INROADS

Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 221–224

Craig Cowan,* Department of Mathematics, Simon Fraser University, Vancouver, BC, Canada. email: ctcowan@sfu.ca

AN ELEMENTARY REMARK ON THE INTERSECTION OF SETS

Abstract

In this paper, we will investigate the following question: Given $C \in (0,1)$ and a sequence $A_n \subseteq [0,1]$ with $\lambda(A_n) = C$, when does there exist a subsequence A_{n_i} such that $\lambda(\cap_i A_{n_i}) > 0$? We will show that the answer to this question can be characterized by the properties of a function g which will be a weak L^1 limit of characteristic functions.

Before we get started, let's mention what notation we will be using. L^p will denote $L^p[0,1]$ with Lebesgue measure λ, and χ_A will denote the indicator function of A. We will use $\rightharpoonup$ to denote weak convergence. All sets will be taken to be Lebesgue measurable.

The main result of this paper is given by the following theorem:

Theorem 1. *Let $C \in (0,1)$ and $A_n \subseteq [0,1]$ with $\lambda(A_n) = C$ for all n. Then the following are equivalent:*
(i) There exists a subsequence of A_n whose intersection has positive measure.
(ii) There exists a subsequence of χ_{A_n} with a weak L^1 limit g and $g = 1$ on a set of positive measure.

Before we prove theorem 1, we will need the following result:

Lemma 1. *Given $B_n \subseteq [0,1]$ with $\lambda(B_n) = C$, there exists B_{n_i} and a measurable function g with $0 \leq g \leq 1$ a.e. such that $\chi_{B_{n_i}} \rightharpoonup g$ in L^1.*

PROOF. By a weak compactness argument in L^2, we see there exists some $g \in L^2$ and $\chi_{B_{n_i}}$ such that $\chi_{B_{n_i}} \rightharpoonup g$ in L^2. Since $L^\infty \subseteq L^2$, we see that $\chi_{B_{n_i}} \rightharpoonup g$ in L^1.

Key Words: Weak L^1 convergence
Mathematical Reviews subject classification: 28A05
Received by the editors January 6, 2006
Communicated by: Paul D. Humke

*Craig Cowan was an undergraduate at Simon Fraser University during the research on this topic.

Define $E := \{x : g(x) < 0\}$ and $F := \{x : g(x) > 1\}$. Then we have

$$0 \leq \int_0^1 \chi_{B_{n_i}} \chi_E \, dx \rightarrow \int_0^1 g\chi_E \, dx,$$

and from this we see $\lambda(E) = 0$. Similarly, we have

$$0 \leq \int_0^1 (1 - \chi_{B_{n_i}})\chi_F \, dx \rightarrow \int_0^1 (1 - g)\chi_F \, dx,$$

and from this we see $\lambda(F) = 0$. Hence, we have $0 \leq g \leq 1$ a.e.

□

Let's now prove theorem 1.

PROOF.
(i) $\Rightarrow$ (ii)
Let $A = \cap_i A_{n_i}$ have positive measure. By lemma 1, there exists a measurable g with $0 \leq g \leq 1$ a.e. and a subsequence of $\chi_{A_{n_i}}$ (which won't be renamed) such that $\chi_{A_{n_i}} \rightharpoonup g$ in L^1. By standard arguments, it can be shown that $\chi_{A_{n_i}} \rightharpoonup g$ in $L^1(A)$.
Since $\chi_{A_{n_i}} = 1$ on A and $0 \leq g \leq 1$ a.e., we see that

$$\begin{aligned}\int_A \left|1 - g\right| dx &= \int_A \left|\chi_{A_{n_i}} - g\right| dx \\ &= \int_A \chi_{A_{n_i}} \, dx - \int_A g \, dx \rightarrow 0 \qquad \text{since } \chi_{A_{n_i}} \rightharpoonup g \text{ in } L^1(A).\end{aligned}$$

Hence, $g = 1$ a.e. on A.
(ii) $\Rightarrow$ (i)
Let $\chi_{A_{n_i}} \rightharpoonup g$ in L^1 where $g = 1$ on A, and A has positive measure. Then we have

$$\int_0^1 \chi_{A_{n_i}} \chi_A \, dx \rightarrow \int_0^1 g\chi_A \, dx \qquad \text{or} \qquad \lambda(A_{n_i} \cap A) \rightarrow \lambda(A),$$

which implies the result.

□

Let's now look at an example and see if it agrees with what theorem 1 says.

Example 1. *Let $C \in (0,1)$ and define*

$$A_n := \bigcup_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k+C}{n}\right].$$

Then $\lambda(A_n) = C$, and it is possible to show that

$$\chi_{A_n} \rightharpoonup C \quad in \quad L^1.$$

If we apply Theorem 1, we see that $\lambda(A) = 0$ where $A := \bigcap_i A_{n_i}$, and A_{n_i} is any subsequence.

Let's now manually check this. Fix $x \in [0,1)$, and let $\epsilon > 0$, but with $x + \epsilon \leq 1$. Then we have

$$\begin{aligned} \lambda\left(A \cap [x, x+\epsilon]\right) &\leq \lambda\left(A_{n_i} \cap [x, x+\epsilon]\right) \\ &= \int_0^1 \chi_{A_{n_i}} \chi_{[x,x+\epsilon]}\, dx \to \int_0^1 C\chi_{[x,x+\epsilon]}\, dx = C\epsilon. \end{aligned}$$

Now divide by ϵ to get

$$\frac{\lambda\left(A \cap [x, x+\epsilon]\right)}{\epsilon} \leq C.$$

So we see

$$\limsup_{\epsilon \to 0^+} \frac{\lambda\left(A \cap [x, x+\epsilon]\right)}{\epsilon} \leq C < 1$$

for all $x \in [0,1)$, and it follows from the Lebesgue Density Theorem that $\lambda(A) = 0$.

References

[1] A. Bruckner, J. Bruckner, B. Thomson, *Real Analysis*, Prentice Hall, Englewood Cliffs, 1996.