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REPRESENTATION OF ABSTRACT AFFINE FUNCTIONS

Abstract

It is known that any subspace $\mathcal{H}$ of the space of continuous functions on a compact set can be represented as the space of affine continuous functions defined on the state space of $\mathcal{H}$. The aim of this paper is to generalize this result for abstract affine functions of various descriptive classes (Borel, Baire etc.). The important step in the proof is to derive results on the preservation of the descriptive properties of topological spaces under perfect mappings. The main results are applied on the space of affine functions on compact convex sets and on approximation of semicontinuous and Baire–one abstract affine functions.

1 Introduction

Let U be a bounded open subset of $\mathbb{R}^n$ and $H(U)$ be the vector space of all continuous functions on the closure $\overline{U}$ of U, which are harmonic on U. For a given continuous function f defined on the boundary ∂U of U, set

$$f^{\complement U} : x \mapsto \int_{\partial U} f \, d\varepsilon_x^{\complement U} \text{ for each } x \in \overline{U}\,.$$

Here $\varepsilon_x^{\complement U}$ denotes the balayage of the Dirac measure ε_x on the complement $\complement U$ of U, so that $\varepsilon_x^{\complement U}$ is the harmonic measure at x for every $x \in U$. The restriction of $f^{\complement U}$ to U is a harmonic function and it yields the solution H^f of the generalized Dirichlet problem for the boundary condition f. If the set U is not regular, the function $f^{\complement U}$ need not be continuous on $\overline{U}$. However, according

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to [5, Proposition 7.1.4] and [8, Theorem 5] it is a Baire–one function on $\overline{U}$. (Recall that a real–valued function on a topological space K is said to be a *Baire–one function* if it is a pointwise limit of a sequence of continuous functions on K.)

In [12, Corollary 6.4] we answer affirmatively a natural question whether the function $f^{\complement U}$ can be pointwise approximated by a sequence of functions from $H(U)$. One of the technique we used, was to represent $f^{\complement U}$ as an affine Baire–one function F on a suitable compact convex set K and then to employ the Mokobodzki approximation theorem (see [13] or [15, Théorème 80]) to get a sequence of continuous affine functions on K which pointwise converges to F.

Let us consider an abstract framework of a *function space* $\mathcal{H}$ on a compact space K. By this we mean a linear subspace of $\mathcal{C}(K)$ (the space of all continuous functions on K) which separates points of K and contains the constant functions. We denote by $\mathbf{S}(\mathcal{H})$ the *state space of* $\mathcal{H}$; i.e., the convex set of all positive functionals φ in the dual space $\mathcal{H}^*$ with $\|\varphi\| = 1$ endowed with the w^*–topology. It is known that the space $\mathcal{H}$ can be isometrically imbedded into the space of affine continuous functions on $\mathbf{S}(\mathcal{H})$ (see e.g. [1, Chapter 2, § 2], [4, Chapter 1, § 4] or [6, Chapter 6, § 29]).

We denote by $\mathcal{H}^\perp$ the space of all signed Radon measures μ on K with $\mu(h) = 0$ for all $h \in \mathcal{H}$. A bounded function f on K, measurable with respect to any Radon measure on K, is said to be *completely $\mathcal{H}$–affine* if $\mu(f) = 0$ for any $\mu \in \mathcal{H}^\perp$. Clearly, any function in $\mathcal{H}$ is completely $\mathcal{H}$–affine. In [12] we prove that $f^{\complement U}$ is a completely $H(U)$–affine function and can be represented as an affine Baire–one function on $\mathbf{S}(H(U))$. A natural question arises. *Is it possible to represent any completely $\mathcal{H}$–affine function as an affine function on $\mathbf{S}(\mathcal{H})$?*

In the sequel we answer this question affirmatively. More precisely, we construct an isometric isomorphism between the space of completely $\mathcal{H}$–affine functions and the space of affine functions on $\mathbf{S}(\mathcal{H})$ satisfying the barycentric formula. Moreover, this isomorphism preserves descriptive properties of completely $\mathcal{H}$–affine functions.

The essential step is the following result. Let φ be a continuous mapping of a compact space K onto a compact space L and let $A \subset L$ be given. We derive descriptive properties of the set A from properties of the set $B := \varphi^{-1}(A)$; e.g. we show that A is Borel if B is Borel.

Thanks to deep theorems of J. Saint–Raymond ([17, Théorème 5]) and J. Jayne and C. Rogers in ([16, Theorem 5.9.13]), this has been already known for Baire sets in compact spaces. In [10, Corollary 15] we generalize their results for Borel sets in compact spaces. Since the proof of our generalization is quite short and easy, we present it for reader's convenience in Section 2 in a simplified form, suitable for our purposes.

As a corollary we obtain that descriptive properties of an affine function defined on a compact convex set K are determined by its behaviour on the closure of extreme points of the set K. This generalizes Corollaire 8 in [17].

As an application of the representation theorem we obtain results on approximation of Baire–one and lower semicontinuous completely $\mathcal{H}$–affine functions (cf. [12, Theorem 5.1] and [1, Corollary I.1.4]).

2 Descriptive Properties of Composed Functions

All topological spaces will be Hausdorff. Let K be a topological space. We recall that $A \subset K$ is an $\mathcal{F}_\sigma$–set if A is a countable union of closed subsets of K. Complements of $\mathcal{F}_\sigma$–sets are called $\mathcal{G}_\delta$–sets.

We denote by Borel(K) the set of all real–valued *Borel functions* on K. If $\mathcal{F}$ is a set of real–valued functions on K, we denote by $\mathcal{F}_b$ the set of all bounded functions of $\mathcal{F}$. Thus the space of all bounded Borel functions on K is denoted by $\text{Borel}_b(K)$. This space will be equipped with the sup–norm $\|f\| = \sup_{x\in K} |f(x)|$. The space of all continuous real–valued functions on a compact space K will be denoted by $\mathcal{C}(K)$.

The space of *Baire functions* on K; i.e., the smallest space of real–valued functions closed under the process of taking pointwise limits of sequences and containing $\mathcal{C}(K)$, will be denoted by Baire(K). We consider the space of continuous functions on K as the space of functions of the Baire class 0. Inductively, for each ordinal α less than the first uncountable ordinal ω_1, we define the space of *Baire–alpha* functions, or *functions of the Baire class* α, to be the space of pointwise limits of sequences of functions contained in the previous classes.

We say that a function $f : K \to (-\infty, \infty]$ is *lower semicontinuous* if $f^{-1}(c, \infty]$ is open for any $c \in \mathbb{R}$. If a function f on K is a pointwise limit of an increasing sequence of continuous functions, we say that f belongs to $\mathcal{C}^\uparrow(K)$. Clearly, any function from $\mathcal{C}^\uparrow(K)$ is lower semicontinuous.

To verify that a lower semicontinuous function f on a normal topological space K is in $\mathcal{C}^\uparrow(K)$ it is enough to check that $f^{-1}(c, \infty]$ is an $\mathcal{F}_\sigma$–set for every real number c. The proof of this assertion can be found in the proof of [7, Problem 1.7.15.(c)].

The space of all signed Radon measures on a compact space K will be denoted by $\mathcal{M}(K)$. We consider the space $\mathcal{M}(K)$ as the dual space to $\mathcal{C}(K)$ equipped with the w^*– topology. We write $\mathcal{M}^1(K)$ for the set of all probability Radon measures on K. For $x \in K$ we write ε_x for the Dirac measure at x;

i.e., $\varepsilon_x(f) = f(x),\ f \in \mathcal{C}(K)$. For $\mu \in \mathcal{M}(K)$, $\operatorname{spt}\mu$ stands for its support. For a μ–integrable function f on K, we simply write $\mu(f)$ instead of $\int_K f\,d\mu$.

A set $A \subset K$ is called *universally measurable* if A is μ–measurable for every $\mu \in \mathcal{M}(K)$. Due to the Jordan decomposition of signed measures, to verify universal measurability of A it is enough to check its measurability with respect to probability measures. A real–valued function f on K is called *universally measurable* if $f^{-1}(U)$ is a universally measurable subset of K for every open set $U \subset \mathbb{R}$.

Let $\varphi : K \to L$ be a continuous mapping of a compact space K into a compact space L and μ be a probability Radon measure on K. Then we define the image $\varphi\mu \in \mathcal{M}^1(L)$ of the measure μ under the continuous mapping φ by the formula $\varphi\mu(g) = \mu(g \circ \varphi)$, $g \in \mathcal{C}(L)$. According to [11, Theorem 12.46] this formula holds also for any bounded universally measurable function g on L.

If the mapping φ is onto, the induced continuous mapping (denoted likewise) $\varphi : \mathcal{M}^1(K) \to \mathcal{M}^1(L)$, assigning to every $\mu \in \mathcal{M}^1(K)$ its image $\varphi\mu \in \mathcal{M}^1(L)$, is also onto.

Let φ be a continuous surjection of a compact space K onto a compact space L and g be a real–valued function on L. The aim of this section is to derive descriptive properties of g from the properties of the function $f := g \circ \varphi$ (e.g. g is Borel if and only if f is Borel). We list results needed in the next sections in the following theorem.

Theorem 2.1. *Let K, L be compact spaces and $\varphi : K \to L$ be a continuous surjection. If g is a real–valued function on L, we set $f := g \circ \varphi$. Then*

i) $f \in \mathcal{C}^{\uparrow}(K)$ if and only if $g \in \mathcal{C}^{\uparrow}(L)$,

ii) f is lower semicontinuous if and only if g is lower semicontinuous,

iii) f is a Baire–alpha function if and only if g is a Baire–alpha function,

iv) f is a Borel function if and only if g is Borel,

v) f is universally measurable if and only if g is universally measurable.

Assertions i), ii) and v) of the theorem are easy to prove, assertion iii) is a consequence of results of [17, Théorème 5] and [16, Theorem 5.9.13 and Theorem 6.1.1].

The fourth assertion of the theorem is proved by an idea contained in Lemma 2.2. Stronger versions of this lemma are used in [10] to prove deeper results on preservation of Borel classes under perfect mappings. (We recall that a closed continuous map from a topological space K to a topological space L is called *perfect* if the fiber $f^{-1}(l)$ is compact for every $l \in L$.) Here we present its simplified modification sufficient for our purposes.

Its idea is to find a set-valued "selection" from the set-valued mapping $\ell \mapsto \varphi^{-1}(\ell), \ell \in L$, which satisfies some auxiliary conditions.

Lemma 2.2. *Let K, L be compact topological spaces and $\Phi : L \to K$ be a set-valued mapping such that $\Phi(\ell)$ is nonempty compact for every $\ell \in L$ and*

$$\Phi^{-1}(F) = \{\ell \in L : \Phi(\ell) \cap F \neq \emptyset\}$$

is closed in L for each closed set F in K. Let $\{F_n\}$ be a sequence of closed subsets of K.

Then there exists a nonempty compact–valued mapping $S : L \to K$ such that:

i) $S(\ell) \subset \Phi(\ell)$ *for every* $\ell \in L$,

ii) $S^{-1}(F_n) \cap S^{-1}(K \setminus F_n) = \emptyset$ *for each* $n \in \mathbb{N}$ *and*

iii) $S^{-1}(F_n)$ *is Borel for each* $n \in \mathbb{N}$.

Proof. Set $F_0 := K$. We will construct by induction a sequence $\{\Phi_n\}_{n=0}^{\infty}$ of mappings from L into K such that, for every $n \geq 0$:

a) $\Phi_n(\ell)$ is a nonempty compact set for each $l \in L$,

b) $\Phi_{n+1}(\ell) \subset \Phi_n(\ell) \subset \Phi_0(\ell)$ for each $\ell \in L$,

c) $\Phi_n^{-1}(F)$ is Borel for every closed set $F \subset K$,

d) $\Phi_n^{-1}(F_n) \cap \Phi_n^{-1}(K \setminus F_n) = \emptyset$.

Thanks to the assumption, by setting $\Phi_0 := \Phi$ we fulfill all conditions needed for Φ_0.

Suppose that Φ_k satisfying the required conditions have been constructed for all $k \leq n$. Set $A := \Phi_n^{-1}(F_{n+1})$. Condition c) ensures that A is Borel. Let Φ_{n+1} be defined as

$$\Phi_{n+1}(\ell) = \begin{cases} \Phi_n(\ell) \cap F_{n+1}, & \ell \in A, \\ \Phi_n(\ell), & \ell \in L \setminus A. \end{cases}$$

Conditions a), b) and d) are obviously satisfied. For a closed subset F of K we have

$$\begin{aligned} \Phi_{n+1}^{-1}(F) = &\big(\{l \in L : \Phi_n(l) \cap F_{n+1} \cap F \neq \emptyset\} \cap A\big) \cup \\ &\big(\{l \in L : \Phi_n(l) \cap F \neq \emptyset\} \cap (L \setminus A)\big), \end{aligned}$$

which yields validity of condition c). Thus the construction is complete.

Set

$$S(\ell) := \bigcap_{n=0}^{\infty} \Phi_n(\ell),\ \ell \in L.$$

Since $\{\Phi_n(\ell)\}$ is a decreasing sequence of nonempty compact subsets of K, $S(\ell)$ is a nonempty compact set contained in $\Phi(\ell)$ for every $\ell \in L$. Since

$$S^{-1}(F_n) \subset \Phi_n^{-1}(F_n) \text{ and } S^{-1}(K \setminus F_n) \subset \Phi_n^{-1}(K \setminus F_n),$$

condition ii) is satisfied and $S^{-1}(F_n) = \Phi_n^{-1}(F_n)$. Thus iii) follows from condition c). □

Remarks. 1. We note that every selection function $s : L \to K$ of S from the lemma above; i.e., $s(\ell) \in S(\ell)$ for $\ell \in L$, fulfils ii) and iii) of the lemma. Thus we may demand that $S(\ell)$ is a singleton for every $\ell \in L$.

2. The following easy fact will be used in the proof of Lemma 2.3. If a nonempty set-valued mapping S from a set L into a set K and a set $A \subset K$ is given, then $S^{-1}(A) \cap S^{-1}(K \setminus A) = \emptyset$ if and only if $S(S^{-1}(A)) \subset A$. (Recall that $S(B) = \bigcup\{S(b) : b \in B\}$.)

Lemma 2.3. *Let S be a set-valued mapping from a set L into a set K with $S^{-1}(K) = L$ and*

$$\mathcal{A} := \{A \subset K : S^{-1}(A) \cap S^{-1}(K \setminus A) = \emptyset\}.$$

Then $\mathcal{A}$ is closed with respect to complements, arbitrary intersections and arbitrary unions. Moreover, for a set $A \in \mathcal{A}$ and a subfamily $\{A_i\}_{i\in I}$ of $\mathcal{A}$ each of the following equalities holds:

i) $L \setminus S^{-1}(A) = S^{-1}(K \setminus A)$,

ii) $S^{-1}\big(\bigcup_{i\in I} A_i\big) = \bigcup_{i\in I} S^{-1}(A_i)$,

iii) $S^{-1}\big(\bigcap_{i\in I} A_i\big) = \bigcap_{i\in I} S^{-1}(A_i)$.

Proof. The family $\mathcal{A}$ is closed with respect to complements from the definition. Let $\{A_i\}_{i\in I}$ be a subfamily of $\mathcal{A}$. From the remark above and the equaltiy

$$S(S^{-1}\bigcup_{i\in I} A_i) = S\Big(\bigcup_{i\in I} S^{-1}(A_i)\Big) = \bigcup_{i\in I} S(S^{-1}(A_i)) \subset \bigcup_{i\in I} A_i$$

it follows that $\mathcal{A}$ is closed with respect to unions and consequently to intersections.

Equality i) follows from the definition and ii) holds for arbitrary sets A_i. From i) and ii) we obtain

$$L \setminus S^{-1}\Big(\bigcap_{i\in I} A_i\Big) = S^{-1}\Big(K \setminus \bigcap_{i\in I} A_i\Big) = S^{-1}\Big(\bigcup_{i\in I}(K \setminus A_i)\Big) = \bigcup_{i\in I} S^{-1}(K \setminus A_i)$$
$$= \bigcup_{i\in I}\big(L \setminus S^{-1}(A_i)\big) = L \setminus \bigcap_{i\in I} S^{-1}(A_i),$$

which proves iii). □

Now we are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. "If" parts of statements of the theorem follows from continuity of the mapping φ.

Assertions i) and ii) follows from the characterization of lower semicontinuous functions (respectively functions from $\mathcal{C}^{\uparrow}(L)$) via level sets and from the fact that φ maps closed sets in K onto closed sets in L.

Statement iii) follows by [16, Theorem 5.9.13] and the Lebesgue–Hausdorff characterization of Baire–alpha functions on completely regular spaces (see [16, Theorem 6.1.1]).

To prove statement iv), it is enough to check the assertion only for the characteristic function of a set $A \subset L$, i.e., we want to prove that A is Borel if $\varphi^{-1}(A)$ is Borel. By transfinite induction, we can find a countable family $\mathcal{F}$ of closed sets in K such that $B := \varphi^{-1}(A)$ is contained in the σ–algebra $\sigma(\mathcal{F})$ generated by the family $\mathcal{F}$. Since the set-valued mapping $\Phi : l \mapsto \varphi^{-1}(l)$ from L into K satisfies the assumptions of Lemma 2.2, we obtain a set-valued mapping S such that for each $\ell \in L$ and $F \in \mathcal{F}$

$$S(\ell) \subset \Phi(\ell),\ S(S^{-1}(F)) \subset F \text{ and } S^{-1}(F) \text{ is Borel.}$$

Let $\mathcal{A}$ be the family of all sets $C \subset K$ with $S(S^{-1}(C)) \subset C$. According to Lemma 2.3, $\sigma(\mathcal{F}) \subset \mathcal{A}$. Set

$$\mathcal{B} := \{F \in \sigma(\mathcal{F}) : S^{-1}(F) \text{ is Borel in } L\}.$$

By Lemma 2.3 the family $\mathcal{B}$ is a σ–algebra and $\mathcal{F} \subset \mathcal{B}$. Thus $\sigma(\mathcal{F}) \subset \mathcal{B}$ and $S^{-1}(B)$ is a Borel subset of L. Since

$$S^{-1}(B) \subset \Phi^{-1}(B) = A \text{ and } S^{-1}(K \setminus B) \subset \Phi^{-1}(K \setminus B) = L \setminus A,$$

we get $S^{-1}(B) = A$. Thus A is a Borel subset of L.

The last assertion remains to be proved. As in the previous paragraph we need to prove that $A \subset L$ is universally measurable if $\varphi^{-1}(A)$ is universally measurable in K. Let ν be a probability measure on L. Since the map $\varphi : \mathcal{M}^1(K) \to \mathcal{M}^1(L)$ is onto, we can find a measure $\mu \in \mathcal{M}^1(K)$ such that $\nu = \varphi\mu$. Since $B := \varphi^{-1}(A)$ is μ–measurable, we can write B as a disjoint union of an $\mathcal{F}_\sigma$–set H and a μ–zero set N. If we enlarge H by $\varphi^{-1}(\varphi(H))$, we may suppose that

$$N = \varphi^{-1}(\varphi(N)) \text{ and } H = \varphi^{-1}(\varphi(H)). \tag{1}$$

Find an $\mathcal{F}_\sigma$–set $F \subset K$ disjoint from N such that $\mu(F) = 1$. Thanks to (1), $\varphi(N) \cap \varphi(F) = \emptyset$. Then

$$\nu(\varphi(F)) = \varphi\mu(\varphi(F)) = \mu(\varphi^{-1}(\varphi(F))) \geq \mu(F) = 1.$$

Thus $\nu(\varphi(N)) = 0$ and A is a disjoint union of an $\mathcal{F}_\sigma$–set $\varphi(H)$ and ν–zero set $\varphi(N)$. Hence A is ν–measurable and the proof is finished. $\square$

3 Application to Affine Functions

In this section we derive some results which are necessary for the proof of Theorem 4.3. At the end we obtain a generalization of the result of J. Saint–Raymond ([17, Corollaire 8]). Here we recall basic definitions and facts needed throughout the section.

A subset L of a vector space is *convex* if L contains the segment joining every pair of points of L. A point x in L is termed *extreme* if x does not lie inside of any nondegenerate segment contained in L. We write $\operatorname{ext} L$ for the set of all extreme points of L. A real–valued function f on a convex set L is called *affine* if $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for every $x, y \in L$ and $\lambda \in [0,1]$. If L is a compact convex subset of a locally convex space, we denote by $A^c(L)$ the space of all continuous affine functions on L. The space of all affine functions on L will be denoted by $A(L)$. According to the previous notation we write $A_b(L)$ for the space of all bounded affine functions on L.

Let L be a compact convex subset of a locally convex space. Let us mention that we write $\mathcal{M}^1(L)$ for the set of all probability Radon measures on L. We recall that $\mathcal{M}^1(L)$ is a convex compact subset of the space of all signed Radon measures on L endowed with the w^*–topology.

A measure $\mu \in \mathcal{M}^1(L)$ is said to *represent* a point $x \in L$, or x is the *barycenter* of μ, if $\mu(h) = h(x)$ for any continuous affine function h on L.

It is well–known that every probability measure has its barycenter. Since affine continuous functions on L separate points of L, the barycenter is uniquely determined. We denote the barycenter of μ by $r(\mu)$.

A bounded function f on L is said to satisfy the *barycentric formula* if f is universally measurable and $\mu(f) = f(r(\mu))$ for any $\mu \in \mathcal{M}^1(L)$.

We write $\mathrm{Bar}(L)$ for the space of all (bounded) functions on L satisfying the barycentric formula.

Remark 1. The definition of functions satisfying the barycentric formula is standard, see e.g. [9, Section 4].

For any $x \in L$ there exists a probability measure μ representing x such that the support $\mathrm{spt}(\mu)$ is contained in $\overline{\mathrm{ext}\, L}$. Moreover, the map which assigns to each $\mu \in \mathcal{M}^1(\overline{\mathrm{ext}\, L})$ its barycenter $r(\mu)$ is a continuous map onto L. It is the main content of the well–known Integral representation theorem (see e.g. [4, Theorem 5.3]).

The following proposition is a particular case of Theorem 4.3. It tells us that any bounded universally measurable function on a compact space L may be regarded as a function on $\mathcal{M}^1(L)$ which satisfies the barycentric formula.

Proposition 3.1. *Let f be a bounded universally mesurable function on a compact space L. Then the function $\mathrm{I}\, f : \mathcal{M}^1(L) \to \mathbb{R}$,*

$$\mathrm{I}\, f(\mu) := \mu(f),\ \ \mu \in \mathcal{M}^1(L)$$

is universally measurable on $\mathcal{M}^1(L)$ and satisfies the barycentric formula.

Moreover, if f is Borel (respectively of the Baire class α, lower semicontinuous, in $\mathcal{C}^{\uparrow}(L)$), then $\mathrm{I}\, f$ is Borel (respectively of the Baire class α, lower semicontinuous, in $\mathcal{C}^{\uparrow}(\mathcal{M}^1(L))$).

Proof. First we will show that the assertion is true for bounded Borel functions on L. Clearly, it is enough to check it for the characteristic functions of Borel subsets of L. Set

$$\mathcal{A} := \{A \subset L : \mathrm{I}\, \chi_A \text{ is Borel on } \mathcal{M}^1(L)\}.$$

Clearly $L \in \mathcal{A}$. If $A, B \in \mathcal{A}$ and $A \subset B$, then $B \setminus A \in \mathcal{A}$. The Levi monotone convergence theorem implies that a countable union of an increasing sequence of sets from $\mathcal{A}$ also belongs to $\mathcal{A}$. Since $\mathcal{A}$ contains all open subsets of L, Dynkin's lemma (see [3, Lemma 8.10]) yields that $\mathcal{A}$ contains any Borel subset of L.

Let f be a bounded Borel function on L and Λ be a probability measure on $\mathcal{M}^1(L)$ with the barycenter s. Define a probability measure λ on L by the formula

$$\lambda(g) := \int_{\mathcal{M}^1(L)} \mu(g)\, d\Lambda(\mu) = \Lambda(\mathrm{I}\, g), \quad g \in \mathcal{C}(L). \tag{2}$$

Since, for $g \in \mathcal{C}(L)$, the function $\mathrm{I}\, g$ is affine and continuous, definition (2) is meaningful and $\lambda(g) = \Lambda(\mathrm{I}\, g) = \mathrm{I}\, g(s) = s(g)$. Thus $s = \lambda$. The standard technique from the proof of Theorem 12.46 in [11] establishes the validity of the formula (2) for any bounded Borel function on L. Hence for $f \in \mathrm{Borel}_b(L)$ we get

$$\begin{aligned}\Lambda(\mathrm{I}\, f) &= \int_{\mathcal{M}^1(L)} \mathrm{I}\, f(\mu)\, d\Lambda(\mu) = \int_{\mathcal{M}^1(L)} \mu(f)\, d\Lambda(\mu) \\ &= \lambda(f) = s(f) = \mathrm{I}\, f(s),\end{aligned} \tag{3}$$

which proves that $\mathrm{I}\, f$ satisfies the barycentric formula on $\mathcal{M}^1(L)$.

To extend the validity of formula (3) for bounded universally measurable functions on L, it is enough to verify it for the characteristic functions of universally measurable subsets of L. Let $A \subset L$ be universally measurable and Λ be a probability measure on $\mathcal{M}^1(L)$ with the barycenter λ. We will show that $\Lambda(\mathrm{I}\, \chi_A) = \mathrm{I}\, \chi_A(\lambda) = \lambda(A)$. Since A is λ–measurable, we can write A as a disjoint union of an $\mathcal{F}_\sigma$–set H and a set N with $\lambda(N) = 0$. Then for $\mu \in \mathcal{M}^1(L)$ we have $\mathrm{I}\, \chi_A(\mu) = \mu(H) + \mu(N)$. Since we have already verified (3) for bounded Borel functions, to finish the reasoning we need to prove that the function $\mathrm{I}\, \chi_N$ is equal to zero Λ–almost everywhere. Find a decreasing sequence $\{G_n\}$ of open subsets of L with $N \subset G_n$ and $\lambda(G_n) \to 0$. For a rational number $q \in (0, 1]$ and $n \in \mathbb{N}$, set $N_q := \{\mu \in \mathcal{M}^1(L) : \mathrm{I}\, \chi_N(\mu) \geq q\}$ and $N_q^n := \{\mu \in \mathcal{M}^1(L) : \mathrm{I}\, \chi_{G_n}(\mu) \geq q\}$. Then, for any $n \in \mathbb{N}$ and $q \in (0, 1] \cap \mathbb{Q}$, formula (3) gives

$$\Lambda(N_q^n) = \int_{N_q^n} 1\, d\Lambda(\mu) \leq \frac{1}{q} \int_{\mathcal{M}^1(L)} \mu(G_n)\, d\Lambda(\mu) = \frac{1}{q} \lambda(G_n).$$

Since $N_q \subset N_q^n$ for every $n \in \mathbb{N}$ and $\lambda(G_n)$ tends to zero as n tends to infinity, N_q is Λ–measurable and $\Lambda(N_q) = 0$. Thus

$$\Lambda\{\mu \in \mathcal{M}^1(L) : \mu(N) > 0\} = \bigcup_{q \in \mathbb{Q} \cap (0,1]} \Lambda(N_q) = 0,$$

which is the desired conclusion.

For the proof of the second assertion, note that $\mathrm{I}\, f$ is continuous whenever f is continuous. The conclusion for Baire functions now follows by transfinite induction and the Lebesgue dominated convergence theorem. If f is lower semicontinuous on L, then $f = \sup\{g \in \mathcal{C}(L) : g \leq f\}$ (see [7, Problem 1.7.15.(a)]). Thus $\mathrm{I}\, f$ is lower semicontinuous by virtue of the general version of the Levi monotone convergence theorem for an upper directed family of continuous functions (see [11, Theorem 9.11]). In case f is in $\mathcal{C}^\uparrow(L)$, we can employ the standard Levi monotone convergence theorem. $\square$

Proposition 3.2. *Let K, L be compact convex sets and φ be a continuous affine map of K onto L. Suppose that g is a real–valued bounded function on L such that $f = g \circ \varphi$ satisfies the barycentric formula on K. Then g satisfies the barycentric formula on L.*

Proof. Let g be a function on L as in the statement. Thanks to Theorem 2.1, the function g is universally measurable on L. It remains to prove the validity of the barycentric formula for g.

To this end, let ν be a probability measure on L. Find a probability measure $\mu \in \mathcal{M}^1(K)$ with $\varphi\mu = \nu$. Then, for every bounded universally measurable function h on L, we have

$$\nu(h) = \mu(h \circ \varphi). \tag{4}$$

Moreover, $\varphi(r(\mu)) = r(\nu)$. Indeed, choose a continuous affine function h on L. Then $h \circ \varphi$ is a continuous affine function on K. Since $A^c(L)$ separates points of L, the equality

$$h(r(\nu)) = \nu(h) = \mu(h \circ \varphi) = (h \circ \varphi)(r(\mu)) = h(\varphi(r(\mu)))$$

finishes the reasoning.

Thanks to the assumption, the function $f = g \circ \varphi$ satisfies the barycentric formula on K. Then the equality (4) gives

$$\nu(g) = \varphi\mu(g) = \mu(g \circ \varphi) = \mu(f) = f(r(\mu)) = g(\varphi(r(\mu))) = g(r(\nu)),$$

which concludes the proof. □

From the preceding propositions we obtain the following corollary. It turns out that we can check the validity of the barycentric formula for a bounded function f on a compact convex set L only by measures supported by the closure of extreme points of L. Moreover, descriptive properties of this function are determined by its behavior on the closure of extreme points of L. The following theorem is a more general version of the result of J. Saint–Raymond (see [17, Corollaire 8]).

Theorem 3.3. *Let L be a compact convex set in a locally convex space and f be a bounded function on L such that, for any $\mu \in \mathcal{M}^1(\overline{\operatorname{ext} L})$, f is μ–measurable and $\mu(f) = f(r(\mu))$. Then f satisfies the barycentric formula on L.*

Moreover, such a function f is Borel (respectively Baire–alpha, lower semicontinuous, in $\mathcal{C}^{\uparrow}(L)$) on L, whenever f is Borel (respectively Baire–alpha, lower semicontinuous, in $\mathcal{C}^{\uparrow}(\overline{\operatorname{ext} L})$) on $\overline{\operatorname{ext} L}$.

PROOF. Set $K := \mathcal{M}^1(\overline{\operatorname{ext} L})$. Let us denote by r the barycentric map from K onto L. Then r is an affine continuous map. The function

$$F : \mu \mapsto \mu(f \restriction \overline{\operatorname{ext} L}),\ \ \mu \in K$$

satisfies the barycentric formula on K due to Proposition 3.1. Pick $\mu \in K$. Thanks to the assumption we have

$$F(\mu) = \mu(f \restriction \overline{\operatorname{ext} L}) = \mu(f) = f(r(\mu)).$$

Thus $F = f \circ r$. Thanks to Theorem 2.1, f is universally measurable on L and the application of Proposition 3.2 yields validity of the barycentric formula for f.

The second part of the statement follows by Proposition 3.1 and Theorem 2.1. □

4 Function Spaces

In this section we study a representation of abstract affine functions. We consider an abstract framework of function spaces. Let K be a compact topological space and $\mathcal{H}$ be a linear subspace of $\mathcal{C}(K)$. We say that $\mathcal{H}$ is a *function space*, if $\mathcal{H}$ contains the constant functions and separates points of K. For $x \in K$, we define the set $\mathcal{M}_x(\mathcal{H})$ of all *$\mathcal{H}$–representing measures* for x by

$$\mathcal{M}_x(\mathcal{H}) = \{\mu \in \mathcal{M}^1(K) : \mu(h) = h(x) \text{ for any } h \in \mathcal{H}\}.$$

Since the Dirac measure ε_x is contained in $\mathcal{M}_x(\mathcal{H})$, this set is nonempty for every $x \in K$. The set of those points x of K, for which ε_x is the only representing measure, is called the *Choquet boundary* of $\mathcal{H}$. We denote it by $\operatorname{Ch}_{\mathcal{H}} K$. A bounded universally measurable function f on K is called *$\mathcal{H}$–affine* if $\mu(f) = f(x)$ for every $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Now we exhibit the most important examples of function spaces.

Continuous functions: Let K be a compact space. Set $\mathcal{H} = \mathcal{C}(K)$. Then the Choquet boundary of $\mathcal{H}$ is equal to K and any function on K is $\mathcal{H}$–affine since the only representing measures are Dirac measures.

Affine functions: Let K be a convex compact subset of a locally convex space and $\mathcal{H}$ be the linear space $A^c(K)$ of all continuous affine functions on K. According to Bauer's characterization of extreme points, see [14, Proposition 1.4], the Choquet boundary $\operatorname{Ch}_{\mathcal{H}} K$ coincides with the set $\operatorname{ext} K$ of all extreme points of K.

Harmonic functions: Let U be a bounded open subset of a Euclidean space $\mathbb{R}^n$ and $\mathcal{H}$ be the linear space $H(U)$ of all continuous functions on $\overline{U}$ which are harmonic on U. Then the Choquet boundary of $H(U)$ coincides with $\partial_{reg}U$, the set of all regular points of U.

Now we introduce a well–known concept of the state space. This notion represents a natural and efficient link between function spaces and convex analysis. Proofs of the properties of the state space and related notions mentioned in the next paragraph are classical and can be found e.g. in [1, Theorem II.2.1], [4, Chapter 1, § 4], [6, Theorem 29.5] or [14, Chapter 6].

Let $\mathcal{H}$ be a function space on a compact space K. We denote by $\mathbf{S}(\mathcal{H})$ the *state space* of $\mathcal{H}$ defined as

$$\mathbf{S}(\mathcal{H}) := \{\varphi \in \mathcal{H}^* : \varphi \geq 0,\ \varphi(1) = 1\} \ .$$

Let $\phi : K \to \mathbf{S}(\mathcal{H})$ be the evaluation mapping defined as $\phi(x) = s_x$, $x \in K$ where $s_x(h) = h(x)$ for $h \in \mathcal{H}$. Further, let $\Phi : \mathcal{H} \to A^c(\mathbf{S}(\mathcal{H}))$ be the mapping defined for $h \in \mathcal{H}$ by $\Phi(h)(s) := s(h)$, $s \in \mathbf{S}(\mathcal{H})$.

The state space $\mathbf{S}(\mathcal{H})$ is a convex compact subset of the dual space $\mathcal{H}^*$ endowed with the w^*–topology. The space $\mathcal{H}^*$ can be identified with the quotient space $(\mathcal{M}(K), w^*)/\mathcal{H}^\perp$ equipped with the quotient (locally convex) topology.

We write π for the quotient mapping from $\mathcal{M}(K)$ onto $\mathcal{H}^*$. A straightforward application of the Hahn-Banach theorem yields

$$\mathbf{S}(\mathcal{H}) = \pi(\mathcal{M}^1(K)). \tag{5}$$

It can be easily verified that $\phi(x) = \pi(\varepsilon_x)$. Moreover, $\phi(\operatorname{Ch}_{\mathcal{H}} K) = \operatorname{ext} \mathbf{S}(\mathcal{H})$. The mapping Φ serves as an isometric isomorphism of $\mathcal{H}$ into the space $A^c(\mathbf{S}(\mathcal{H}))$, and Φ is onto if and only if the function space $\mathcal{H}$ is (uniformly) closed in $\mathcal{C}(K)$. In this case the inverse mapping is realized by

$$\Phi^{-1}(F) = F \circ \phi, \quad F \in A^c(\mathbf{S}(\mathcal{H})). \tag{6}$$

We call a bounded universally measurable function f on K *completely $\mathcal{H}$-affine* if $\mu(f) = 0$ for each $\mu \in \mathcal{H}^\perp$. The space of all completely $\mathcal{H}$-affine functions on K will be denoted by $\mathbf{A}(\mathcal{H})$.

Remark 2. Completely $\mathcal{H}$–affine functions are termed *"fonctions qui vérifient la calcul barycentrique modulo $\mathcal{H}$"* by M. Rogalski in [15, Définition 40]. These functions were also considered by E. Alfsen and M. Hirsberg (see [2, Definition 2.1]).

A continuous function h is completely $\mathcal{H}$-affine if and only if $h \in \overline{\mathcal{H}}$. This is an easy consequence of the Hahn-Banach theorem. Clearly, every completely

$\mathcal{H}$–affine function is $\mathcal{H}$–affine. In [12, Example 5.1] we present an example of an $\mathcal{H}$–affine function which is not completely $\mathcal{H}$–affine.

The mapping Φ provides a representation of functions from $\mathcal{H}$ as continuous affine functions on $\mathbf{S}(\mathcal{H})$. The following definition is an extension of the mapping Φ.

Let $\mathcal{H}$ be a function space on a compact space K. We define a map T from $\mathbf{A}(\mathcal{H})$ into $A_b(\mathbf{S}(\mathcal{H}))$ by the formula

$$\mathrm{T}\, f(s) := \mu(f),\ f \in \mathbf{A}(\mathcal{H}),\ \mu \in \pi^{-1}(s),\ s \in \mathbf{S}(\mathcal{H}).$$

Note that the definition of T is correct since for a completely $\mathcal{H}$–affine function f and any two measures $\mu_1, \mu_2 \in \pi^{-1}(s)$ holds $\mu_1(f) = \mu_2(f)$.

In order to investigate properties of the mapping T we need some preliminary results. The first one is a well–known assertion on uniform density of functionals in the space of affine continuous functions on a compact convex set.

Lemma 4.1. *Let $\mathcal{H}$ be a function space on a compact space K. Then the space $\Phi(\mathcal{H})$ is uniformly dense in $A^c(\mathbf{S}(\mathcal{H}))$.*

PROOF. See e.g. [1, Corollary I.1.5], [4, Corollary 4.8] or [14, Proposition 4.5]. □

Lemma 4.2. *Let $\mathcal{H}$ be a function space on a compact space K and μ be a probability measure on K. Then $r(\phi\mu) = \pi(\mu)$.*

PROOF. Pick $\mu \in \mathcal{M}^1(K)$ and $h \in \mathcal{H}$. Then

$$\begin{aligned}\Phi(h)\big(r(\phi\mu)\big) &= \int_{\mathbf{S}(\mathcal{H})} \Phi(h)(s)\, d(\phi\mu)(s) = \int_K \big(\Phi(h) \circ \phi\big)(x)\, d\mu(x) \\ &= \int_K h(x)\, d\mu(x) = \pi(\mu)(h) = \Phi(h)\big(\pi(\mu)\big).\end{aligned} \tag{7}$$

Since $A^c(\mathbf{S}(\mathcal{H}))$ separates points of $\mathbf{S}(\mathcal{H})$ and $\Phi(\mathcal{H})$ is its dense subspace (see Lemma 4.1), the equality (7) implies $r(\phi\mu) = \pi(\mu)$. □

Theorem 4.3. *Let $\mathcal{H}$ be a function space on a compact space K. Then T is an isometric isomorphism between $\mathbf{A}(\mathcal{H})$ and $\mathrm{Bar}(\mathbf{S}(\mathcal{H}))$ such that $\mathrm{T} = \Phi$ on $\mathcal{H}$. Its inverse is given by*

$$\mathrm{T}^{-1} = F \circ \phi,\ F \in \mathrm{Bar}(\mathbf{S}(\mathcal{H})). \tag{8}$$

This isomorphism T maps the space of completely $\mathcal{H}$–affine Borel (respectively Baire–alpha, lower semicontinuous or $\mathcal{C}^{\uparrow}(K)$) functions onto the space of all bounded Borel (respectively Baire–alpha, lower semicontinuous or $\mathcal{C}^{\uparrow}(\mathbf{S}(\mathcal{H}))$) functions on $\mathbf{S}(\mathcal{H})$ satisfying the barycentric formula.

PROOF. Linearity of T is obvious. Let f be a completely $\mathcal{H}$–affine function on K. According to Lemma 3.1, the function $\mathrm{I}\, f : \mathcal{M}^1(K) \to \mathbb{R}$ defined as $\mathrm{I}\, f(\mu) := \mu(f)$, $\mu \in \mathcal{M}^1(K)$ inherits descriptive properties from the function f and satisfies the barycentric formula on $\mathcal{M}^1(K)$. Notice that for $\mu \in \mathcal{M}^1(K)$ we have $\mathrm{I}\, f(\mu) = \mathrm{T}\, f(\pi(\mu))$. Thus $\mathrm{T}\, f$ is contained in $\mathrm{Bar}(\mathbf{S}(\mathcal{H}))$ thanks to Proposition 3.2 (put $K := \mathcal{M}^1(K)$, $L := \mathbf{S}(\mathcal{H})$, $\varphi := \pi$, $g := \mathrm{T}(f)$ and $f := \mathrm{I}\, f$).

Let $F \in \mathrm{Bar}(\mathbf{S}(\mathcal{H}))$ be given. Set $f(x) := F(\phi(x)), x \in K$. We want to show that $f \in \mathbf{A}(\mathcal{H})$ and $\mathrm{T}\, f = F$. It is easy to see that f is universally measurable. Pick $\mu_1, \mu_2 \in \mathcal{M}^1(K)$ such that $\mu_1 - \mu_2 \in \mathcal{H}^\perp$. Then $\pi(\mu_1) = \pi(\mu_2)$. According to Lemma 4.2, $r(\phi\mu_1) = r(\phi\mu_2)$. Hence we obtain

$$\begin{aligned}\mu_1(f) &= \mu_1(F \circ \phi) = \phi\mu_1(F) = F\big(r(\phi\mu_1)\big) \\ &= F\big(r(\phi\mu_2)\big) = \phi\mu_2(F) = \mu_2(F \circ \phi) = \mu_2(f)\end{aligned}$$

and f is completely $\mathcal{H}$–affine. Further, for $\mu \in \mathcal{M}^1(K)$ we get

$$\mathrm{T}\, f(\pi(\mu)) = \mu(f) = \mu(F \circ \phi) = \phi\mu(F) = F\big(r(\phi\mu)\big) = F(\pi(\mu)),$$

which proves the formula (8). For $h \in \mathcal{H}$, the equality

$$\mathrm{T}\, h(\pi(\mu)) = \mu(h) = \pi(\mu)(h) = \Phi(h)\big(\pi(\mu)\big)$$

gives $\mathrm{T} = \Phi$ on $\mathcal{H}$.

To check that T is an isometry, it is enough to verify

$$\begin{aligned}\|\mathrm{T}\, f\| &= \sup_{s \in \mathbf{S}(\mathcal{H})} |\mathrm{T}\, f(s)| = \sup_{\mu \in \mathcal{M}^1(K)} |\mathrm{T}\, f(\pi(\mu))| = \sup_{\mu \in \mathcal{M}^1(K)} |\mu(f)| \\ &= \sup_{x \in K} |f(x)| = \|f\|.\end{aligned}$$

Since for $\mu \in \mathcal{M}^1(K)$ we have $\mathrm{T}\, f(\pi(\mu)) = \mathrm{I}\, f(\mu)$, preservation of the descriptive properties of f follows by Theorem 2.1 and Proposition 3.1. Surjectivity of T on coresponding classes of functions easily follows from the inverse formula (8) and the fact that ϕ is a homeomorphism. This observation finishes the proof. $\square$

Since we have obtained a representation of lower semicontinuous (respectively Baire–one) completely $\mathcal{H}$–affine functions as lower semicontinuous (respectively Baire–one) affine functions on $\mathbf{S}(\mathcal{H})$, we can employ known results on approximation of affine functions on a compact convex set to obtain analogous assertions for completely $\mathcal{H}$–affine functions. The next theorem collects the results for affine functions on a compact convex set.

Theorem 4.4. *Let f be an affine function on a compact convex set K.*

i) If f is a Baire–one function, then there exists a bounded sequence $\{h_n\}$ of continuous affine functions on K such that $f = \lim_{n\to\infty} h_n$.

ii) If f is lower semicontinuous function, then f is bounded and

$$f = \sup\{h : h < f,\ h \in A^c(K)\}$$

where the set $\{h : h < f,\ h \in A^c(K)\}$ is upward directed.

iii) If $f \in \mathcal{C}^\uparrow(K)$, then there exists a bounded increasing sequence $\{h_n\}$ of affine continuous functions on K such that $f = \lim_{n\to\infty} h_n$.

PROOF. The proof of the first assertion depends on Choquet's theorem on validity of the barycenytric formula for affine Baire–one functions. G. Mokobodzki in [13] used it for the proof of i). This result can be also found in [15, Théorème 80]. Assertions ii) and iii) are well–known. Their proofs only need the Hahn–Banach separation theorem, see e.g. [1, Corollary I.1.4]. □

For function spaces we cannot use the previous methods, namely it is impossible to deal with the Hahn–Banach separation theorem. However, Theorem 4.3 allows us to avoid this difficulty. Precise statements are listed below.

Theorem 4.5. *Let $\mathcal{H}$ be a function space on a compact space K and f be a completely $\mathcal{H}$–affine function on K.*

i) If f is of the first Baire class, then there exists a bounded sequence $\{h_n\}$ of functions from $\mathcal{H}$ such that $f = \lim_{n\to\infty} h_n$.

ii) If f is lower semicontinuous, then

$$f = \sup\{h : h < f,\ h \in \mathcal{H}\}$$

where the set $\{h : h < f,\ h \in \mathcal{H}\}$ is upward directed.

iii) If $f \in \mathcal{C}^\uparrow(K)$, then there exists a bounded increasing sequence $\{h_n\}$ of functions from $\mathcal{H}$ such that $f = \lim_{n\to\infty} h_n$.

PROOF. The first assertion can be found in [12, Theorem 5.1]. We prove it for the sake of completness.

i) Let f be a Baire–one completely $\mathcal{H}$–affine function on K. According to Theorem 4.3, $\mathrm{T} f$ is an affine Baire–one function on $\mathbf{S}(\mathcal{H})$. An appeal to Theorem 4.4 i) yields the existence of a bounded sequence $\{F_n\}$ of affine continuous functions on $\mathbf{S}(\mathcal{H})$ such that $F_n \to \mathrm{T}\, f$. Since $\Phi(\mathcal{H})$ is dense in $A^c(\mathbf{S}(\mathcal{H}))$ (see Lemma 4.1), we may find functions $F_n \in \Phi(\mathcal{H})$ converging pointwise to $\mathrm{T}\, f$. Set $h_n := \mathrm{T}^{-1}(F_n) = F_n \circ \phi$. Then $\{h_n\}$ is bounded sequence of functions from $\mathcal{H}$ and for $x \in K$,

$$f(x) = \mathrm{T}\, f(\pi(\varepsilon_x)) = \mathrm{T}\, f(\phi(x)) = \lim_{n\to\infty} \Phi(h_n)(\phi(x)) = \lim_{n\to\infty} \phi(x)(h_n)$$
$$= \lim_{n\to\infty} h_n(x).$$

ii) Let f be a lower semicontinuous completely $\mathcal{H}$–affine function on K. Due to Theorem 4.4 ii), $\mathrm{T}\, f = \sup\{H : H \in A^c(\mathbf{S}(\mathcal{H})), H < \mathrm{T}\, f\}$. Using density of $\Phi(\mathcal{H})$ in $A^c(\mathbf{S}(H))$, we obtain that $\mathrm{T}\, f = \sup\{\Phi(h) : h \in \mathcal{H}, \Phi(h) < \mathrm{T}\, f\}$. Thus $f = \sup\{h : h \in \mathcal{H}, h < f\}$. It remains to prove that the set $A := \{h : h \in \mathcal{H}, h < f\}$ is upward directed. Let $h_1, h_2 \in A$ be given. It follows from the definition of T that $\mathrm{T}\, h_i < \mathrm{T}\, f,\ i = 1, 2$ on $\mathbf{S}(\mathcal{H})$. Theorem 4.4 ii) ensures the existence of an affine continuous function H on $\mathbf{S}(\mathcal{H})$ such that $\sup(\mathrm{T}\, h_1, \mathrm{T}\, h_2) \le H < \mathrm{T}\, f$. By adding an appropriate constant we may achieve that $\sup(\mathrm{T}\, h_1, \mathrm{T}\, h_2) < H < \mathrm{T}\, f$. Moreover, using density of $\Phi(\mathcal{H})$ in $A^c(\mathbf{S}(\mathcal{H}))$ and lower semicontinuity of $\mathrm{T}\, f$, we may suppose that $H = \Phi(h)$ for some $h \in \mathcal{H}$. Then $\sup(h_1, h_2) < h < f$ and we are done.

iii) The proof of the last assertion is similar. For $f \in \mathbf{A}(\mathcal{H}) \cap \mathcal{C}^{\uparrow}(K)$, the function $\mathrm{T}\, f$ is in $\mathcal{C}^{\uparrow}(\mathbf{S}(\mathcal{H}))$. Theorem 4.4 iii) yields the existence of a bounded sequence $\{F_n\}$ of affine continuous functions on $\mathbf{S}(\mathcal{H})$ such that $\{F_n\}$ converge pointwise to $\mathrm{T}\, f$ and $F_n < F_{n+1}$ for every $n \in \mathbb{N}$. Again, by an application of a density argument we may find functions $H_n \in \Phi(\mathcal{H})$ with $F_n < H_n < F_{n+1}$ for every $n \in \mathbb{N}$. By setting $h_n := \mathrm{T}^{-1}(H_n) = H_n \circ \phi$ we finish the proof. $\square$

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