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A TYPE OF PATH DERIVATIVE

Abstract

In this paper a type of path derivative, which is not based on the non-empty path intersection, is introduced. Such derivatives share some basic properties with sequencial derivatives but there exist some sharp contrasts between them. The path derivative here restricts the speed of convergence in the definition of limits naturally and has a sharp contrast with the classical Dini derivatives for the typical continuous function.

Path differentiation was introduced by Bruckner, O'Malley and Thomson in [2], where many properties of functions and their derivatives were found under the condition of nonempty path intersection. The path derivative is a unified approach to certain generalized derivatives. In [1] (pp. 115), Bruckner introduced a type of generalized derivative by requiring only that there exists a sequence $\{h_n\} \to 0$ ($h_n \neq 0$) such that

$$\lim_{n\to 0} \frac{F(x+h_n)-F(x)}{h_n} = f(x)$$

for every $x \in \mathbb{R}$. Following this he asked, given f and $\{h_n\}$, under what circumstances does there exist a function F such that the above equality holds, and also variants of this problem by letting $\{h_n\}$ depend on f or even on the point x, or by requiring F to meet some extra condition such as being continuous. Laczkovich and Petruska in [3] introduced the sequential derivative and discussed the first problem deeply. Here we would like to discuss a variant of the first problem and therefore introduce a type of path derivative which is not based on the condition of nonempty intersection. The derivative we will define is different from the sequential derivative though in general they have some similar properties. The resulting derived numbers, which we will define, of a typical continuous function have a property contrasting sharply with the

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property of classical derived numbers shown by V. Jarník ([1]) that a typical continuous function in the space $C[0,1]$ of continuous functions with the sup norm has every extended real number as a derived number at every point of $[0,1]$. Now let us first give the definition of the restrictive derived numbers under our conditions and their properties.

Definition 1. Let $f:[0,1]\to\mathbb{R}$, $\{h_n\}$ and $\{k_n\}$ be two sequences which are strictly decreasing to 0. A number α is called an (h_n,k_n)-parasequential derived number of the function f at a point $x\in[0,1]$ if there is a sequence $\{a_n\}$ contained in $[0,1]$ such that there is a positive integer N so that $a_n\in[x-h_n,x-h_{n+1}]$ or $a_n\in[x+k_{n+1},x+k_n]$ for all $n>N$, and

$$\frac{f(a_n)-f(x)}{a_n-x}\to\alpha.$$

(For the endpoints 0 and 1 we just consider one-sided limit.)

Theorem 1. *Given any two sequences (h_n,k_n) which are strictly decreasing to 0, a typical continuous function in $C[0,1]$ has no finite (h_n,k_n)-parasequential derived numbers at any point $x\in[0,1]$.*

PROOF. Let m and q be positive integers. Let

$$A^-(m,q)=\left\{f\in C[0,1]:\begin{array}{c}\text{there exist } x_f\in[0,1] \text{ and}\\ a_f\in[x_f-h_q,x_f-h_{q+1}] \text{ such that}\\ -m\le\frac{f(a_f)-f(x_f)}{a_f-x_f}\le m.\end{array}\right\}.$$

and

$$A^+(m,q)=\left\{f\in C[0,1]:\begin{array}{c}\text{there exist } x_f\in[0,1] \text{ and}\\ a_f\in[x_f+k_{n+1},x_f+k_n] \text{ such that}\\ -m\le\frac{f(a_f)-f(x_f)}{a_f-x_f}\le m.\end{array}\right\}.$$

Let

$$A=\left\{f\in C[0,1]:\begin{array}{c}\text{there exists a } x_f\in[0,1] \text{ such that } f \text{ has a finite}\\ (h_n,k_n)\text{-parasequential derived number at } x_f.\end{array}\right\}.$$

Then

$$A\subseteq\bigcup_{m=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcap_{q=l}^{\infty}\Big[A^+(m,q)\bigcup A^-(m,q)\Big].$$

In fact, for any $f\in A$, there exist a point x_f and a sequence $\{a_n\}$ contained in $[0,1]$ such that there is a positive integer N so that $a_n\in[x_f-h_n,x_f-h_{n+1}]$

or $a_n \in [x_f + k_{n+1}, x_f + k_n]$ for all $n \geq N$ and $\frac{f(a_n)-f(x_f)}{a_n - x_f}$ approaches a finite number. Thus there is an integer m such that

$$-m \leq \frac{f(a_n) - f(x_f)}{a_n - x_f} \leq m$$

for all sufficiently large $n \geq N$ and $a_n \in [x_f - h_n, x_f - h_{n+1}]$ or $a_n \in [x_f + k_{n+1}, x_f + k_n]$. Therefore

$$f \in \bigcap_{q=p}^{\infty} \Big[A^+(m,q) \bigcup A^-(m,q) \Big]$$

where p is some number bigger than N. So

$$f \in \bigcup_{m=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{q=l}^{\infty} \Big[A^+(m,q) \bigcup A^-(m,q) \Big].$$

Now we show that $A^+(m,q)$ is closed for each m and q. For any Cauchy sequence $\{f_i\} \subseteq A^+(m,q)$, $f_i \to f \in C[0,1]$. There exist x_{f_i} and $a_{f_i} \in [x_{f_i} + k_{q+1}, x_{f_i} + k_q]$ such that

$$-m \leq \frac{f_i(a_{f_i}) - f_i(x_{f_i})}{a_{f_i} - x_{f_i}} \leq m.$$

By the Borel-Bolzano Theroem there exist $x_0,\ a_f \in [0,1]$ and subsequences $\{x_{f_{i_j}}\}$, $\{a_{f_{i_j}}\}$ such that $x_{f_{i_j}} \to x_0$ and $a_{f_{i_j}} \to a_f$ as $j \to \infty$. Since $a_{f_{i_j}} \in [x_{f_{i_j}} k_{q+1}, x_{f_{i_j}} + k_q]$ and $b_{f_{i_j}} \in [x_{f_{i_j}} + k_{q+1}, x_{f_{i_j}} + k_q]$, we have $a_f \in [x_0 + k_{q+1}, x_0 + k_q]$. Notice that

$$-m \leq \frac{f_{i_j}(a_{f_{i_j}}) - f_{i_j}(x_{f_{i_j}})}{a_{f_{i_j}} - x_{f_{i_j}}} \leq m.$$

Letting $j \to \infty$, we have

$$-m \leq \frac{f(a_f) - f(x_0)}{a_f - x_0} \leq m.$$

That is, $f \in A^+(m,q)$ and so $A^+(m,q)$ is closed. Using a similar method we can show that $A^-(m,q)$ is also closed. Hence $\bigcap_{q=l}^{\infty} \Big[A^+(m,q) \bigcup A^-(m,q) \Big]$ is closed for any positive integer l.

Now we show that the set $\bigcap_{q=l}^{\infty} \Big[A^+(m,q) \bigcup A^-(m,q) \Big]$ is nowhere dense in $C[0,1]$ for each positive integer l. For any ball $B(f,\epsilon) \subseteq C[0,1]$, we can choose a saw-like function $g \in B(f, \epsilon)$ so that all the slopes of segments of g are bigger than m or less than $-m$. Then, at each point $x \in [0,1]$, for any sequence $\{a_n\}$ contained in $[0,1]$ with the condition $a_n \in [x - h_n, x - h_{n+1}]$ or $a_n \in [x + k_{n+1}, x + k_n]$, we have $\left|\dfrac{g(a_n) - g(x)}{a_n - x}\right| > m$ for sufficiently large n. Thus $g \notin \bigcap_{q=l}^{\infty} \left[A^+(m,q) \bigcup A^-(m,q)\right]$ for any positive integer l. So $\bigcap_{q=l}^{\infty} A^+(m,q) \bigcup A^-(m,q)$ is closed and nowhere dense in $C[0,1]$ for any positive integer l. Hence the set A is of the first category. $\square$

Now we are ready to give the definition of our path derivative. In the following, unless otherwise specified, $\{h_n\}$ and $\{k_n\}$ denote any two fixed sequences which are strictly decreasing to 0.

Definition 2. A continuous function $f : [0,1] \to \mathbb{R}$ is said to have an (h_n, k_n)-parasequential derivative g on $[0,1]$ if and only for every $x \in [0,1]$ there exist sequences $\{a_n\}$ and $\{b_n\}$ contained in $[0,1]$ such that there is an integer N so that $a_n \in [x - h_n, x - h_{n+1}]$ and $b_n \in [x + k_{n+1}, x + k_n]$ for all $n > N$ and

$$\lim_{n\to\infty} \frac{f(a_n) - f(x)}{a_n - x} = \lim_{n\to\infty} \frac{f(b_n) - f(x)}{b_n - x} = g(x).$$

For the endpoints 0 and 1, only one-sided limits are required.

Theorem 2. *Let g be an (h_n, k_n)-parasequential derivative of a continuous function f on $[0,1]$. The the following properties hold.*

(i) If $g(x) \geq 0$, f is increasing on $[0,1]$.

(ii) If $g(x) = 0$, f is constant on $[0,1]$.

(iii) g possesses Darboux property.

PROOF. If $g(x) \geq 0$, the upper Dini derivate $\overline{D}f(x) \geq 0$ everywhere; so the function f is increasing (see Theorem 7.2 in [4]). Therefore (i) holds and (ii) is an immediate consequence. We now show (iii). From the definition of the (h_n, k_n)-parasequential derivative $g(x) = 0$ holds at every local extremum of f. So the standard argument applies. $\square$

Theorem 3. *If a continuous function $f : [0,1] \to \mathbb{R}$ has an (h_n, k_n)-parasequential derivative at every point $x \in [0,1]$, then there is a dense open set V on which f is differentiable for almost all x in V.*

PROOF. Let g be an (h_n, k_n)-parasequential derivative of the function f. Then

$$\bigcup_{m=-\infty}^{\infty} \bigcup_{l=1}^{\infty} B_{m,l} = [0,1],$$

where

$$B_{m,l} = \left\{ x \in [0,1] : \begin{array}{l} \text{there exist sequences } \{a_n\}, \{b_n\} \subseteq [0,1] \text{ such that} \\ a_n \in [x - h_n, x - h_{n+1}],\ b_n \in [x + k_{n+1}, x + k_n], \\ \lim_{n\to\infty} \frac{f(a_n)-f(x)}{a_n - x} = \lim_{n\to\infty} \frac{f(b_n)-f(x)}{b_n - x} = g(x) \\ \text{and if } n > l,\ \frac{f(a_n)-f(x)}{a_n-x},\ \frac{f(b_n)-f(x)}{b_n-x} > m. \end{array} \right\}.$$

The end points $0, 1$ could be included in $B_{m,l}$ if the corresponding conditions are satisfied. Therefore,

$$\bigcup_{m=-\infty}^{\infty} \bigcup_{l=1}^{\infty} F_{m,l} = [0,1],$$

where $F_{m,l}$ is the closure of the set $B_{m,l}$. On each set $F_{m,l}$ the upper Dini derivate $\overline{D}(f) \geq m$. In fact, for any $x \in F_{m,l}$, there exists a sequence of points $\{x_i\} \subseteq B_{m,l}$ such that $x_i \to x$. For each x_i there exist two sequences $\{a_{i,n}\}$ and $\{b_{i,n}\}$ contained in $[0,1]$ such that $a_{i,n} \in [x_i - h_n, x_i - h_{n+1}]$, $b_{i,n} \in [x_i + k_{n+1}, x_i + k_n]$, and if $n > l$,

$$\frac{f(a_{i,n}) - f(x_i)}{a_{i,n} - x_i} > m \text{ and } \frac{f(b_{i,n}) - f(x_i)}{b_{i,n} - x_i} > m.$$

For any fixed n, the sequences $\{a_{i,n}\}$ and $\{b_{i,n}\}$ have subsequences that converge to two numbers a_n and b_n respectly. For convenience we still use $\{a_{i,n}\}$ and $\{b_{i,n}\}$ to denote the two subsequences. Then $a_n \in [x - h_n, x - h_{n+1}]$ and $b_n \in [x + k_{n+1}, x + k_n]$. Also, by the continuity of f,

$$\frac{f(a_n) - f(x)}{a_n - x} \geq m \text{ and } \frac{f(b_n) - f(x)}{b_n - x} \geq m.$$

Thus $\overline{D}(f)(x) \geq m$. For any closed interval $[a,b] \subseteq [0,1]$, application of the Baire category theorem to the sequence of sets $F_{m,l} \cap [a,b]$ guarantees the existence of an open interval (c,d) and intergers M, L such that $(c,d) \subseteq F_{M,L}$. Thus for every $x \in (c,d)$, the upper Dini derivate $\overline{D}(f)(x) \geq M$. Therefore the function $f(x) - Mx$ on (c,d) is increasing and f is thus differentiable at almost all $x \in (c,d)$. □

Theorem 4. *Let g be an (h_n, k_n)-parasequential derivative of a continuous function f. If g is bounded on $[0,1]$, then f is a Lipchitz function.*

PROOF. Let M be a constant such that $|g(x)| \leq M$ for all $x \in [0,1]$. Then the upper Dini derivate of the function $f(x) - Mx$ is less than or equal to zero on $[0,1]$ and the lower Dini derivate of the function $f(x) + Mx$ is bigger than or equal to zero. Thus $f(x) - Mx$ is decreasing on $[0,1]$ and $f(x) + Mx$ is increasing on $[0,1]$. Therefore, for any $x, y \in [0,1]$, $|f(x) - f(y)| \leq M|x - y|$. So the function f is a Lipchitz function. □

One might expect that the (h_n, k_n)-parasequential derivative of a continuous function is always a Baire one function. The next example below shows that, in general, the (h_n, k_n)-parasequential derivative of a continuous function may not be Baire one or even Borel measurable. This contrasts sharply with the Baire one property of the sequential derivative (see [3]).

Theorem 5. *There exists a continuous function f with a $(\frac{1}{3^n}, \frac{1}{3^n})$-parasequential derivative that is Borel measurable but not Baire one, and another $(\frac{1}{3^n}, \frac{1}{3^n})$-parasequential derivative that is non-Borel measurable.*

PROOF. We will construct a continuous fuction whose $(\frac{1}{3^n}, \frac{1}{3^n})$-parasequential derivative is not Baire one by using the structure of the Cantor ternary set C.

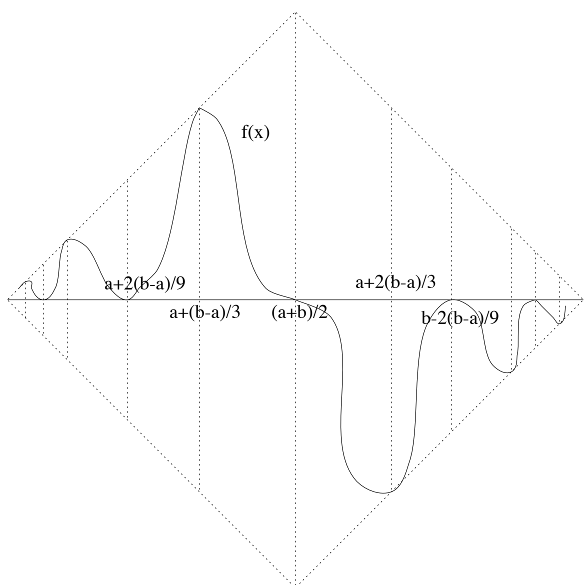

Figure 1: The construction of the function f.

We define a fuction f that takes zero on every point of the Cantor set C. Let (a, b) be a component interval of the complement of the Cantor set C. We define the function f on (a, b) as follows.

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{a+b}{2} \\ \frac{b-a}{3^i} & \text{if } x = a + \frac{b-a}{3^i}, i = 1, 2, \cdots \\ 0 & \text{if } x = a + \frac{2(b-a)}{3^i}, i = 2, 3, \cdots \\ -\frac{b-a}{3^i} & \text{if } x = b - \frac{b-a}{3^i}, i = 1, 2, \cdots \\ 0 & \text{if } x = b - \frac{2(b-a)}{3^i}, i = 2, 3, \cdots \end{cases}$$

and then connect the above points in the graph of the function f to make f differentiable on (a, b) and monotone on any subinterval with the projections of two neighbouring points on the x-axis as its endpoints. A continuous function f on $[0, 1]$ is defined. Its graph is in Figure 1.

We will show that one $(\frac{1}{3^n}, \frac{1}{3^n})$-parasequential derivative of the function f can take any value on any point of the Cantor set. Let x be a point of the Cantor set. For any interval $[x + \frac{1}{3^{n+1}}, x + \frac{1}{3^n}] \subset [0, 1]$ (only $[x - \frac{1}{3^n}, x - \frac{1}{3^{n+1}}]$ to be considered if $x = 1$), from the structure of the Cantor set there is a removed interval $[a, b]$ whose intersection $[c, d]$ with $[x + \frac{1}{3^{n+1}}, x + \frac{1}{3^n}]$ has length $d - c \geq \frac{1}{3^{n+3}}$ and $[c, d]$ contains at least two zero points of the function f. So, for $0 < \alpha < (\frac{1}{4} \cdot \frac{1}{3^{n+3}})/(\frac{1}{3^n}) < \frac{1}{3^4}$, a line throught the point x and with slope α must intersect the graph of the function f on $[c, d]$ in at least one point because of the construction of f. Thus we can have a $(\frac{1}{3^n}, \frac{1}{3^n})$-parasequential derivative g of the function f, which takes values 0 at endpoint of each removed interval from the construction of the Cantor set C and $\frac{1}{3^6}$ at other points of the Cantor set C. So g is Borel measurable but not Baire one. If we require that g take values $\frac{1}{3^7}$ on a non-Borel measurable subset B of the Cantor set and $\frac{1}{3^8}$ on its complement set $C \setminus B$, the function g is non-Borel measurable. $\square$

Remark. From the construction of the function f and the choice of the sequences $\{a_n\}$ and $\{b_n\}$ in the proof we see that the $(\frac{1}{3^n}, \frac{1}{3^n})$-parasequential derivative of the continuous function f is not unique. In general, the (h_n, k_n)-parasequential derivative of a continuous function is unique almost everywhere on a dense open set (see the proof of Theorem 3). Further, if the (h_n, k_n)-parasequential derivative g of a continuous function f is bounded, it is the derivative of the function f almost everywhere (see Theorem 4) and so is unique up to a set of measure zero. We do not know how large is the set of points on which the (h_n, k_n)-parasequential derivative of a continuous function is not unique.

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