= u(a) V_N R^N \frac{(1-|a|^2)^N}{(1+R|a|)^N}.\end{aligned} \tag{4}$$

Whence

$$u(a)(1-|a|^2)^\alpha \le \frac{1}{V_N R^N}\frac{(1+R|a|)^N}{(1-|a|^2)^{N-\alpha}}\int_{E(a,R)} u(x)\,dx.$$

As $1-|a|^2 = (1-R^2|a|^2)\left(\dfrac{\operatorname{Vol} E(a,R)}{V_N R^N}\right)^{\frac{2}{N+1}}$ according to Lemma 4, we obtain

$$\begin{aligned} u(a)(1-|a|^2)^\alpha &\le \frac{(1+R|a|)^N}{V_N R^N}\frac{(V_N R^N)^{\frac{2(N-\alpha)}{N+1}}}{(1-R^2|a|^2)^{N-\alpha}[\operatorname{Vol} E(a,R)]^{\frac{2(N-\alpha)}{N+1}}}\int_{E(a,R)} u(x)\,dx \\ &= \frac{(1-R^2|a|^2)^\alpha}{(1-R|a|)^N}\frac{(V_N R^N)^{\frac{N-1-2\alpha}{N+1}}}{[\operatorname{Vol} E(a,R)]^{\frac{2(N-\alpha)}{N+1}}}\int_{E(a,R)} u(x)\,dx. \end{aligned}$$

Let the function $g : [0,1[\to [0,+\infty[$ be defined by $g(t) = \frac{(1-t^2)^\alpha}{(1-t)^N}$. When $0<\alpha\le N$, g is increasing on $[0,1[$ so that $g(R|a|)\le g(R)$ $\forall a\in B_N$ $\forall R\in [0,1[$. When $\alpha > N$, a study of the derivative g' shows that g is increasing on $[0,\tau[$ with $\tau = \frac{N}{2\alpha-N}$ and decreasing on $]\tau,1[$, with maximum $g(\tau) = \left(\frac{2}{2\alpha-N}\right)^{2\alpha-N}(\alpha-N)^{\alpha-N}\alpha^\alpha$. Hence $g(R|a|)\le g(R)\le g(\tau)$ $\forall a\in B_N$ $\forall R\in [0,\tau]$ and $g(R|a|)\le g(\tau)$ $\forall a\in B_N$ $\forall R\in[\tau,1[$. □

Corollary 2. *Given $\alpha>0$, let $u\ge 0$ be a subharmonic function in B_N such that $L_{\alpha,R}(u) < +\infty$ $\forall R\in]0,1[$.*

(i) If $L_{\alpha,R}(u)\le C(1-R)^{N+\varepsilon}$ $\forall R\in]0,1[$ (for some constants $C>0$ and $\varepsilon>0$), then u is the function identically zero in B_N.

(ii) Let $\mu = \frac{2N(\alpha+1)}{N+1} - N$. If $L_{\alpha,R}(u)\le CR^{\mu+\varepsilon}$ $\forall R\in]0,1[$ (for some constants $C>0$ and $\varepsilon>0$), then u is the function identically zero in B_N.

Proof. (*i*) Since $G_\alpha(u)\le C(V_N R^N)^{-\frac{\mu}{N}}(1-R)^\varepsilon$ $\forall R\in]0,1[$, the result follows as $R\to 1^-$

(*ii*) Since $G_\alpha(u)\le C\frac{(V_N)^{-\frac{\mu}{N}}}{(1-R)^N}R^\varepsilon$ $\forall R\in]0,1[$, the result follows by letting $R\to 0^+$. □

The converse of Theorem 2 does not hold for all $u\in\mathcal{B}_\alpha$. The function u of Proposition 1 produces a counterexample.

Proposition 1. *Given $\alpha>0$ and $R\in]0,1[$, the function u defined by $u(x) = \frac{1}{(1-|x|^2)^\alpha}$ ($\forall x\in B_N$) belongs to $\mathcal{B}_\alpha$ but*

$$\sup_{a\in B_N}\frac{1}{[\operatorname{Vol} E(a,R)]^{2\frac{N-\alpha}{N+1}}}\int_{E(a,R)} u(x)\,dx = +\infty.$$

Proof. The subharmonicity of u follows from $\Delta u(x) = g''(r) + \frac{N-1}{r} g'(r) \geq 0$ where $r = |x|$ (see [2, p.26]) and $g(r) = \frac{1}{(1-r^2)^\alpha}$ ($r \in [0,1[$).

Let $a \in B_N$. Since φ_a is a $\mathcal{C}^1$–diffeomorphism of B_N onto itself (Lemma 2), the change of variable $x = \varphi_a(y)$ leads to

$$\begin{aligned}\int_{E(a,R)} u(x)\,dx &= \int_{B(0,R)} \frac{1}{(1-|\varphi_a(y)|^2)^\alpha} \left(\frac{\sqrt{1-|a|^2}}{1-\langle y,a\rangle}\right)^{N+1} dy \\ &= \int_{|y|<R} \frac{(1-\langle y,a\rangle)^{2\alpha-(N+1)}}{(1-|a|^2)^{\alpha-\frac{N+1}{2}}(1-|y|^2)^\alpha}\,dy.\end{aligned}$$

From the Cauchy–Schwarz inequality $1-R \leq 1-R|a| \leq 1-\langle y,a\rangle \leq 1+R|a| \leq 1+R \leq \frac{1}{1-R}$. Thus $(1-\langle y,a\rangle)^{2\alpha-N-1} \geq (1-R)^{|2\alpha-N-1|}$. Let $d\sigma$ denote the area element on the unit sphere S_N of $\mathbb{R}^N$. With $y = r\eta$, where $r = |y|$ and $\eta \in S_N$, we have $\int_{|y|<R} \frac{dy}{(1-|y|^2)^\alpha} = \int_0^R \int_{S_N} \frac{d\sigma(\eta) r^{N-1}\,dr}{(1-r^2)^\alpha}$, so that

$$\int_{E(a,R)} u(x)\,dx \geq (1-|a|^2)^{\frac{N+1}{2}-\alpha}(1-R)^{|2\alpha-N-1|}\sigma_N \int_0^R \frac{r^{N-1}\,dr}{(1-r^2)^\alpha} \tag{5}$$

with $\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ the area of S_N ([2, p.29]). Now, Lemma 4 (Section 6) provides

$$\begin{aligned}[\mathrm{Vol}\,E(a,R)]^{2\frac{N-\alpha}{N+1}} &= (V_N R^N)^{2\frac{N-\alpha}{N+1}} \left(\frac{1-|a|^2}{1-R^2|a|^2}\right)^{N-\alpha} \\ &\leq (1-|a|^2)^{N-\alpha} \frac{(V_N R^N)^{2\frac{N-\alpha}{N+1}}}{(1-R^2)^{|N-\alpha|}}\end{aligned}$$

since $1-R^2 \leq 1-R^2|a|^2 \leq 1 \leq \frac{1}{1-R^2}$ implies $(1-R^2|a|^2)^{N-\alpha} \geq (1-R^2)^{|N-\alpha|}$. Finally

$$\frac{1}{[\mathrm{Vol}\,E(a,R)]^{2\frac{N-\alpha}{N+1}}} \int_{E(a,R)} u(x)\,dx \geq C(N,\alpha,R) \frac{1}{(1-|a|^2)^{\frac{N-1}{2}}}$$

for some constant $C(N,\alpha,R)$ independant of $a \in B_N$. □

When $\mathrm{Vol}\,E(a,R)$ is considered with the same exponent $\frac{N-\alpha}{N}$ as $\mathrm{Vol}\,B(a,R_a)$ in Theorem 1, instead of the exponent $2\frac{N-\alpha}{N+1}$, we also obtain the next assertion.

Proposition 2. *Let $\alpha \geq N$ and $R \in]0,1[$. If a subharmonic function $u \geq 0$ in B_N satisfies*

$$P_{\alpha,R}(u) = \sup_{a \in B_N} \frac{1}{[\mathrm{Vol}\,E(a,R)]^{\frac{N-\alpha}{N}}} \int_{E(a,R)} u(x)\,dx < +\infty, \tag{6}$$

then $u \in \mathcal{B}_\alpha$. But the converse is not valid, the same function u as in Proposition 1 also serves as a counterexample here.

Proof. It is enough to show that

$$\frac{1}{[\operatorname{Vol} E(a,R)]^{2\frac{N-\alpha}{N+1}}} \leq (V_N)^{\frac{(\alpha-N)(N-1)}{N(N+1)}} \frac{1}{[\operatorname{Vol} E(a,R)]^{\frac{N-\alpha}{N}}}$$

This is a consequence of Lemma 4

$$[\operatorname{Vol} E(a,R)]^{(N-\alpha)\left(\frac{1}{N}-\frac{2}{N+1}\right)} = [\operatorname{Vol} E(a,R)]^{(\alpha-N)\frac{N-1}{N(N+1)}} =$$

$$= \left[V_N R^N \left(\frac{1-|a|^2}{1-R^2|a|^2}\right)^{\frac{N+1}{2}}\right]^{\frac{(\alpha-N)(N-1)}{N(N+1)}}.$$

Now, $R < 1$, $\frac{1-|a|^2}{1-R^2|a|^2} \leq 1$ and $(\alpha - N)\frac{N-1}{N(N+1)} \geq 0$, hence the majorization above.

On one hand, if (6) holds, then Theorem 2 applies, thus $u \in \mathcal{B}_\alpha$. On the other hand, for the function u from Proposition 1, (6) does not hold: the "sup" in (6) is infinite.

Proposition 3. *Let $0 < \alpha < N$ and $R \in]0,1[$. If a subharmonic function $u \geq 0$ in B_N satisfies (6), then $u \in \mathcal{B}_\nu$ with $\nu = N + \frac{(\alpha-N)(N+1)}{2N}$. But the converse is not valid.*

Proof. First suppose that u satisfies (6). Let $a \in B_N$. According to (4) and Lemma 4

$$\begin{aligned}
&\frac{1}{[\operatorname{Vol} E(a,R)]^{\frac{N-\alpha}{N}}} \int_{E(a,R)} u(x)\,dx \\
&\geq u(a) V_N R^N \frac{(1-|a|^2)^N}{(1+R|a|)^N} (V_N R^N)^{\frac{\alpha-N}{N}} \left(\frac{1-|a|^2}{1-R^2|a|^2}\right)^{\frac{(N+1)(\alpha-N)}{2N}} \qquad (7)\\
&\geq u(a) (V_N R^N)^{\frac{\alpha}{N}} \frac{(1-|a|^2)^N}{(1+R)^N} \left(\frac{1-|a|^2}{1-R^2}\right)^{\frac{(N+1)(\alpha-N)}{2N}}
\end{aligned}$$

since $1 + R|a| \leq 1 + R$, $1 - R^2|a|^2 \geq 1 - R^2$ and $\frac{(N+1)(\alpha-N)}{2N} < 0$. Note that $\nu = \frac{N-1}{2} + \alpha\frac{N+1}{2N} > \alpha$ because $\nu - \alpha = \frac{N-1}{2} + \alpha\frac{1-N}{2N} = \frac{N-1}{2}(1 - \frac{\alpha}{N}) > 0$.

Next consider the function u from Proposition 1. Then $u \in \mathcal{B}_\alpha$. Hence $u \in \mathcal{B}_\nu$ ($\mathcal{B}_\alpha \subset \mathcal{B}_\nu$ since $\alpha \leq \nu$). Let $a \in B_N$. From (5) together with

$$[\mathrm{Vol}\, E(a,R)]^{\frac{N-\alpha}{N}} = (V_N R^N)^{\frac{N-\alpha}{N}} \left(\frac{1-|a|^2}{1-R^2|a|^2}\right)^{\frac{(N+1)(N-\alpha)}{2N}} \leq (V_N)^{\frac{N-\alpha}{N}} \left(\frac{1-|a|^2}{1-R^2}\right)^{\frac{(N+1)(N-\alpha)}{2N}}$$

(since $R < 1$ and $N - \alpha \geq 0$), it follows that

$$\frac{1}{[\mathrm{Vol}\, E(a,R)]^{\frac{N-\alpha}{N}}} \int_{E(a,R)} u(x)\, dx \geq K \frac{(1-|a|^2)^{\frac{N+1}{2}-\alpha}}{(1-|a|^2)^{\frac{(N+1)(N-\alpha)}{2N}}} = K(1-|a|^2)^{\varepsilon}$$

with $\varepsilon = \frac{N+1}{2} - \alpha - \frac{(N+1)(N-\alpha)}{2N} = -\alpha + \frac{(N+1)\alpha}{2N} = \alpha \frac{1-N}{2N} < 0$ and $K = K(N,\alpha,R)$ a constant independant of $a \in B_N$. Finally

$$\sup_{a \in B_N} \frac{1}{[\mathrm{Vol}\, E(a,R)]^{\frac{N-\alpha}{N}}} \int_{E(a,R)} u(x)\, dx = +\infty. \qquad \square$$

Corollary 3. *Given $\alpha > 0$, let ν be defined as in Proposition 3 and $u \geq 0$ be a subharmonic function in B_N, such that $P_{\alpha,R}(u) < +\infty$ $\forall R \in]0,1[$.*

(i) If there exist constants $C > 0$ and $\varepsilon > 0$ such that $P_{\alpha,R}(u) \leq CR^{\alpha+\varepsilon}$ $\forall R \in]0,1[$, then $u \equiv 0$ in B_N.

(ii) If $P_{\alpha,R}(u) \leq C(1-R)^{|N-\nu|+\varepsilon}$ $\forall R \in]0,1[$ (for some constants $C > 0$ and $\varepsilon > 0$), then $u \equiv 0$ in B_N.

Proof. Given $a \in B_N$, the first inequality of (7) is valid for all $\alpha > 0$. Since $\frac{1}{1-R^2} \geq 1 - R^2|a|^2 \geq 1 - R^2$, it follows that $(1-R^2|a|^2)^{N-\nu} \geq (1-R^2)^{|N-\nu|}$. Hence

$$P_{\alpha,R}(u) \geq u(a)(1-|a|^2)^{\nu}(V_N)^{\frac{\alpha}{N}} \frac{R^\alpha}{(1+R)^N}(1-R^2)^{|N-\nu|} \quad \forall R \in]0,1[.$$

Proof of (i). Since $u(a)(1-|a|^2)^{\nu}(V_N)^{\frac{\alpha}{N}} \frac{(1-R^2)^{|N-\nu|}}{(1+R)^N} \leq CR^{\varepsilon}$ $\forall R \in]0,1[$, the result $u(a) = 0$ follows when $R \to 0^+$.

Proof of (ii). Now $u(a)(1-|a|^2)^{\nu}(V_N)^{\frac{\alpha}{N}} R^\alpha (1+R)^{|N-\nu|-N} \leq C(1-R)^{\varepsilon}$ $\forall R \in]0,1[$. Letting $R \to 1^-$, we obtain *(ii)*. $\square$

4 Another Characterization of $\mathcal{B}_\alpha$.

Theorem 3. *Given $\alpha > 0$ and $R \in]0,1[$, a non–negative subharmonic function u in B_N belongs to $\mathcal{B}_\alpha$ if and only if $\sup\limits_{a \in B_N} \int_{B(a,R_a)} u(x)(1-|x|^2)^{\alpha-N}\, dx < +\infty$.*

PROOF. Since $[\mathrm{Vol}\, B(a,R_a)]^{\frac{\alpha}{N}-1} = (V_N)^{\frac{\alpha-N}{N}} \left[\frac{R(1-|a|^2)}{1+R|a|}\right]^{\alpha-N}$ and $\frac{1-|x|^2}{2} \leq 1-|a|^2 \leq \frac{1+R}{1-R}(1-|x|^2)$ $\forall x \in B(a,R_a)$ (Lemmas 1 and 5, Section 6), it follows that

$$\left(\frac{R(1-|x|^2)}{2(1+R)}\right)^{\alpha-N} \leq \left[\frac{\mathrm{Vol}\, B(a,R_a)}{V_N}\right]^{\frac{\alpha-N}{N}} \leq \left(\frac{R(1+R)(1-|x|^2)}{1-R}\right)^{\alpha-N} \quad \text{when } \alpha \geq N$$

and

$$\left(\frac{R(1+R)(1-|x|^2)}{1-R}\right)^{\alpha-N} \leq \left[\frac{\mathrm{Vol}\, B(a,R_a)}{V_N}\right]^{\frac{\alpha-N}{N}} \leq \left(\frac{R(1-|x|^2)}{2(1+R)}\right)^{\alpha-N} \quad \text{when } \alpha < N.$$

Now $u(x) \geq 0$, so that for all $x \in B(a,R_a)$,

$$D \cdot u(x)(1-|x|^2)^{\alpha-N} \leq [\mathrm{Vol}\, B(a,R_a)]^{\frac{\alpha}{N}-1} u(x) \leq D'.u(x)(1-|x|^2)^{\alpha-N}$$

where constants $D = D(N,\alpha,R)$ and $D' = D'(N,\alpha,R)$ are independant of x and a. Hence Theorem 3 follows from our characterization (2). □

Theorem 4. *Let $\omega : [0,1[\to [0,+\infty[$ be a decreasing function. Given $\alpha > 0$ and $p \leq \alpha - N$, if a non–negative subharmonic function u in B_N satisfies $S_{p,\omega}(u) < +\infty$, then $u \in \mathcal{B}_\alpha$.*

PROOF. Given $a \in B_N$, the following holds for all $R \in]0,1[$ since $u(x) \geq 0$ $\forall x \in B_N$.

$$\begin{aligned}\int_{B_N} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\, dx &\geq \int_{B(a,R_a)} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\, dx \\ &\geq \int_{B(a,R_a)} u(x)(1-|x|^2)^{\alpha-N} \omega(|\varphi_a(x)|)\, dx \\ &\quad (\text{since } (1-|x|^2)^p \geq (1-|x|^2)^{\alpha-N}) \\ &\geq \omega(R) \int_{B(a,R_a)} u(x)(1-|x|^2)^{\alpha-N}\, dx\end{aligned}$$

since ω decreases and $B(a,R_a) \subset E(a,R)$ from Lemma 3; hence $|\varphi_a(x)| < R$ $\forall x \in B(a,R_a)$. With R fixed, the result "$u \in \mathcal{B}_\alpha$" follows from Theorem 3.

The converse of Theorem 4 is not necessarily valid.

Proposition 4. *With ω as in Definition 3, $\alpha > 0$ and $p < \alpha - \frac{N+1}{2}$, the function u from Proposition 1 belongs to $\mathcal{B}_\alpha$ but $S_{p,\omega}(u) = +\infty$.*

PROOF. Given $a \in B_N$, the change of variable $y = \varphi_a(x)$ (see Lemma 2, Section 6) leads to

$$\begin{aligned}\int_{B_N} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\,dx &= \int_{B_N} (1-|x|^2)^{p-\alpha}\omega(|\varphi_a(x)|)\,dx \\ &= \int_{B_N} (1-|\varphi_a(y)|^2)^{p-\alpha}\omega(|y|)\left(\frac{1-|\varphi_a(y)|^2}{1-|y|^2}\right)^{\frac{N+1}{2}} dy \\ &= \int_{B_N} \left[\frac{1-|a|^2}{(1-\langle y,a\rangle)^2}\right]^{p-\alpha+\frac{N+1}{2}} (1-|y|^2)^{p-\alpha}\omega(|y|)\,dy.\end{aligned}$$

Now $|\langle y,a\rangle| \le \frac{|a|}{2} < \frac{1}{2}$ if $y \in B_N$ satisfies $|y| \le \frac{1}{2}$. Hence $1 - \langle y,a\rangle \ge \frac{1}{2}$ for such y. Since $p - \alpha + \frac{N+1}{2} < 0$, we obtain

$$\int_{B_N} u(x)(1-|x|^2)^p\omega(|\varphi_a(x)|)\,dx \ge [4(1-|a|^2)]^{p-\alpha+\frac{N+1}{2}} \int_{|y|\le\frac{1}{2}} (1-|y|^2)^{p-\alpha}\omega(|y|)\,dy.$$

The result "$S_{p,\omega}(u) = +\infty$" follows from $\sup_{a\in B_N} (1-|a|^2)^{p-\alpha+\frac{N+1}{2}} = +\infty$ (the exponent being strictly negative).

Theorem 5. *Let function $\omega : [0,1[\to [0,+\infty[$ satisfy (3). Given $\alpha > 0$ and $p \ge \alpha - \frac{N+1}{2}$, the inclusion $\mathcal{B}_\alpha \subset \mathcal{SH}(p,\omega)$ holds.*

PROOF. Let $u \in \mathcal{B}_\alpha$. Thus $u(x) \le \frac{G_\alpha(u)}{(1-|x|^2)^\alpha}$ $\forall x \in B_N$. Hence

$$\begin{aligned}\int_{B_N} u(x)(1-|x|^2)^p\omega(|\varphi_a(x)|)\,dx &\le G_\alpha(u)\int_{B_N}(1-|x|^2)^{p-\alpha}\omega(|\varphi_a(x)|)\,dx \\ &= G_\alpha(u)\int_{B_N}(1-|\varphi_a(y)|^2)^{p-\alpha+\frac{N+1}{2}}\omega(|y|)\frac{dy}{(1-|y|^2)^{\frac{N+1}{2}}} \\ &\le G_\alpha(u)\int_{B_N}\frac{\omega(|y|)}{(1-|y|^2)^{\frac{N+1}{2}}}\,dy = G_\alpha(u)\sigma_N\int_0^1 \frac{\omega(r)r^{N-1}}{(1-r^2)^{\frac{N+1}{2}}}\,dr \;\forall a \in B_N\end{aligned}$$

with the same notations and changes of variables as in the proof of Proposition 1. We have majorized $(1-|\varphi_a(y)|^2)^{p-\alpha+\frac{N+1}{2}}$ by 1 since $p - \alpha + \frac{N+1}{2} \ge 0$. Finally $S_{p,\omega}(u) \le G_\alpha(u)\sigma_N\Omega$. □

Proposition 5. *With ω and $\alpha > 0$ as in Theorem 5, let $p > \alpha - \frac{N+1}{2}$ and $\alpha < \beta \le p + \frac{N+1}{2}$. Then the function u defined by $u(x) = \frac{1}{(1-|x|^2)^\beta}$ $\forall x \in B_N$ belongs to $\mathcal{SH}(p,\omega)$ but not to $\mathcal{B}_\alpha$.*

Proof. Since $\Delta u \geq 0$ can be verified as in the proof of Proposition 1, $u \notin \mathcal{B}_\alpha$ is a consequence of $\sup_{x \in B_N} (1-|x|^2)^{\alpha-\beta} = +\infty$. Given $a \in B_N$, we obtain

$$\begin{aligned}\int_{B_N} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\, dx &= \int_{B_N} (1-|\varphi_a(y)|^2)^{p-\beta+\frac{N+1}{2}} \frac{\omega(|y|)}{(1-|y|^2)^{\frac{N+1}{2}}}\, dy \\ &\leq \sigma_N \Omega\end{aligned}$$

in the same way as in the previous proof. Hence $S_{p,\omega}(u) < +\infty$. □

Proposition 6. *If* $p > -\frac{N+1}{2}$ *and the function* $\omega : [0,1[\to [0,+\infty[$ *satisfies* (3)*, then*

$$\max\{\alpha > 0 : \mathcal{B}_\alpha \subset \mathcal{SH}(p,\omega)\} = p + \frac{N+1}{2}.$$

Proof. Theorem 5 already asserts $\mathcal{B}_\alpha \subset \mathcal{SH}(p,\omega)$ $\forall \alpha \in]0, p+\frac{N+1}{2}]$. For $\alpha > p + \frac{N+1}{2}$, $\mathcal{B}_\alpha \not\subset \mathcal{SH}(p,\omega)$ follows from Proposition 4.

5 Gap Subharmonic Functions.

Definition 4. Let $\mathcal{G}$ be the set of all functions u defined on B_N by $u(x) = f(|x|)$ $\forall x \in B_N$, where $f(r)$ is the sum of some power series with coefficients $c_k \geq 0$ ($k \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$) of the kind

$$f(r) = \sum_{k \in \mathbb{N}^*} c_k r^{2^k} \tag{8}$$

which converges for all $r \in [0,1[$.

Remark 4. Such functions u are non–negative and subharmonic in B_N since $\Delta u(x) = f''(r) + \frac{N-1}{r} f'(r)$ (with $r = |x|$, see [2, p.26]) and $f'(r) \geq 0$, $f''(r) \geq 0$ $\forall r \in [0,1[$.

Theorem 6. *Given* $p > -\frac{N+3}{4}$ *and* $\omega : [0,1[\to [0,+\infty[$ *a measurable function such that*

$$\Omega' := \int_0^1 \frac{[\omega(r)]^2 r^{N-1}}{(1-r^2)^{\frac{N+1}{2}}}\, dr < +\infty, \tag{9}$$

let $u \in \mathcal{G}$ *with gap development* (8)*. If* $\sum_{k \in \mathbb{N}} c_{k+1}^2 2^{-2k(p+\frac{N+3}{4})} < +\infty$*, then* $u \in \mathcal{SH}(p,\omega)$*.*

Example. The function ω defined by $\omega(r) = \left(\log \frac{1}{r}\right)^s$ with $s > \frac{N-1}{4}$ fulfills condition (9).

Proof. Given $a \in B_N$, Cauchy–Schwarz' inequality leads to

$$\int_{B_N} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\,dx = \int_{B_N} u(x)(1-|x|^2)^{p+\frac{N+1}{4}} \frac{\omega(|\varphi_a(x)|)}{(1-|x|^2)^{\frac{N+1}{4}}}\,dx$$
$$\le \left(\int_{B_N} [u(x)]^2(1-|x|^2)^{2p+\frac{N+1}{2}}\,dx\right)^{\frac{1}{2}} \left(\int_{B_N} \frac{[\omega(|\varphi_a(x)|)]^2}{(1-|x|^2)^{\frac{N+1}{2}}}\,dx\right)^{\frac{1}{2}}.$$

Now, the change of variable $y=\varphi_a(x)$ turns the second integral into

$$\int_{B_N} \frac{[\omega(|\varphi_a(x)|)]^2}{(1-|x|^2)^{\frac{N+1}{2}}}\,dx = \int_{B_N} \frac{[\omega(|y|)]^2}{(1-|\varphi_a(y)|^2)^{\frac{N+1}{2}}} \left(\frac{1-|\varphi_a(y)|^2}{1-|y|^2}\right)^{\frac{N+1}{2}} dy$$
$$= \sigma_N \int_0^1 \frac{[\omega(r)]^2}{(1-r^2)^{\frac{N+1}{2}}} r^{N-1}\,dr = \sigma_N \Omega' \quad \forall a \in B_N.$$

Besides that

$$\int_{B_N} [u(x)]^2(1-|x|^2)^{2p+\frac{N+1}{2}}\,dx = \sigma_N \int_0^1 [f(r)]^2(1-r^2)^{2p+\frac{N+1}{2}} r^{N-2} r\,dr$$
$$\le \frac{\sigma_N}{2} \int_0^1 [g(t)]^2(1-t)^{2p+\frac{N+1}{2}}\,dt \text{ since } r^{N-2} \le 1$$

with $g(t) = f(\sqrt{t}) = \sum_{k\in\mathbb{N}^*} c_k t^{2^{k-1}} = \sum_{k\in\mathbb{N}} c_{k+1} t^{2^k}$. From Lemma 6 (Section 6), with $\alpha = 2p+\frac{N+1}{2}+1 = 2p+\frac{N+3}{2} > 0$, $\beta = 2$, $s_k = c_{k+1}$, the above integral is majorized by $K \sum_{k\in\mathbb{N}} c_{k+1}^2 2^{-k(2p+\frac{N+3}{2})}$. Finally

$$\int_{B_N} u(x)(1-|x|^2)^p \omega(|\varphi_a(x)|)\,dx \le \sqrt{\sigma_N \Omega'} \sqrt{\frac{\sigma_N}{2} K} \sqrt{\sum_{k\in\mathbb{N}} c_{k+1}^2 2^{-k(2p+\frac{N+3}{2})}}.$$

□

Theorem 7. *Given $p \in \mathbb{R}$, $s \in \mathbb{R}$ satisfying $p+s+1>0$ and $\omega : [0,1[\to [0,+\infty[$ a measurable function for which there exists a constant $C>0$ such that $\omega(r) \ge C(1-r^2)^s$ $\forall r \in [0,1[$, let $u \in \mathcal{G}$ with gap development* (8). *If $u \in \mathcal{SH}(p,\omega)$, then $\sum_{k\in\mathbb{N}} c_{k+1} 2^{-k(p+s+1)} < +\infty$.*

Example. The function ω defined by $\omega(r) = \left(\log \frac{1}{r}\right)^s$ with $s \ge 0$ satisfies $\omega(r) \ge (1-r)^s \ge \frac{1}{2^s}(1-r^2)^s$.

Proof. For $a = 0$, we have $|\varphi_a(x)| = |x|$. Hence

$$S_{p,\omega}(u) \geq \int_{B_N} u(x)(1-|x|^2)^p \omega(|x|)\, dx \geq C \int_{B_N} u(x)(1-|x|^2)^{p+s}\, dx$$
$$= C\sigma_N \int_0^1 f(r)(1-r^2)^{p+s} r^{N-1}\, dr = \frac{C\sigma_N}{2} \int_0^1 f(\sqrt{t})t^{\frac{N}{2}-1}(1-t)^{p+s}\, dt.$$

Let $k_0 \in \mathbb{N}$ such that $\frac{N}{2} \leq 2^{k_0}$. Hence $1 + \dfrac{\frac{N}{2}-1}{2^k} \leq 2^{k_0}$ $\forall k \in \mathbb{N}$, in other words $2^k + \frac{N}{2} - 1 \leq 2^{k+k_0}$. Thus $t^{2^k + \frac{N}{2}-1} \geq t^{2^{k+k_0}}$ $\forall t \in [0,1[$ and

$$f(\sqrt{t})t^{\frac{N}{2}-1} \geq h(t) := \sum_{k\in\mathbb{N}} c_{k+1} t^{2^{k+k_0}} = \sum_{k\geq k_0} c_{k+1-k_0} t^{2^k}.$$

Finally

$$S_{p,\omega}(u) \geq \frac{C\sigma_N}{2} \int_0^1 h(t)(1-t)^{p+s}\, dt \geq \frac{C\sigma_N}{2K} \sum_{k\geq k_0} c_{k+1-k_0} 2^{-k(p+s+1)}$$
$$= 2^{-k_0(p+s+1)} \frac{C\sigma_N}{2K} \sum_{k\in\mathbb{N}} c_{k+1} 2^{-k(p+s+1)}$$

from Lemma 6 applied with $\alpha = p+s+1$, $\beta = 1$, $s_k = c_{k+1-k_0}$ $\forall k \geq k_0$ and $s_k = 0$ $\forall k \in \{0,1,2,\ldots,k_0-1\}$. (Here, K does not have the same value as in the previous proof.) $\square$

Proposition 7. *Let p, s and ω be defined as in Theorem 7. Then $\mathcal{G} \cap \mathcal{SH}(p,\omega) \subset \mathcal{B}_\alpha$ for any $\alpha \geq p+s+1$.*

Example. When ω is decreasing, this inclusion in $\mathcal{B}_\alpha$ follows from Theorem 4 for $\alpha \geq p+N$, thus Proposition 7 brings some new information in the case $0 \leq s < N-1$.

Proof. Let $u \in \mathcal{G} \cap \mathcal{SH}(p,\omega)$, with gap development (8). According to Theorem 7, the series $\sum_{k\in\mathbb{N}} c_{k+1} 2^{-k(p+s+1)}$ converges. Thus $\lim_{k\to+\infty} c_{k+1} 2^{-k(p+s+1)} = 0$. For k sufficiently large, $c_{k+1} 2^{-k(p+s+1)} \leq 1$. Now

$$c_{k+1} 2^{-(k+1)\alpha} = 2^{-\alpha} c_{k+1} 2^{-k\alpha} \leq 2^{-\alpha} c_{k+1} 2^{-k(p+s+1)} \quad \forall k \in \mathbb{N}.$$

Hence $\sup_{k\geq 1} c_k 2^{-k\alpha} < \infty$ and Lemma 7 (Section 6) implies $u \in \mathcal{B}_\alpha$. (It could even be verified that $\lim_{k\to+\infty} c_k 2^{-k\alpha} = 0$.) $\square$

Remark 5. Under the conditions of Theorem 7, the inclusion $\mathcal{G} \cap \mathcal{B}_\alpha \subset \mathcal{SH}(p,\omega)$ does not hold for $\alpha \geq p+s+1$. For instance, the function $u \in \mathcal{G}$, with development (8) defined by $c_k = 2^{k\alpha}$ $\forall k \in \mathbb{N}^*$, belongs to $\mathcal{B}_\alpha$ but not to $\mathcal{SH}(p,\omega)$, since $\sup\limits_{k\geq 1} c_k 2^{-k\alpha} < +\infty$ and

$$\sum_{k\in\mathbb{N}} c_{k+1} 2^{-k(p+s+1)} = 2^\alpha \sum_{k\in\mathbb{N}} 2^{k(\alpha-p-s-1)} = +\infty.$$

Proposition 8. *Let p and ω be defined as in Theorem 6. Then $\mathcal{G} \cap \mathcal{B}_\alpha \subset \mathcal{SH}(p,\omega)$ for any $\alpha < p + \frac{N+3}{4}$.*

Example. When $\omega(r) = \left(\log \frac{1}{r}\right)^s$ with $\frac{N-1}{4} < s \leq \frac{N-1}{2}$, Theorem 5 cannot be used because (3) does not hold, but Proposition 8 can be applied.

Proof. Let $u \in \mathcal{G} \cap \mathcal{B}_\alpha$, with gap development (8). Since $c_{k+1}2^{-(k+1)\alpha} = 2^{-\alpha} c_{k+1} 2^{-k\alpha}$ $\forall k \in \mathbb{N}$, Lemma 7 (Section 6) leads to $\sup\limits_{k\geq 1} c_{k+1}2^{-k\alpha} < +\infty$. The radius of convergence of the power series $\sum\limits_{k\in\mathbb{N}} c_{k+1}^2 z^{2k}$ $(z \in \mathbb{C})$ thus is $\geq 2^{-\alpha}$. Otherwise, the sequence $\left(c_{k+1}^2 2^{-2k\alpha}\right)_{k\in\mathbb{N}}$ would be unbounded according to Abel's Lemma. Now $2^{-\alpha} > 2^{-(p+\frac{N+3}{4})}$. Hence $\sum\limits_{k\in\mathbb{N}} c_{k+1}^2 2^{-2k(p+\frac{N+3}{4})}$ converges and $u \in \mathcal{SH}(p,\omega)$ from Theorem 6. □

Remark 6. Under the conditions of Theorem 6, the inclusion $\mathcal{G} \cap \mathcal{SH}(p,\omega) \subset \mathcal{B}_\alpha$ does not hold for $\alpha < p + \frac{N+3}{4}$. For instance, the function $u \in \mathcal{G}$ with development (8) defined by $c_k = k2^{k\alpha}$ $\forall k \in \mathbb{N}^*$, belongs to $\mathcal{SH}(p,\omega)$ but not to $\mathcal{B}_\alpha$, since $\sup\limits_{k\geq 1} c_k 2^{-k\alpha} = +\infty$ and

$$\sum_{k\in\mathbb{N}} c_{k+1}^2 2^{-2k(p+\frac{N+3}{4})} = 2^{2\alpha} \sum_{k\in\mathbb{N}} (k+1)^2 2^{-2k(p+\frac{N+3}{4}-\alpha)} < +\infty.$$

6 Appendix: Some Technical Results

Lemma 1. *Given $a \in B_N$ and $R \in [0,1[$, we have $1-|x|^2 \geq \dfrac{1-R}{1+R}(1-|a|^2)$ for any $x \in B(a, R_a)$.*

Proof. We have $|x| \leq |a| + R_a = \dfrac{|a| + R|a|^2 + R - R|a|^2}{1+R|a|} = \dfrac{|a|+R}{1+R|a|} < 1$, since $|a| + R - 1 - R|a| = (1-|a|)(R-1) < 0$. Hence

$$1-|x|^2 \geq 1 - \left(\frac{|a|+R}{1+R|a|}\right)^2 = \frac{1+2R|a|+R^2|a|^2 - (|a|^2+R^2+2R|a|)}{(1+R|a|)^2} =$$

$$= \frac{(1-|a|^2)(1-R^2)}{(1+R|a|)^2} \geq \frac{(1-|a|^2)(1-R^2)}{(1+R)^2}.$$

□

Lemma 2. *Given $a \in B_N$, the function $\varphi_a : B_N \to B_N$ is an involutive bijection and*

$$1 - |\varphi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{(1-\langle x, a\rangle)^2} \quad \forall x \in B_N.$$

Let $J_a(x)$ stand for the determinant of matrix $\left(\frac{\partial \varphi_{a,i}}{\partial x_j}(x)\right)_{1\leq i,j\leq N}$ where $\varphi_{a,1}$, $\varphi_{a,2}$, ..., $\varphi_{a,N}$ are the N components of map φ_a. Then

$$J_a(x) = (-1)^N \left(\frac{\sqrt{1-|a|^2}}{1-\langle x, a\rangle}\right)^{N+1} = (-1)^N \left(\frac{1-|\varphi_a(x)|^2}{1-|x|^2}\right)^{\frac{N+1}{2}}.$$

Proof. See [5, pp.25–26] and [1, p.115] for properties of map φ_a and [6] for the computation of $J_a(x)$.

Lemma 3. *For any $a \in B_N$ and any $R \in [0, 1[$, the ellipsoid $E(a, R)$ contains $B(a, R_a)$, with merely $E(0, R) = B(0, R)$ when $a = 0$.*

Proof. See [6]. □

Lemma 4. *For any $a \in B_N$ and any $R \in [0, 1[$, the volume of the ellipsoid $E(a, R)$ is*

$$\operatorname{Vol} E(a, R) = V_N R^N \left(\frac{1-|a|^2}{1-R^2|a|^2}\right)^{\frac{N+1}{2}}.$$

Proof. The same changes of variables as in the proof of Proposition 1 lead to

$$\begin{aligned}\operatorname{Vol} E(a, R) &= \int_{E(a,R)} dx = \int_{B(0,R)} \left(\frac{\sqrt{1-|a|^2}}{1-\langle y, a\rangle}\right)^{N+1} dy \\ &= (1-|a|^2)^{\frac{N+1}{2}} \int_0^R \int_{S_N} \frac{d\sigma(\eta) r^{N-1}\, dr}{(1-r\langle \eta, a\rangle)^{N+1}}.\end{aligned}$$

Without restriction, we may assume $a \neq 0$ and $a = |a|(1, 0, \ldots, 0)$. Polar coordinates in $\mathbb{R}^N$ provide $\eta_1 = \cos\theta_1$ and

$$d\sigma = (\sin\theta_1)^{N-2}(\sin\theta_2)^{N-3}\ldots(\sin\theta_{N-2})d\theta_1 d\theta_2 \ldots d\theta_{N-1}$$

with $\theta_1, \theta_2, \ldots, \theta_{N-2} \in]0,\pi[$ and $\theta_{N-1} \in]0, 2\pi[$ (see [11, p.15]).

It is clear for $N \geq 3$ that $(\sin\theta_2)^{N-3}(\sin\theta_3)^{N-4}\ldots(\sin\theta_{N-2})d\theta_2 d\theta_3\ldots d\theta_{N-1}$ is the area element on S_{N-1}. Since $\sigma_1 = 2$, we have for $N \geq 3$ and for $N = 2$

$$\begin{aligned}\operatorname{Vol} E(a,R) &= (1-|a|^2)^{\frac{N+1}{2}} \int_0^R \int_0^\pi \frac{\sigma_{N-1}(\sin\theta_1)^{N-2} d\theta_1}{(1-r|a|\cos\theta_1)^{N+1}} r^{N-1}\, dr \\ &= (1-|a|^2)^{\frac{N+1}{2}} \sigma_{N-1} \int\int_H \frac{t^{N-2}}{(1-|a|s)^{N+1}}\, ds\, dt\end{aligned}$$

where $s = r\cos\theta_1$, $t = r\sin\theta_1$ and $H = \{(s,t) \in \mathbb{R}^2 : t \geq 0, s^2+t^2 \leq R^2\}$ is a half–disk.

Since $N+1 \notin -\mathbb{N}$, using [10, p. 53] yields

$$\frac{t^{N-2}}{(1-|a|s)^{N+1}} = \sum_{n\geq 0} \frac{\Gamma(n+N+1)}{n!\Gamma(N+1)} |a|^n s^n t^{N-2}.$$

This series converges normally on H, since $|a| < 1$. Hence $\int\int_H \frac{t^{N-2}}{(1-|a|s)^{N+1}}\, ds\, dt$ $= \sum_{n\geq 0} \frac{\Gamma(n+N+1)}{n!\Gamma(N+1)} |a|^n J_n$ with $J_n = \int\int_H s^n t^{N-2}\, ds\, dt$. When n is odd, $J_n = 0$. For even n ($n = 2k$) $J_n = \frac{R^{2k+N}}{N-1} \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{N+1}{2})}{\Gamma(k+\frac{N}{2}+1)}$ using Euler's identity for the Beta function (see [4, pp. 67–68]). Whence

$$\iint_H \frac{t^{N-2}\, ds\, dt}{(1-|a|s)^{N+1}} = \frac{R^N}{N-1} \frac{\Gamma(\frac{N+1}{2})}{\Gamma(N+1)} \sum_{k\geq 0} \frac{\Gamma(2k+N+1)}{\Gamma(k+\frac{N}{2}+1)} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(2k+1)} (R^2|a|^2)^k$$

$$= \frac{R^N}{N-1} \sqrt{\pi} \frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2}+1)} \sum_{k\geq 0} \frac{\Gamma(k+\frac{N+1}{2})}{k!\Gamma(\frac{N+1}{2})} (R^2|a|^2)^k = R^N \frac{V_N}{\sigma_{N-1}} \left(\frac{1}{1-R^2|a|^2}\right)^{\frac{N+1}{2}}$$

by the duplication formula $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$ for the Gamma function ([4, p. 45]), applied successively with $z = k + \frac{N}{2} + \frac{1}{2}$, $z = k + \frac{1}{2}$ and $z = \frac{N+1}{2}$. □

Lemma 5. *For all $a \in B_N$ and $R \in [0,1[$, we have $1-|x|^2 \leq 2(1-|a|^2)$ $\forall x \in B(a, R_a)$.*

PROOF. If $|a| \leq \frac{1}{\sqrt{2}}$, then $1-2|a|^2 \geq 0$. Hence $1-|x|^2 \leq 1 \leq 1+(1-2|a|^2) = 2(1-|a|^2)$ $\forall x \in B_N$. If $|a| > \frac{1}{\sqrt{2}}$, then $R_a \leq |a|$ $\forall R \in [0,1[$ since

$$|a| - R_a = \frac{|a|(1+R|a|) - R(1-|a|^2)}{1+R|a|} = \frac{|a| + (2|a|^2-1)R}{1+R|a|} \geq 0 \ \ \forall R \in [0,1[.$$

Thus $|x| \geq |a| - R_a \geq 0$ for any $x \in B(a, R_a)$. Hence

$$\begin{aligned}
1 - |x|^2 \leq 1 - (|a| - R_a)^2 &= 1 - \left[|a| - \frac{R(1-|a|^2)}{1+R|a|} \right]^2 \\
&= 1 - \left[|a|^2 - \frac{2|a|R}{1+R|a|}(1-|a|^2) + \frac{R^2}{(1+R|a|)^2}(1-|a|^2)^2 \right] \\
&= (1-|a|^2)\left[1 + \frac{2|a|R}{1+R|a|} - \frac{R^2(1-|a|^2)}{(1+R|a|)^2} \right] \\
&\leq (1-|a|^2)\left[1 + \frac{2|a|R}{1+R|a|} \right] \leq 2(1-|a|^2)
\end{aligned}$$

because $R|a| \leq 1$; thus $2R|a| \leq 1 + R|a|$. □

Lemma 6. *(see [3]). Given $\alpha > 0$, $\beta > 0$ and a power series $g(t) = \sum_{n \in \mathbb{N}^*} b_n t^n$ (convergent for $|t| < 1$) with non–negative coefficients b_n ($n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$), let $s_k = \sum_{n \in I_k} b_n$ where $I_k = \{n \in \mathbb{N}^* : 2^k \leq n < 2^{k+1}\}$ $\forall k \in \mathbb{N}$. There exists a constant K, depending only on $\alpha > 0$ and $\beta > 0$, such that*

$$\frac{1}{K} \sum_{k \in \mathbb{N}} 2^{-k\alpha} s_k^\beta \leq \int_0^1 (1-t)^{\alpha-1} [g(t)]^\beta dt \leq K \sum_{k \in \mathbb{N}} 2^{-k\alpha} s_k^\beta.$$

Lemma 7. *Given $\alpha > 0$ and a convergent power series of sum $f(r)$ and coefficients $c_k \geq 0$ as in* (8)*, we have*

$$\sup_{0 \leq r < 1} (1-r^2)^\alpha f(r) < +\infty \quad \Longleftrightarrow \quad \sup_{k \geq 1} c_k 2^{-k\alpha} < +\infty.$$

Proof. Since $(1-r)^\alpha \leq (1-r^2)^\alpha \leq 2^\alpha (1-r)^\alpha$ $\forall r \in [0, 1[$, we will prove as in [7]

$$G := \sup_{0 \leq r < 1} (1-r)^\alpha f(r) < +\infty \Longleftrightarrow \sup_{k \geq 1} c_k 2^{-k\alpha} < +\infty.$$

$\Longrightarrow$ Given $k \in \mathbb{N}^*$, Cauchy's formula in $\mathbb{C}$ yields $c_k = \dfrac{1}{2i\pi} \displaystyle\int_{|z|=r} \frac{f(z)}{z^{1+2^k}} dz$ whatever $r \in]0, 1[$, hence: $|c_k| \leq \dfrac{1}{r^{2^k}} \displaystyle\sup_{|z|=r} |f(z)|$. Here $|f(z)| \leq f(|z|)$ $\forall z \in \mathbb{C}$, $|z| < 1$, since f has non–negative Taylor coefficients at the origin. Thus $0 \leq c_k \leq \frac{1}{r^{2^k}} f(r) \leq \frac{G}{r^{2^k}(1-r)^\alpha}$ $\forall r \in]0, 1[$. The choice $r = 1 - \frac{1}{2^k}$ leads to $c_k \leq G 2^{k\alpha} \left(1 - \dfrac{1}{2^k}\right)^{-2^k}$. Since $\displaystyle\lim_{k \to +\infty} \left(1 - \frac{1}{2^k}\right)^{2^k} = 1/e$, the conclusion $\displaystyle\sup_{k \geq 1} c_k 2^{-k\alpha} < +\infty$ holds.

$\Longleftarrow$ There exists some constant $L \geq 0$ such that $c_k \leq L2^{k\alpha}$ $\forall k \in \mathbb{N}^*$. Hence $0 \leq f(r) \leq L \sum_{k\in\mathbb{N}^*} 2^{k\alpha} r^{2^k}$ $\forall r \in [0,1[$. Besides that

$$\frac{1}{(1-r)^\alpha} = \sum_{n\geq 0} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} r^n \quad \forall r \in [0,1[$$

since $\alpha \notin -\mathbb{N}$. Stirling's formula (see [4, p.59]) implies $\frac{\Gamma(n+\alpha)}{n!} \sim n^{\alpha-1}$ as $n \to +\infty$. There is thus some constant $M > 1$ (depending only on α) such that $n^{\alpha-1} \leq M\frac{\Gamma(n+\alpha)}{n!}$ $\forall n \in \mathbb{N}^*$. We will soon prove that

$$\sum_{k\in\mathbb{N}^*} 2^{k\alpha} r^{2^k} \leq 2^{\alpha+1} \sum_{n\geq 1} n^{\alpha-1} r^n \quad \forall r \in [0,1[. \tag{10}$$

This will lead to $f(r) \leq \frac{L2^{\alpha+1}M}{(1-r)^\alpha}\Gamma(\alpha)$ $\forall r \in [0,1[$ and the conclusion will follow.

Let us now establish (10). With I_k defined as in Lemma 6, $\displaystyle\sum_{n\geq 1} n^{\alpha-1} r^n = \sum_{k\geq 0}\sum_{n\in I_k} n^{\alpha-1} r^n$. Since $0 \leq r < 1$, $r^n \geq r^{2^{k+1}}$ $\forall n < 2^{k+1}$ and $n^\alpha \geq 2^{k\alpha}$ $\forall n \geq 2^k$. Hence

$$\sum_{n\in I_k} n^{\alpha-1} r^n \geq r^{2^{k+1}} \sum_{n\in I_k} n^{\alpha-1} \geq r^{2^{k+1}} 2^{k\alpha} \sum_{n\in I_k} \frac{1}{n}.$$

The last sum contains 2^k terms, each of which $\geq \frac{1}{2^{k+1}}$, so that

$$\sum_{n\in I_k} n^{\alpha-1} r^n \geq r^{2^{k+1}} 2^{k\alpha} \frac{1}{2} = \frac{1}{2^{1+\alpha}} r^{2^{k+1}} 2^{(k+1)\alpha}.$$

Finally $\displaystyle\sum_{n\geq 1} n^{\alpha-1} r^n \geq \frac{1}{2^{1+\alpha}} \sum_{k\geq 0} r^{2^{k+1}} 2^{(k+1)\alpha}$ and (10) follows.

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