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DIFFERENCE FUNCTIONS OF PERIODIC L_p FUNCTIONS

Abstract

We examine for which sets H of the circle group \mathbb{R}/\mathbb{Z} can the difference functions $f(x+h) - f(x)$ of a measurable or L_p function f belong to an L_q class for every $h \in H$ without f itself being in L_q . Tamás Keleti conjectured in [2] that these sets are the N -sets; that is, the sets of absolute convergence of Fourier-series. We prove this conjecture for $q \leq 2$. For $q = 2$, as a quantitative analogue of this statement, we prove a minimax theorem.

1 Introduction

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the circle group and let L_0 be the class of measurable complex valued functions on \mathbb{T} . Let $\mathcal{G} \subset \mathcal{F} \subset L_0$ be two classes of functions on \mathbb{T} . Following the notations in [2] we denote by $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ those subsets H of \mathbb{T} for which there exists a function $f \in \mathcal{F} \setminus \mathcal{G}$ with difference functions $\Delta_h f(x) = f(x+h) - f(x)$ belonging to \mathcal{G} for every $h \in H$. That is,

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \{H \subset \mathbb{T} : (\exists f \in \mathcal{F} \setminus \mathcal{G}) (\forall h \in H) \Delta_h f \in \mathcal{G}\}.$$

For the general properties of $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ see [1].

Let L_p ($0 < p < \infty$) denote the class of those measurable functions on \mathbb{T} for which $\|f\|_{L_p} = (\int_{\mathbb{T}} |f|^p)^{\frac{1}{p}} < \infty$. It is clear that $q \leq p$ implies $L_p \subset L_q$. Let

$$L_{p<} = \bigcap_{0 < q < p} L_q.$$

Then clearly $L_p \subset L_{p<}$.

Tamás Keleti observed in [2] that for these function classes the classes $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ are related to some classes of thin sets in harmonic analysis. He proved

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that $\mathfrak{H}(L_p, L_q)$ is a proper F_σ subgroup of \mathbb{T} , that every pseudo-Dirichlet set is contained in $\mathfrak{H}(L_p, L_q)$ and he also realized the importance of the non-ejectivity property of the sets in $\mathfrak{H}(L_p, L_q)$. (For the precise definitions see [5] and below.) Since the class of sets of absolute convergence of Fourier-series is exactly the closure of the class of compact non-ejective sets under taking generated subgroup, Tamás Keleti conjectured that $\mathfrak{H}(L_p, L_q)$ is exactly the class of N -sets; that is, the sets of absolute convergence of Fourier-series. We prove this for $q \leq 2$.

Tamás Keleti also observed ([1], Theorem 4.10) that if $H \subset \mathbb{T}$ is not an N -set, $f : \mathbb{T} \rightarrow \mathbb{C}$ is a measurable function with $\Delta_h f \in L_\infty$ for every $h \in H$, then $f \in L_{\infty <}$. We prove the analogous statement for every L_p class.

Now we collect here those classes which arise in our proofs and results. For a detailed study on these classes of thin sets see [5].

Definition 1.1. A set $H \subset \mathbb{T}$ is an N -set if there exists a trigonometric series on \mathbb{T} which is absolutely convergent on H but not absolutely convergent everywhere. The class of N -sets on \mathbb{T} will be denoted by \mathfrak{N} .

Definition 1.2. A Borel set $H \subset \mathbb{T}$ is a *weak-Dirichlet set* if for every probability measure μ supported on H , $\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n)| = 1$, where $\hat{\mu}(n) = \int_{\mathbb{T}} e^{-2\pi int} d\mu(t)$ denotes the Fourier-transform of μ . The class of weak-Dirichlet sets on \mathbb{T} will be denoted by \mathfrak{wD} , while \mathfrak{cwD} will stand for the class of compact weak-Dirichlet sets.

Remark 1.3. An easy computation shows that we get an equivalent definition of weak-Dirichlet sets by requiring $\inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi int} - 1|^2 d\mu(t) = 0$ for every probability measure μ supported by H .

With this notation our results can be summarized as follows.

Theorem 1.4. 1. For every $0 \leq p < q \leq 2$, $\mathfrak{H}(L_p, L_q) = \mathfrak{N}$.

2. For every $0 \leq p < q < \infty$, $\mathfrak{H}(L_p, L_{q <}) = \mathfrak{N}$.

Corollary 1.5. Let $0 \leq p < \infty$ and $H \subset \mathbb{T}$ be not an N -set. If $f : \mathbb{T} \rightarrow \mathbb{C}$ is a measurable function with $\Delta_h f \in L_p$ for every $h \in H$, then $f \in L_{p <}$.

2 Preliminaries

For proving that the classes in question are contained in \mathfrak{N} we use Fourier transforms. For $0 < p < \infty$ let l_p denote the Banach-space of the two-way

infinite sequences $c = (c_n)_{n=-\infty}^{\infty}$ of complex numbers for which

$$\|c\|_{l_p} = \left(\sum |c_n|^p \right)^{\frac{1}{p}} < \infty.$$

For an $f : \mathbb{T} \rightarrow \mathbb{C}$ integrable function let $\mathfrak{F}f(n) = \int_{\mathbb{T}} f(t)e^{2\pi int} dt$ denote the Fourier transform of f . The Plancherel Theorem says that $f \in L_2$ if and only if $\mathfrak{F}f \in l_2$, moreover every sequence of l_2 can be obtained as the Fourier transform of a function belonging to the L_2 class. According to the Parseval formula $\|f\|_{L_2} = \|\mathfrak{F}f\|_{l_2}$.

The following two theorems show the close relation of N -sets and weak-Dirichlet sets. (See [5], page 190, Theorem 1.5. and Corollary 1.6. or [4], page 49.)

Theorem 2.1. *Any increasing union of compact, weak-Dirichlet sets is an N -set.*

Theorem 2.2. *Any N -set can be obtained as an increasing countable union of compact weak-Dirichlet sets.*

Finally we state a result, which will be crucial for proving that \mathfrak{N} is contained in the classes we examine. For an $h \in \mathbb{T}$ and $A \subset \mathbb{T}$ we introduce the notation

$$\Delta_h A = ((A + h) \setminus A) \cup (A \setminus (A + h)).$$

The following notion was introduced by M. Laczkovich and I. Ruzsa in [3].

Definition 2.3. Let λ denote the normalized Lebesgue measure on \mathbb{T} . A compact set $H \subset \mathbb{T}$ is non-ejective if for every $0 \leq x \leq 1$,

$$\inf_{\substack{A \subset \mathbb{T} \\ \lambda(A)=x}} \sup_{h \in H} \lambda(\Delta_h A) = 0.$$

M. Laczkovich and I. Ruzsa stated the following result. (See [3], page 162. Theorem 4.2 and Remark 4.3.)

Theorem 2.4. *Let $H \subset \mathbb{T}$ be a compact set. Then H is weak-Dirichlet if and only if H is non-ejective.*

Unfortunately, the proof of this theorem has not been published yet. Theorem 4.2 and Remark 4.3 in [3] prove only that non-ejectivity implies the weak-Dirichlet property.

In the sequel we will use the implication weak-Dirichlet \rightarrow non-ejective. The (easy) direction non-ejective \rightarrow weak-Dirichlet will follow from our results, too. We also present the idea of Imre Ruzsa for the proof of the other implication.

In the proofs \mathbb{N} and \mathbb{Z} will stand for the set of nonnegative integers and integers respectively. Translation on \mathbb{T} by $h \in \mathbb{T}$ will be denoted by γ_h .

For a function $f : \mathbb{T} \rightarrow \mathbb{C}$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ let

$$[a \leq f < b] = \{x \in \mathbb{T} : a \leq f(x) < b\}.$$

For a set $K \subset \mathbb{T}$, χ_K denotes the characteristic function of K .

3 $\mathfrak{H}(L_1, L_2) = \mathfrak{N}$

Our main goal in this section is to show that $\mathfrak{H}(L_1, L_2) = \mathfrak{N}$. The extension of this result for other classes requires mainly computation and will be discussed in the next section.

3.1 Upper Bound

The proof of the upper bound uses an idea of de Bruijn proving the weak difference property for the L_2 class (see [6]). The trick is to average the Fourier coefficients of the functions $\Delta_h f$ for different h 's.

Theorem 3.1. *For every $1 \leq p < 2$, $\mathfrak{H}(L_p, L_2) \subset \mathfrak{N}$.*

PROOF. Let $H \subset \mathbb{T}$ not be an N -set and suppose that for a function $f \in L_p$ we have $\Delta_h f \in L_2$ for every $h \in H$. We show that $f \in L_2$. For $0 < M$ let $A_M = \{h \in \mathbb{T} : \|\Delta_h f\|_{L_2}^2 \leq M\}$. It is easy to see that A_M is compact and for $M < N$ we have $A_M \subset A_N$. Since according to our assumption the increasing union $H \subset \bigcup_{M \in \mathbb{N}} A_M$ is not an N -set, we have from Theorem 2.1 that for an $M \in \mathbb{N}$ the set A_M is not weak-Dirichlet.

Let $A_M \notin \mathfrak{wD}$. This means that there exists a probability measure μ supported on A_M for which $\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n)| < 1$. Thus for an $\eta > 0$ sufficiently small and an n_0 we have

$$\eta < |\hat{\mu}(n) - 1| \tag{1}$$

for every $|n| > n_0$.

For every $h \in A_M$ we have

$$\begin{aligned} N(h) &= \|\Delta_h f\|_{L_2}^2 = \|\mathfrak{F}\Delta_h f\|_{l_2}^2 = \sum |\mathfrak{F}(f \circ \gamma_h)(n) - \mathfrak{F}f(n)|^2 \\ &= \sum |(\mathfrak{F}f)(n) (e^{-2\pi i n h} - 1)|^2. \end{aligned}$$

As a function of h , $N(h)$ is measurable and bounded by M on the support of μ . So by the Hölder inequality and the Fubini Theorem,

$$\begin{aligned} \|(\mathfrak{F}f)(\hat{\mu} - 1)\|_{l_2}^2 &= \sum |\mathfrak{F}f|^2(n) |\hat{\mu} - 1|^2(n) \\ &= \sum |\mathfrak{F}f|^2(n) \left| \int_{\mathbb{T}} (e^{-2\pi inh} - 1) d\mu(h) \right|^2 \\ &\leq \sum |\mathfrak{F}f|^2(n) \int_{\mathbb{T}} |e^{-2\pi inh} - 1|^2 d\mu(h) \\ &= \int_{\mathbb{T}} \sum |(\mathfrak{F}f)(n) (e^{-2\pi inh} - 1)|^2 d\mu(h) = \int_{\mathbb{T}} N(h) d\mu(h) \leq M. \end{aligned}$$

From this and (1) we have $\|\mathfrak{F}f\|_{l_2}^2 \leq \sum_{j=-n_0}^{n_0} |\mathfrak{F}f|^2(j) + \frac{M}{\eta^2}$. Thus $\mathfrak{F}f \in l_2$, which implies $f \in L_2$ by the Plancherel Theorem. \square

3.2 Lower Bounds

Definition 3.2. For a measurable function $f : \mathbb{T} \rightarrow \mathbb{C}$,

$$\mathcal{L}(f) = \sup \{0 \leq p : f \in L_p\}$$

denotes the *class of f* .

The lower bounds are based on the non-ejectivity property of weak-Dirichlet sets; namely we prove first that every non-ejective set is contained in our classes.

Lemma 3.3. *For every compact, non-ejective set $H \subset \mathbb{T}$ and $0 \leq p < \infty$ fixed, there is a function $f : \mathbb{T} \rightarrow \mathbb{R}$ such that $\mathcal{L}(f) = p$ and $\mathcal{L}(\Delta_h f) = \infty$ with*

$$\|\Delta_h f\|_{L_q} \leq M(q)$$

for every $h \in H$ and $1 \leq q$, where $M(q)$ is a constant depending only on q (for fixed p).

PROOF. By the non-ejectivity of H , for every fixed $n \in \mathbb{N}$ and for every $0 < \varepsilon$ there exists a set $K_n(\varepsilon)$ such that

$$\lambda(K_n(\varepsilon)) = \frac{1}{2^n} \tag{2}$$

and $\lambda(\Delta_h K_n(\varepsilon)) \leq \varepsilon$, $\forall h \in H$. Let $f_n = 2^{\frac{n}{p}} \chi_{K_n(\frac{1}{2^n})}$ and $f = \sum_{n=1}^{\infty} f_n$, as a point-wise limit of measurable functions. We show that $f < \infty$ almost

everywhere, $\mathcal{L}(f) = p$ and $\mathcal{L}(\Delta_h f) = \infty$ with $\|\Delta_h f\|_{L_q} \leq M(q)$ for every $h \in H$ with an appropriate constant $M(q)$ for every $1 \leq q$. By (2) we have $\sum_{n=1}^{\infty} \lambda\left(K_n\left(\frac{1}{2^{2^n}}\right)\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$, so the first statement follows from the Borel-Cantelli Lemma. Since $f_n \leq f$ for every $n \in \mathbb{N}$ and for every $p < \alpha$, we have

$$\|f_n\|_{L_\alpha} = \frac{1}{2^{\frac{n}{\alpha}}} 2^{\frac{n}{p}} = 2^{n(\frac{1}{p} - \frac{1}{\alpha})} \rightarrow \infty$$

if $n \rightarrow \infty$, we can conclude $\mathcal{L}(f) \leq p$.

If a $t \in \mathbb{T}$ satisfies $t \notin K_m(\frac{1}{2^{2^m}})$ for every $m > n$, then

$$f(t) \leq \sum_{j=1}^n 2^{\frac{j}{p}} = \frac{2^{\frac{n+1}{p}} - 2^{\frac{1}{p}}}{2^{\frac{1}{p}} - 1} \leq C_p 2^{\frac{n}{p}}$$

with a constant $C_p = \frac{2^{\frac{1}{p}}}{2^{\frac{1}{p}} - 1}$ depending only on p . Since

$$\lambda\left(\bigcup_{m=n+1}^{\infty} K_m\left(\frac{1}{2^{2^m}}\right)\right) \leq \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n},$$

we have $\lambda\left(\left[C_p 2^{\frac{n}{p}} \leq f\right]\right) \leq \frac{1}{2^n}$. So if $C_p 2^{\frac{n}{p}} \leq x < C_p 2^{\frac{n+1}{p}}$ for some $n \in \mathbb{N}$, then $\lambda([x \leq f]) \leq \frac{1}{2^n}$ which implies that $\lambda([x \leq f]) < \frac{2C_p^p}{x^p}$. Thus for an $\alpha \leq p$,

$$\begin{aligned} \|f\|_{L_\alpha}^\alpha &= \int_{\mathbb{T}} f^\alpha = \int_0^\infty \lambda([y \leq f^\alpha]) dy \leq C_p + \int_{C_p}^\infty \lambda([y \leq f^\alpha]) dy \\ &= C_p + \int_{C_p}^\infty \alpha x^{\alpha-1} \lambda([x \leq f]) dx \leq C_p + \int_{C_p}^\infty 2C_p^p \alpha x^{\alpha-1} x^{-p} dx \\ &= C_p + \int_{C_p}^\infty 2C_p^p \alpha x^{\alpha-1-p} dx < \infty \end{aligned}$$

if $\alpha < p$. This proves $p \leq \mathcal{L}(f)$.

Now let $1 \leq q$ be fixed. From $f < \infty$ almost everywhere we have that the series of the difference functions $\sum_{n=1}^{\infty} \Delta_h f_n$ converges almost everywhere to $\Delta_h f$. For every $N, M \in \mathbb{N}$ with $N < M$ we have

$$\left\| \sum_{n=N}^M \Delta_h f_n \right\|_{L_q} \leq \sum_{n=N}^M \|\Delta_h f_n\|_{L_q} \leq \sum_{n=N}^M \left(2^{\frac{n}{p}q} \frac{1}{2^{2^n}} \right)^{\frac{1}{q}} = \sum_{n=N}^M 2^{\frac{n}{p} - \frac{2^n}{q}} \rightarrow 0$$

as $N \rightarrow \infty$. Hence the series $\sum_{n=1}^{\infty} \Delta_h f_n$ is Cauchy in L_q , so it is convergent point-wise and in L_q too, so the point-wise limit $\Delta_h f$ belongs to L_q with

$$\|\Delta_h f\|_{L_q} = \left\| \sum_{n=1}^{\infty} \Delta_h f_n \right\|_{L_q} \leq \sum_{n=1}^{\infty} \|\Delta_h f_n\|_{L_q} \leq \sum_{n=1}^{\infty} 2^{\frac{n}{p} - \frac{2^n}{q}},$$

which completes the proof. □

Corollary 3.4. *For every $H \in \text{cw}\mathfrak{D}$ and $0 \leq p < \infty$ fixed, there is a function $f : \mathbb{T} \rightarrow \mathbb{R}$ such that $\mathcal{L}(f) = p$ and $\mathcal{L}(\Delta_h f) = \infty$ with $\|\Delta_h f\|_{L_q} \leq M(q)$ for every $h \in H$ and $1 \leq q$ with a constant $M(q)$ depending only on q (for fixed p).*

PROOF. Since by Theorem 2.4 every compact weak-Dirichlet set is non-ejective, the statement follows from the previous lemma. □

Now we consider general N -sets. On the account of Theorem 2.2, the remaining task is to examine the behavior of $\mathfrak{H}(L_1, L_2)$ for compact increasing union.

Lemma 3.5. *Let $1 \leq p < q \leq r < \infty$ and suppose we have an increasing sequence of sets (H_i) such that for every $i \in \mathbb{N}$ there is a function $f_i \in L_p \setminus L_q$ such that for every $h \in H_i$ we have $\|\Delta_h f_i\|_{L_r} \leq K_i$. Then for $H = \bigcup_{i=1}^{\infty} H_i$ there is a function $f \in L_p \setminus L_q$ such that $\|\Delta_h f\|_{L_r} < \infty$ for every $h \in H$.*

PROOF. By taking absolute value we can suppose that $0 \leq f_i$ for every $i \in \mathbb{N}$. For every $M \in \mathbb{R}$ and $i \in \mathbb{N}$, with the notation $f_i \wedge M = \min(f_i, M)$, we have $\|f_i \wedge M\|_{L_p} \leq \|f_i\|_{L_p}$ and $\|\Delta_h(f_i \wedge M)\|_{L_r} \leq \|\Delta_h f_i\|_{L_r} \leq K_i$ for any $h \in H_i$. On the other hand, $\|f_i \wedge M\|_{L_q} \rightarrow \infty$ for $M \rightarrow \infty$, so we can find an $M_i \in \mathbb{R}$ such that $2^i \max(K_i, \|f_i\|_{L_p}) \leq \|f_i \wedge M_i\|_{L_q}$. Let

$$f = \sum_{i=1}^{\infty} \frac{f_i \wedge M_i}{2^i \max(K_i, \|f_i\|_{L_p})}, \tag{3}$$

as a point-wise limit of measurable functions.

It is obvious that $f \in L_p \setminus L_q$, in particular $f < \infty$ almost everywhere. Thus the series (3) is absolutely convergent almost everywhere, so for every $h \in H$ the series $\sum_{i=1}^{\infty} \Delta_h \frac{f_i \wedge M_i}{2^i \max(K_i, \|f_i\|_{L_p})}$ converges point-wise to $\Delta_h f$ almost

everywhere. If $h \in H_j$ for a $j \in \mathbb{N}$, then for every $N, M \in \mathbb{N}$ with $j < N < M$ we have

$$\begin{aligned} \left\| \sum_{i=N}^M \Delta_h \left(\frac{f_i \wedge M_i}{2^i \max(K_i, \|f_i\|_{L_p})} \right) \right\|_{L_r} &\leq \sum_{i=N}^M \left\| \Delta_h \left(\frac{f_i \wedge M_i}{2^i \max(K_i, \|f_i\|_{L_p})} \right) \right\|_{L_r} \\ &= \sum_{i=N}^M \frac{\|\Delta_h(f_i \wedge M_i)\|_{L_r}}{2^i \max(K_i, \|f_i\|_{L_p})} \\ &\leq \sum_{i=N}^M \frac{1}{2^i} \leq \frac{1}{2^{N-1}}, \end{aligned}$$

so the series $\sum_{i=1}^{\infty} \Delta_h \frac{f_i \wedge M_i}{2^i \max(K_i, \|f_i\|_{L_p})}$ is Cauchy and hence convergent in L_r . So we have that $\Delta_h f$ itself is in L_r for every $h \in H$. □

Theorem 3.6. *For every $1 \leq p < q < \infty$,*

1. $\mathfrak{N} \subset \mathfrak{H}(L_p, L_{q<})$.
2. $\mathfrak{N} \subset \mathfrak{H}(L_p, L_q)$.

PROOF. We prove the two statements simultaneously. Let $H \in \mathfrak{N}$. By Theorem 2.2 there is an increasing sequence of sets $(H_i) \subset \mathfrak{CW}\mathfrak{D}$ with $H = \bigcup_{i=1}^{\infty} H_i$. Let $1 \leq p < q < \infty$ be fixed. By Corollary 3.4, for any $i = 1, 2, \dots$ we have a function f_i fulfilling $\mathcal{L}(f_i) = \frac{p+q}{2}$ and $\mathcal{L}(\Delta_h f_i) = \infty$ with $\|\Delta_h f_i\|_{L_q} \leq M(q)$ for some $M(q) \in \mathbb{R}$ fixed, for every $h \in H_i$. Thus by Lemma 3.5 there is a function $f \in L_p \setminus L_{\frac{p+3q}{4}} \subset L_p \setminus L_{q<}$ for which $\Delta_h f \in L_q$ for every $h \in H$. This shows $H \in \mathfrak{H}(L_p, L_{q<}) \cap \mathfrak{H}(L_p, L_q)$ so the proof is complete. □

Theorem 3.1 and Theorem 3.6 together prove $\mathfrak{H}(L_1, L_2) = \mathfrak{N}$.

4 Inclusions of Classes

The arguments in the previous section used the Banach-space structure of L_p for $1 \leq p$. Hence for $p < 1$ the proofs don't work.

In this section we prove the following two inclusion theorems between the classes $\mathfrak{H}(L_p, L_q)$ and $\mathfrak{H}(L_r, L_{s<})$.

Theorem 4.1. *For every $0 \leq p < q < \infty$, $0 \leq r < s < \infty$ with $q \leq s$, $\mathfrak{H}(L_p, L_q) \subset \mathfrak{H}(L_r, L_s)$.*

Theorem 4.2. *For every $0 \leq p < q < \infty$ the classes $\mathfrak{H}(L_p, L_{q<})$ coincide.*

We also finish the proof of our main theorem. The proofs of these theorems use some properties of powers of functions. These properties are described in Lemma 4.3, Lemma 4.5 and Lemma 4.7.

4.1 Powers Less Than 1

The following lemma describes the effect of taking the power f^α for $\alpha \leq 1$.

Lemma 4.3. *Let $0 \leq p \leq r < \infty$ and $h \in \mathbb{T}$. If $f \in L_p$, $0 \leq f$ and $\Delta_h f \in L_r$ then for every $0 < \alpha \leq 1$ we have $f^\alpha \in L_{\frac{p}{\alpha}}$ and $\Delta_h f^\alpha \in L_{\frac{r}{\alpha}}$ with $\|\Delta_h f^\alpha\|_{L_{\frac{r}{\alpha}}} \leq \|\Delta_h f\|_{L_r}^\alpha$. We also have $\mathcal{L}(f^\alpha) = \frac{\mathcal{L}(f)}{\alpha}$.*

PROOF. The first and third statement being obvious we prove only the second one. It is easy to see that for $0 < \alpha \leq 1$ and $a, b \in \mathbb{C}$, $\|a|^\alpha - |b|^\alpha\| \leq |a - b|^\alpha$. From this with $a = f(x + h)$, $b = f(x)$ we have

$$|f^\alpha(x + h) - f^\alpha(x)|^{\frac{r}{\alpha}} \leq |f(x + h) - f(x)|^{\frac{r}{\alpha}} = |f(x + h) - f(x)|^r,$$

so $\Delta_h f^\alpha \in L_r$ and $\|\Delta_h f^\alpha\|_{L_{\frac{r}{\alpha}}} \leq \|\Delta_h f\|_{L_r}^\alpha$. \square

4.2 Powers Greater Than 1

Now we examine f^α for $1 < \alpha$. As we shall see, this power does not behave as well as it did for $\alpha \leq 1$, there are more technical difficulties for a lesser result.

Lemma 4.4. *Let $0 < \delta < 1$, $1 \leq b \leq b + b^\delta \leq a$ be reals. Then for every $1 \leq \gamma$ we have $(a^\gamma - b^\gamma) \leq 2^\gamma (a - b)^{\gamma + \frac{(1-\delta)(\gamma-1)}{\delta}}$.*

PROOF. We show that the value of the fraction $\frac{a^\gamma - b^\gamma}{(a - b)^\gamma}$ is maximal when $a = b + b^\delta$. This is true since $\left(\frac{x^\gamma - b^\gamma}{(x - b)^\gamma}\right)' = (x - b)^{-1-\gamma} \gamma b (b^{\gamma-1} - x^{\gamma-1}) \leq 0$ for $b \leq x$ and $1 \leq \gamma$, so the fraction $\frac{x^\gamma - b^\gamma}{(x - b)^\gamma}$ is a decreasing function of x for $b \leq x$. For every $1 \leq \gamma$ and $0 \leq t \leq 1$ we have

$$(1 + t)^\gamma \leq 1 + 2^\gamma t, \quad (4)$$

since the left-hand side of the expression is convex while the right-hand side is linear in t , and the inequality holds for $t = 0$ and $t = 1$. Applying (4) with $t = \frac{1}{y}$ we have $(1 + y)^\gamma - y^\gamma \leq 2^\gamma y^{\gamma-1}$, and with $y = b^{1-\delta}$ we get

$$\begin{aligned} \frac{a^\gamma - b^\gamma}{(a - b)^\gamma} &\leq \frac{(b + b^\delta)^\gamma - b^\gamma}{(b + b^\delta - b)^\gamma} = (1 + b^{1-\delta})^\gamma - b^{(1-\delta)\gamma} \\ &\leq 2^\gamma b^{(1-\delta)(\gamma-1)} \leq 2^\gamma (a - b)^{\frac{(1-\delta)(\gamma-1)}{\delta}} \end{aligned}$$

as stated. □

Lemma 4.5. *Let $0 < q < r < \infty$, $H \subset \mathbb{T}$ and $f : \mathbb{T} \rightarrow \mathbb{C}$ such that $\mathcal{L}(f) = q$ and $\Delta_h f \in L_r$ for every $h \in H$. Then there is a continuous function $\rho : (0, q) \rightarrow \mathbb{R}$ such that $p < \rho(p)$ and for every $0 < p < q$ there exists a function f_p such that $\mathcal{L}(f_p) = p$ and $\rho(p) \leq \mathcal{L}(\Delta_h f_p)$ for every $h \in H$.*

PROOF. Let $f_p = |f|^{\frac{q}{p}}$. Again, it is obvious that $\mathcal{L}(f_p) = p$. Fix an $h \in H$ and let $0 < \delta < 1$, $p < s$ be arbitrary. The value of δ will be chosen later. Let

$$a(x) = \max(|f(x+h)|, |f(x)|),$$

$$b(x) = \min(|f(x+h)|, |f(x)|),$$

$$A = \{x \in \mathbb{T} : b(x) + b^\delta(x) \leq a(x)\}, \quad A' = \{x \in \mathbb{T} : 1 \leq b(x)\}.$$

Then

$$\begin{aligned} \|\Delta_h f_p\|_{L_s}^s &= \int_{\mathbb{T}} \left| |f(x+h)|^{\frac{q}{p}} - |f(x)|^{\frac{q}{p}} \right|^s dx \\ &= \int_{\mathbb{T}} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx = \int_{A \cap A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx \\ &\quad + \int_{(\mathbb{T} \setminus A) \cap A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx + \int_{\mathbb{T} \setminus A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx. \end{aligned}$$

We estimate the integrals separately.

For the first term, by applying Lemma 4.4 with $\gamma = \frac{q}{p}$, we get

$$\int_A \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx \leq 2^{\frac{q}{p}} \int_{\mathbb{T}} (a(x) - b(x))^{s\frac{q}{p} + s\frac{(1-\delta)(\frac{q}{p}-1)}{\delta}} dx. \tag{5}$$

Since $\Delta_h f \in L_r$ means

$$\int_{\mathbb{T}} |f(x+h) - f(x)|^r dx = \int_{\mathbb{T}} (a(x) - b(x))^r dx < \infty,$$

(5) is finite if $s\frac{q}{p} + s\frac{(1-\delta)(\frac{q}{p}-1)}{\delta} \leq r$; that is, if

$$s \leq \frac{rp}{q + \frac{(1-\delta)(q-p)}{\delta}}. \tag{6}$$

For the second term, again from Lemma 4.4 with $\gamma = \frac{q}{p}$, we get

$$\begin{aligned} \int_{(\mathbb{T} \setminus A) \cap A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx &\leq \int_{\mathbb{T}} \left((b + b^\delta)^{\frac{q}{p}} - b^{\frac{q}{p}} \right)^s dx \\ &\leq 2^{s \frac{q}{p}} \int_{\mathbb{T}} b^{s \delta \left(\frac{q}{p} + \frac{(1-\delta)(\frac{q}{p}-1)}{\delta} \right)} dx \\ &\leq 2^{s \frac{q}{p}} \int_{\mathbb{T}} b^{s \left(\delta + \frac{q}{p} - 1 \right)} dx \\ &\leq 2^{s \frac{q}{p}} \int_{\mathbb{T}} |f|^{s \left(\delta + \frac{q}{p} - 1 \right)} dx, \end{aligned}$$

which is finite if $s \left(\delta + \frac{q}{p} - 1 \right) < q$; that is, if

$$s < \frac{p}{1 - (1 - \delta) \frac{p}{q}}. \tag{7}$$

To estimate the third term we use that if $0 \leq b \leq 1$, $b \leq a$, then

$$a^\alpha \leq 2^\alpha (a - b)^\alpha + 2^\alpha,$$

for $0 \leq a \leq 2$ by the second term, for $2 \leq a$ by the first term. Applying this with $\alpha = s \frac{q}{p}$, for the third term we have

$$\int_{\mathbb{T} \setminus A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s dx \leq \int_{\mathbb{T} \setminus A'} a^{s \frac{q}{p}} dx \leq \int_{\mathbb{T} \setminus A'} 2^{s \frac{q}{p}} \left((a - b)^{s \frac{q}{p}} + 1 \right) dx,$$

which is finite whenever $s \frac{q}{p} \leq r$; that is, if

$$s \leq \frac{pr}{q}. \tag{8}$$

With the unique $0 < \delta < 1$ fulfilling $\frac{1-\delta}{\delta}(q-p) = \frac{r-q}{2}$, the right-hand side of (6) and (7) is bigger than p . Fixing this δ let

$$\rho = \min \left\{ \frac{rp}{q + \frac{(1-\delta)(q-p)}{\delta}}, \frac{p}{1 - (1 - \delta) \frac{q}{p}}, \frac{pr}{q} \right\}.$$

Then $p < \rho$, and (6), (7) and (8) are satisfied for $s < \rho$. This proves the statement. \square

4.3 A Small Power

To put a function f with $\mathcal{L}(f) = 0$ in a L_p class for $0 < p$ it must be composed by a function which increases more slowly than x^α for any $0 < \alpha$. This is done in the following lemmas. They will help to deal with the $p = 0$ case.

Lemma 4.6. *If $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a concave function with $\Phi(1) \leq 1$, $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and $\Delta_h f \in L_p$ for some $0 \leq p < \infty$ and $h \in \mathbb{T}$, then $\Delta_h \Phi(|f|) \in L_p$ with $\|\Delta_h \Phi(|f|)\|_{L_p}$ bounded by a function of $\|\Delta_h f\|_{L_p}$.*

PROOF. Since $0 \leq \Phi$, the concavity of Φ and $\Phi(1) \leq 1$ imply $\Phi(x) \leq x + 1$. Using the concavity and $\Phi(x) \leq x + 1$ we get

$$\begin{aligned} |\Phi(|f(x+h)|) - \Phi(|f(x)|)| &\leq \Phi(|f(x+h)| - |f(x)|) - \Phi(0) \\ &\leq |f(x+h)| - |f(x)| + 1 \\ &\leq |f(x+h) - f(x)| + 1, \end{aligned}$$

from which the statement follows. □

Lemma 4.7. *Let $(\lambda_i)_{i=0}^\infty$ be a given sequence of nonnegative reals with*

$$\sum_{i=0}^\infty \lambda_i = 1, \tag{9}$$

$$\sum_{i=1}^\infty i \lambda_i = \infty. \tag{10}$$

Then there is a $\Phi : [0, \infty) \rightarrow [0, \infty)$ concave function with $\Phi(1) \leq 1$ such that

$$\sum_{i=0}^\infty \Phi(i) \lambda_i < \infty, \tag{11}$$

$$\sum_{i=0}^\infty \Phi^3(i) \lambda_i = \infty. \tag{12}$$

PROOF. From (9) we have an increasing sequence $(i_k)_{k=0}^\infty$ of integers with

$$\sum_{i_k \leq i} \lambda_i \leq \frac{1}{2^k}. \tag{13}$$

We define recursively a strictly increasing sequence $(a_k)_{k=0}^\infty$ of integers fulfilling

$$\max\{k, i_k\} \leq a_k, \tag{14}$$

$$a_k - a_{k-1} \leq a_{k+1} - a_k \tag{15}$$

and

$$\frac{1}{k^2} \leq \sum_{a_k \leq i < a_{k+1}} \left(k + \frac{i - a_k}{a_k - a_{k-1}} \right) \lambda_i \tag{16}$$

for every $k = 1, 2, \dots$. Let $a_0 = 0, a_1 = i_1$. If a_k is already defined, we define a_{k+1} simply by taking its value large enough. Conditions (14) and (15) can easily be satisfied, while (10) shows that (16) also holds for a large a_{k+1} , since the coefficients in (16) grow linearly in i .

If there is a given strictly increasing sequence $(a_k)_{k=0}^\infty$ of integers with $a_0 = 0$ and a sequence $(b_k)_{k=0}^\infty$ of reals, we denote by $\Phi_{(a_k), (b_k)}$ the function for which $\Phi(a_k) = b_k$ and linear on the intervals $[a_k, a_{k+1}]$. With this notation let $\Phi = \Phi_{(a_k), (k)}$, a concave function by (15) and $\Phi(1) \leq 1$ by (14).

Since

$$\begin{aligned} \sum_{i=0}^\infty \Phi(i)\lambda_i &= \sum_{k=0}^\infty \sum_{a_k \leq i < a_{k+1}} \Phi(i)\lambda_i \leq \sum_{k=0}^\infty \sum_{a_k \leq i < a_{k+1}} (k+1)\lambda_i \leq \sum_{k=0}^\infty \sum_{a_k \leq i} (k+1)\lambda_i \\ &\leq \sum_{k=0}^\infty \sum_{i_k \leq i} (k+1)\lambda_i \leq \sum_{k=0}^\infty \frac{(k+1)}{2^k} < \infty, \end{aligned}$$

we have property (11). If we knew

$$\frac{1}{k^2} \leq \sum_{a_k \leq i} \Phi(i)\lambda_i \tag{17}$$

for infinitely many k 's, then since $k \leq \Phi(i)$ for $a_k \leq i$, we would have $1 \leq \sum_{a_k \leq i} \Phi^3(i)\lambda_i$ for infinitely many k 's, which would show that the series of (12) cannot converge. So if (17) holds only for a finite set of k 's, then we modify Φ to have it for infinitely many k 's.

If $\sum_{a_k \leq i} \Phi(i)\lambda_i < \frac{1}{k^2}$, then there is a $0 < d$ such that with the sequence $b_l = l$, if $l \leq k$ and $b_l = l + d$, if $k < l$, we have

$$\sum_{a_k \leq i} \Phi_{(a_k), (b_k)}(i)\lambda_i = \frac{1}{k^2} \tag{18}$$

and the function $\Phi_{(a_k), (b_k)}$ is still concave. This follows from (16), which says that if Φ would increase on $[a_k, a_{k+1}]$ as is does on $[a_{k-1}, a_k]$, then $\frac{1}{k^2} \leq \sum_{a_k \leq i < a_{k+1}} \Phi(i)\lambda_i$. Moreover, if this modification is done for a k bigger than 2, then also $\Phi(1) \leq 1$ remains true.

We repeat this modification step finitely or infinitely many times to get, as a monotone increasing point-wise limit of concave functions, a concave function Φ for which (17) holds for infinitely many k 's, so we certainly have property (12). We show that we did not loose property (11). Either there were finitely or infinitely many modification steps, in the j th step, say at a_{k_j} , the value of the sum $\sum_{i=1}^{\infty} \Phi(i)\lambda_i$ could increase by at most $\frac{1}{k_j^2}$. Since

$$\sum_{j=1}^{\infty} \frac{1}{k_j^2} \leq \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty, \sum_{i=0}^{\infty} \Phi(i)\lambda_i$$

must converge; so the proof is complete. □

As an application, we can prove the last lemma of this section.

Lemma 4.8. *Let $0 < s < \infty$, $H \subset \mathbb{T}$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ be such that $\mathcal{L}(f) = 0$ and $\Delta_h f \in L_s$ for every $h \in H$. Then there is a function $0 \leq f_\star$ such that $f_\star \in L_1 \setminus L_3$ and $\Delta_h f_\star \in L_4$ for every $h \in H$ with $\|\Delta_h f_\star\|_{L_4}$ bounded by some function of $\|\Delta_h f\|_{L_s}$.*

PROOF. If $s < 4$, then let $\tilde{f} = \lceil |f|^{\frac{s}{4}} \rceil$, otherwise we set $\tilde{f} = \lceil |f| \rceil$, where $\lceil \cdot \rceil$ denotes the integer part. In the first case, by Lemma 4.3, we have $\Delta_h \tilde{f} \in L_4$ with

$$\|\Delta_h \tilde{f}\|_{L_4} \leq \|\Delta_h f\|_{L_s} + 2 \tag{19}$$

for every $h \in H$. In the second case (19) holds obviously for every $h \in H$. Since $\mathcal{L}(f) = 0$, we have $\mathcal{L}(\tilde{f}) = 0$ in both cases. By applying Lemma 4.7 with $\lambda_i = \lambda \left(\lceil \tilde{f} = i \rceil \right)$, we get the concave function Φ for which $f_\star = \Phi \circ \tilde{f} \in L_1 \setminus L_3$. According to Lemma 4.6, $\Delta_h f_\star \in L_4$ with $\|\Delta_h f_\star\|_{L_4}$ bounded by some function of $\|\Delta_h f\|_{L_s}$ for every $h \in H$. □

Corollary 4.9. *Let $0 < s < \infty$, $H \subset \mathbb{T}$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ be such that $\mathcal{L}(f) = 0$ and $\Delta_h f \in L_s$ for every $h \in H$. Then $H \in \mathfrak{H}(L_u, L_v) \cap \mathfrak{H}(L_u, L_{v<})$ for every $0 \leq u < v < \infty$.*

PROOF. Let f_\star be as in the previous Lemma. If $\mathcal{L}(f_\star) \leq \frac{u+v}{2}$ or $4 \leq v$, we apply Lemma 4.3 for $f = f_\star$ with $r = 4$ and $\alpha = \min \left\{ \frac{4}{v}, \frac{2\mathcal{L}(f_\star)}{u+v} \right\}$. From $\alpha \leq \frac{2\mathcal{L}(f_\star)}{u+v}$ we get that the function f_\star^α satisfies $u < \mathcal{L}(f_\star^\alpha) < v$, while from $\alpha \leq \frac{4}{v}$ we have $\Delta_h f_\star^\alpha \in L_v$ for every $h \in H$. If $\frac{u+v}{2} < \mathcal{L}(f_\star)$ and $v < 4$, we can apply Lemma 4.5 for $f = f_\star$ with $q = \mathcal{L}(f_\star)$, $r = 4$. Since the function ρ

is continuous and $p < \rho(p)$ for every $p \in (0, q)$, there is a \bar{p} with $u < \bar{p} < v$ for which $v < \rho(\bar{p})$. For this \bar{p} from $\mathcal{L}(f_{\bar{p}}) = \bar{p}$, $v < \rho(\bar{p}) \leq \mathcal{L}(\Delta_h f_{\bar{p}})$ we have $u < \mathcal{L}(f_{\bar{p}}) < v$ and $\Delta_h f_{\bar{p}} \in L_v$ for every $h \in H$. \square

4.4 The Inclusion Theorems

Now we can prove the two inclusion theorems and Theorem 1.4.

PROOF OF THEOREM 4.1. Suppose that for a set $H \subset \mathbb{T}$ we have $H \in \mathfrak{H}(L_p, L_q)$; that is, there is a function $f \in L_p \setminus L_q$ with $\Delta_h f \in L_q$ for every $h \in H$. Then by Lemma 4.3 with $\alpha = \frac{q}{s}$ we have $\tilde{f} = |f|^{\frac{q}{s}} \in L_{\frac{ps}{q}} \setminus L_s$ and $\Delta_h \tilde{f} \in L_s$ for every $h \in H$. So $H \in \mathfrak{H}(L_r, L_s)$ if $r < \mathcal{L}(\tilde{f})$.

Suppose now that $\mathcal{L}(\tilde{f}) \leq r < s$. If $0 < \mathcal{L}(\tilde{f})$, then by applying Lemma 4.3 with $\alpha = \frac{2\mathcal{L}(\tilde{f})}{r+s}$ we have a function $\bar{f} \in L_r \setminus L_s$ with $\Delta_h \bar{f} \in L_{s \frac{r+s}{2\mathcal{L}(\tilde{f})}} \subset L_s$ for every $h \in H$. Thus $H \in \mathfrak{H}(L_r, L_s)$.

If $\mathcal{L}(\tilde{f}) = 0$, then the statement follows from Corollary 4.9. \square

PROOF OF THEOREM 4.2. It is enough to prove that $\mathfrak{H}(L_p, L_{q<}) \subset \mathfrak{H}(L_r, L_{s<})$ for every $0 \leq p < q < \infty, 0 \leq r < s < \infty$. Let $H \in \mathfrak{H}(L_p, L_{q<})$. We distinguish two cases.

Case 1. ($q \leq s$) We proceed as in the proof of Theorem 4.1. By Lemma 4.3 with $\alpha = \frac{q}{s}$ we have $\tilde{f} = |f|^{\frac{q}{s}} \in L_{\frac{ps}{q}} \setminus L_{s<}$ and $\Delta_h \tilde{f} \in L_{s<}$ for every $h \in H$, and we are done if $r < \mathcal{L}(\tilde{f})$.

Now if $0 < \mathcal{L}(\tilde{f}) \leq r < s$, then again by Lemma 4.3 with $\alpha = \frac{2\mathcal{L}(\tilde{f})}{r+s}$ we have a function $\bar{f} \in L_r \setminus L_{s<}$ with $\Delta_h \bar{f} \in L_{(s \frac{r+s}{2\mathcal{L}(\tilde{f})})<} \subset L_s$ for every $h \in H$ which shows $H \in \mathfrak{H}(L_r, L_{s<})$.

If $\mathcal{L}(\tilde{f}) = 0$, then the statement follows from Corollary 4.9.

Case 2. ($s < q$) Let us consider first $0 < \mathcal{L}(f)$. By Lemma 4.5 we have a pair u, v satisfying $0 < u < v < s$ and a function \tilde{f} with $\mathcal{L}(\tilde{f}) = u$ and $\Delta_h \tilde{f} \in L_{v<}$ for every $h \in H$. This \tilde{f} shows that $H \in \mathfrak{H}(L_u, L_{v<})$. Now we are in the situation of *Case 1.*, so we have $H \in \mathfrak{H}(L_r, L_{s<})$.

Now if $\mathcal{L}(f) = 0$, then again the statement follows from Corollary 4.9. \square

PROOF OF THEOREM 1.4. Let $0 \leq p < q < \infty$. From Theorem 3.6, Theorem 4.1 and Theorem 3.1 we have

$$\mathfrak{N} \subset \mathfrak{H}(L_1, L_{2<}) \subset \bigcup_{1 < r < 2} \mathfrak{H}(L_1, L_r) \subset \mathfrak{H}(L_1, L_2) \subset \mathfrak{N},$$

so from Theorem 4.2, $\mathfrak{H}(L_p, L_{q<}) = \mathfrak{H}(L_1, L_{2<}) = \mathfrak{N}$. This proves $\mathfrak{H}(L_p, L_{q<}) = \mathfrak{N}$.

If $q \leq 2$, then from the preceding and from Theorem 4.1 and Theorem 3.1 we have

$$\mathfrak{N} \subset \mathfrak{H}(L_p, L_{q<}) \subset \bigcup_{p < r < q} \mathfrak{H}(L_p, L_r) \subset \mathfrak{H}(L_p, L_q) \subset \mathfrak{H}(L_1, L_2) \subset \mathfrak{N}. \quad \square$$

5 Ejectivity

As a corollary of Lemma 3.3 and Theorem 3.1 we get that every non-ejective set is an N -set, in particular every compact, non-ejective set is a compact N -set, hence a compact weak-Dirichlet set. (For the last implication see [5].) This proves Theorem 2.4 in one direction. However, as Imre Ruzsa showed, the averaging technique of Theorem 3.1 can be applied directly to ejective sets, which gives a direct proof (see [7]). In this section we prove a minimax theorem and we discuss how it can be applied for proving Theorem 2.4.

5.1 A Minimax Theorem

In Corollary 3.4 we have proved in particular that for a compact, weak-Dirichlet set H there is a function $f \in L_2 \setminus L_3$ such that every $h \in H$ the difference functions $\Delta_h f$ are so small that they may belong to any fixed higher class. The following theorem shows a quantitative analogue of this result in the class L_2 .

For a set $H \subset \mathbb{T}$, $\mathcal{M}(H)$ denotes the set of probability measures supported by H . For local use let

$$\mathcal{F} = \left\{ f : \mathbb{T} \rightarrow \mathbb{C} : f \in L_2, \|f\|_{L_2} = 1, \int_{\mathbb{T}} f = 0 \right\},$$

while ${}_{\mathbb{R}}l_p \subset l_p$ stands for the *real* Banach-space of two-way infinite p -summable sequences.

Theorem 5.1. *Let $H \subset \mathbb{T}$ be a compact set. Then*

$$\inf_{f \in \mathcal{F}} \sup_{h \in H} \|\Delta_h f\|_{L_2}^2 = \sup_{\mu \in \mathcal{M}(H)} \inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 d\mu(h).$$

PROOF. If for an $f \in \mathcal{F}$ we have $\sup_{h \in H} \|\Delta_h f\|_{L_2}^2 \leq \varepsilon^2$, then by the Parseval formula $\sum |\mathfrak{F}f(n)|^2 = \|f\|_{L_2}^2 = 1$ and

$$\sum |\mathfrak{F}f(n)|^2 |e^{-2\pi i n h} - 1|^2 = \|\mathfrak{F}\Delta_h f\|_{l_2}^2 = \|\Delta_h f\|_{L_2}^2 \leq \varepsilon^2.$$

Thus for every $\mu \in \mathcal{M}(H)$ we have by the Fubini Theorem

$$\begin{aligned} & \sum |\mathfrak{F}f(n)|^2 \int_{\mathbb{T}} |e^{-2\pi inh} - 1|^2 d\mu(h) \\ &= \int_{\mathbb{T}} \sum |\mathfrak{F}f(n)|^2 |e^{-2\pi inh} - 1|^2 d\mu(h) \leq \varepsilon^2. \end{aligned}$$

Since $\mathfrak{F}f(0) = 0$ for our $f \in \mathcal{F}$, we get $\inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi inh} - 1|^2 d\mu(h) \leq \varepsilon^2$, which proves

$$\sup_{\mu \in \mathcal{M}(H)} \inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi inh} - 1|^2 d\mu(h) \leq \inf_{f \in \mathcal{F}} \sup_{h \in H} \|\Delta_h f\|_{L_2}^2.$$

For the other direction let $\varepsilon^2 = \inf_{f \in \mathcal{F}} \sup_{h \in H} \|\Delta_h f\|_{L_2}^2$. We denote by $(c_0, \|\cdot\|_\infty)$ the real Banach space of two-way infinite null-convergent sequences; that is, $(c_n)_{n=-\infty}^\infty \in c_0$ if and only if $\lim_{n \rightarrow \infty} |c_n| + |c_{-n}| = 0$. For every positive $n \in \mathbb{N}$ let

$$e_n = (\dots, 0, \underbrace{1, \dots, 1}_n, 1, 0, \underbrace{1, 1, \dots, 1}_n, 0, \dots) \in l_\infty;$$

that is, for a $j \in \mathbb{Z}$, the j^{th} coordinate of e_n is 1 if $1 \leq |j| \leq n$, otherwise it is 0. For an $h \in \mathbb{T}$, let $a(h) = (a_n(h))_{n=-\infty}^\infty = (e^{-2\pi inh} - 1)_{n=-\infty}^\infty$, and for every $Q \in \mathbb{N}$ let

$$W_Q = \left\{ P_m \left((|a_n(h)|^2) \mid m \in \mathbb{N}, Q \leq m, h \in H \right) \right\},$$

where $P_m : l_\infty \rightarrow c_0$, defined by

$$P_m \left((c_n)_{n=-\infty}^\infty \right) = (\dots, 0, c_{-m}, c_{-m+1}, \dots, c_0, \dots, c_{m-1}, c_m, 0, \dots),$$

is projection on the central $2m + 1$ coordinates. Let K_Q denote the closure of the convex hull of W_Q in the $\|\cdot\|_\infty$ norm. We show that for every $0 < \delta \leq \varepsilon^2$ and $n, Q \in \mathbb{N}$, $(\varepsilon^2 - \delta)e_n \in K_Q$.

Suppose that this is not true for a Q and n and consider the Minkowski sum K of K_Q and the segment $[0, \frac{\delta}{2}]e_n$. Let $\gamma = \sup\{0 < \tau < 1 : \tau e_n \in K\}$. We have $\frac{\delta}{2} \leq \gamma \leq \varepsilon^2 - \frac{\delta}{2}$. Since K is a convex subset of the locally convex Banach space c_0 and γe_n is one point on its boundary, by the Hahn Banach Theorem there is a functional $\phi \in c_0^*$ with norm 1 such that

$$\phi(c) \leq \gamma, \quad \forall c \in K \tag{20}$$

and $\phi(e_n) = 1$. Since $c_0^* = {}_{\mathbb{R}}l_1$, the functional ϕ can be represented as a $(z_n)_{n=-\infty}^{\infty} \in {}_{\mathbb{R}}l_1$ sequence, for which we have

$$\sum_{j=-\infty}^{\infty} |z_j| = \|(z_n)\|_{l_1} = 1 = \phi(e_n) = \sum_{\substack{j=-n \\ j \neq 0}}^n z_j;$$

that is, $0 \leq z_j$ and $z_m = 0$ for $n < |m|$ and for $m = 0$. Let $(d_n)_{n=-\infty}^{\infty} \in {}_{\mathbb{R}}l_2$ be the sequence for which $0 \leq d_n = \sqrt{z_n}$. Then, from (20),

$$\|(a_n(h)d_n)\|_{l_2}^2 = \phi(P_n(|a_n(h)|^2)) \leq \gamma$$

for every $h \in H$.

Consider now $f = \mathfrak{F}((d_n)_{n=-\infty}^{\infty})$. By the Parseval formula and $d_0 = \sqrt{z_0} = 0$ we have that $\|f\|_{L_2}^2 = 1$ and $\int_{\mathbb{T}} f = 0$, so $f \in \mathcal{F}$. On the other hand, for every $h \in H$,

$$\|\Delta_h f\|_{L_2}^2 = \|\mathfrak{F}(\Delta_h f)\|_{l_2}^2 = \|(a_n(h)d_n)\|_{l_2}^2 \leq \gamma \leq \varepsilon^2 - \frac{\delta}{2},$$

which is a contradiction.

So we have that for every $0 < \delta \leq \varepsilon^2$ the point $(\varepsilon^2 - \delta)e_n$ is in K_Q ; that is, it can be obtained as the limit of a sequence of convex combinations of elements of W_Q . This means that for every $0 < \delta \leq \varepsilon^2$ and $Q \in \mathbb{N}$ there is a convex combination $(w_n^Q)_{n=-\infty}^{\infty}$ of certain elements of W_Q such that $(\varepsilon^2 - \delta) \leq w_j^Q$ for every $0 \neq |j| \leq Q$. By reformulation, for every $Q \in \mathbb{N}$ there is a convex combination μ_Q of Dirac measures on \mathbb{T} such that for every $|j| \leq Q, j \neq 0$,

$$\varepsilon^2 - \delta \leq \left(\int_{\mathbb{T}} P_Q(|a(h)|^2) d\mu_Q(h) \right) (j) = P_Q \left(\int_{\mathbb{T}} |a(h)|^2 d\mu_Q(h) \right) (j);$$

that is,

$$\varepsilon^2 - \delta \leq \int_{\mathbb{T}} |a_j|^2(h) d\mu_Q(h) = \int_{\mathbb{T}} |e^{-2\pi i j h} - 1|^2 d\mu_Q(h)$$

for every $|j| \leq Q, j \neq 0$. Moreover, μ_Q is a probability measure. The Banach space of measures on \mathbb{T} being the dual space of the continuous functions on \mathbb{T} , we can take a subsequence μ_{Q_j} weakly converging to a μ probability measure, and since $\text{supp } \mu_Q \subset H$ for every $Q \in \mathbb{N}$ and H is compact, we have $\text{supp } \mu \subset H$ also. By the definition of weak limit we get

$$\varepsilon^2 - \delta \leq \lim_{j \rightarrow \infty} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 d\mu_{Q_j}(h) = \int_{\mathbb{T}} |e^{-2\pi i n t} - 1|^2 d\mu(h)$$

for every $n \neq 0$. Taking $\delta \rightarrow 0$ we obtain

$$\inf_{f \in \mathcal{F}} \sup_{h \in H} \|\Delta_h f\|_{L_2}^2 \leq \sup_{\mu \in \mathcal{M}(H)} \inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 d\mu(h),$$

so the proof is complete. \square

The argument of Imre Ruzsa for proving that every compact weak-Dirichlet set is non-ejective goes as follows. If the set $H \subset \mathbb{T}$ is a compact weak-Dirichlet set, then by Remark 1.3,

$$\sup_{\mu \in \mathcal{M}} \inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 d\mu(h) = 0.$$

So by Theorem 5.1, for every $0 < \varepsilon$ there is a function $f^{(\varepsilon)} \in L_2$ with $\|f^{(\varepsilon)}\|_{L_2} = 1$ and $\int_{\mathbb{T}} f^{(\varepsilon)} = 0$ such that $\|\Delta_h f^{(\varepsilon)}\|_{L_2} \leq \varepsilon$ for every $h \in H$. (Imre Ruzsa proved this corollary of Theorem 5.1 in a different way, independently of our work.)

For a sequence $(\varepsilon_n) \in \{-1, +1\}^{\mathbb{Z}}$ let $f_{(\varepsilon_n)}^\varepsilon = \mathfrak{F}^{-1}((\varepsilon_n(\mathfrak{F}f)(n)))$; that is, the function obtained from f^ε by signing its Fourier coefficients with (ε_n) . By averaging on signs, Imre Ruzsa showed that there is a sequence $(\varepsilon_n) \in \{-1, +1\}^{\mathbb{Z}}$ and a $u \in \mathbb{R}$ such that for the sub-level set $A = [f_{(\varepsilon_n)}^\varepsilon < u]$ we get $\sup_{h \in H} \lambda(\Delta_h A) \leq \varepsilon \lambda(A)$, from which the non-ejectivity of H follows easily.

There is no reason to think that the conjecture of Tamás Keleti fails for the class $\mathfrak{H}(L_p, L_q)$ if $2 < q$. Our proof brakes down for $2 < q$ only because the Fourier transform happens to be an isometric isomorphism only on the L_2 class. We leave as a problem to prove the upper bound for $2 < q$.

For $2 < q$ we have no Theorem 5.1 like the statement for the L_q class. But from a function $f \notin L_q$ with $\Delta_h f \in L_q$ for every $h \in H$ one could construct a function for which a sub-level set works in the same way. A difficulty can be that not every function has appropriate sub-level sets even in the L_2 case, and for $2 < q$ we cannot average on the Fourier coefficients.

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