INROADS

Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 205–212

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A CLASSIFICATION OF BAIRE ONE STAR FUNCTIONS

Abstract

We present a new classification of Baire one star functions and examine sums of functions from the defined classes.

The letter $\mathbb{R}$ denotes the real line. The symbols ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. The word *function* denotes a mapping from a subset of $\mathbb{R}$ into $\mathbb{R}$. The symbol $\mathcal{C}(f)$ stands for the set of points of continuity of a function f.

Let $A \subset \mathbb{R}$. We use the symbols $\operatorname{int} A$ and $\operatorname{cl} A$ to denote the interior and the closure of A, respectively. If A is closed, then for each $\alpha < \omega_1$, we denote by $A^{(\alpha)}$ the α^{th} Cantor-Bendixson derivative of A; i.e.,

$$A^{(\alpha)} \stackrel{\text{df}}{=} \begin{cases} A & \text{if } \alpha = 0, \\ \left(A^{(\beta)}\right)' & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta<\alpha} A^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal,} \end{cases}$$

where B' is the set of all accumulation points of B. Clearly, $A^{(\alpha)} \supset A^{(\beta)}$ whenever $\alpha < \beta < \omega_1$.

If $f \colon \mathbb{R} \to \mathbb{R}$, then for every ordinal α, we define

$$\mathcal{U}_\alpha(f) \stackrel{\text{df}}{=} \operatorname{int}\Big(\bigcup_{\beta<\alpha} \mathcal{U}_\beta(f) \cup \mathcal{C}\big(f\restriction\mathbb{R} \setminus \bigcup_{\beta<\alpha} \mathcal{U}_\beta(f)\big)\Big).$$

(Clearly $\mathcal{U}_\alpha(f) \subset \mathcal{U}_\beta(f)$ for all ordinals $\alpha < \beta$.) For each $\alpha < \omega_1$, we denote

$$\mathcal{S}_\alpha \stackrel{\text{df}}{=} \big\{f \colon \mathbb{R} \to \mathbb{R}; \mathcal{U}_\alpha(f) = \mathbb{R}\big\}.$$

Key Words: Baire one star function, sum of functions
Mathematical Reviews subject classification: Primary 26A15; Secondary 26A21
Received by the editors June 24, 2004
Communicated by: Udayan B. Darji
*Partially supported by University in Bydgoszcz.

Observe that, in particular, $\mathcal{S}_1$ is the class $\mathcal{B}_1^{**}$ defined by R.J. Pawlak [7].

We say that $f\colon \mathbb{R} \to \mathbb{R}$ is a *Baire one star function* [6] if for each nonempty closed set $P \subset \mathbb{R}$, there is a nonempty portion $P' \overset{\text{df}}{=} P \cap (a,b)$ of P such that $f{\restriction}P'$ is continuous. We denote the family of all Baire one star functions by $\mathcal{B}_1^*$.

Recall the following theorem proved by B. Kirchheim [3, Theorem 2.3].

Theorem 1. *For every function $f\colon \mathbb{R} \to \mathbb{R}$, the following are equivalent*:

(i) $f \in \mathcal{B}_1^*$;

(ii) *for each $a \in \mathbb{R}$, both $f^{-1}\big((-\infty,a]\big)$ and $f^{-1}\big([a,\infty)\big)$ are F_σ-sets.*

The next lemma is easy to prove.

Lemma 2. *Let $U \subset \mathbb{R}$ be an open set, $f\colon \mathbb{R} \to \mathbb{R}$, and $\alpha < \omega_1$. The following conditions are equivalent*:

(i) $U \subset \mathcal{U}_\alpha(f)$;

(ii) *the restriction $f{\restriction}U \setminus \bigcup_{\beta<\alpha} \mathcal{U}_\beta(f)$ is continuous.*

Now we can prove the first main result.

Theorem 3. $\bigcup_{\alpha<\omega_1} \mathcal{S}_\alpha = \mathcal{B}_1^*$.

PROOF. First let $f \in \mathcal{B}_1^*$, and suppose that $\mathcal{U}_\alpha(f) \neq \mathbb{R}$ for each $\alpha < \omega_1$. Since $\langle \mathcal{U}_\alpha(f); \alpha < \omega_1\rangle$ is an ascending transfinite sequence of open subsets of $\mathbb{R}$, there is an $\alpha < \omega_1$ such that $\mathcal{U}_\alpha(f) = \mathcal{U}_{\alpha+1}(f)$. (We use the Cantor–Baire stationary principle; see, e.g., [4, Theorem 2, p. 146].) Then $P \overset{\text{df}}{=} \mathbb{R} \setminus \mathcal{U}_\alpha(f)$ is a nonempty closed set. So, by definition, there is an open interval (a,b) such that $P' \overset{\text{df}}{=} P \cap (a,b) \neq \emptyset$, and $f{\restriction}P'$ is continuous. Since $P' = (a,b) \setminus \mathcal{U}_\alpha(f)$, by Lemma 2, we obtain $(a,b) \subset \mathcal{U}_{\alpha+1}(f)$. Hence, $P' \subset \mathcal{U}_{\alpha+1}(f) \setminus \mathcal{U}_\alpha(f) = \emptyset$, which is impossible.

Now let $f \in \mathcal{S}_\alpha$ for some $\alpha < \omega_1$. Fix an $a \in \mathbb{R}$. We have

$$f^{-1}\big((-\infty,a]\big) = \bigcup_{\gamma\le\alpha} \big\{x \in \mathcal{U}_\gamma(f) \setminus \textstyle\bigcup_{\beta<\gamma} \mathcal{U}_\beta(f); f(x) \le a\big\} = \displaystyle\bigcup_{\gamma\le\alpha} K_\gamma,$$

where

$$K_\gamma \overset{\text{df}}{=} \big(f{\restriction}\mathcal{U}_\gamma(f) \setminus \textstyle\bigcup_{\beta<\gamma} \mathcal{U}_\beta(f)\big)^{-1}\big((-\infty,a]\big).$$

Let $\gamma \le \alpha$. Put $V_\gamma \overset{\text{df}}{=} \mathcal{U}_\gamma(f) \setminus \bigcup_{\beta<\gamma} \mathcal{U}_\beta(f)$. Since $f{\restriction}V_\gamma$ is continuous, the set K_γ is closed in V_γ. Let F_γ be a closed subset of $\mathbb{R}$ such that $K_\gamma = F_\gamma \cap V_\gamma$. Then

$$f^{-1}\big((-\infty,a]\big) = \bigcup_{\gamma\le\alpha} (F_\gamma \cap V_\gamma)$$

is a countable union of F_σ-sets, and hence, it too is an F_σ set. Analogously, we can show that $f^{-1}\big([a,\infty)\big)$ is an F_σ-set. By Theorem 1, $f \in \mathcal{B}_1^*$. □

Theorem 4. *For each $\alpha < \omega_1$, we have $\mathcal{S}_\alpha \setminus \bigcup_{\beta<\alpha} \mathcal{S}_\beta \neq \emptyset$.*

Proof. Let $A \subset \mathbb{R}$ be a countable, compact set such that $A^{(\alpha+1)} = \emptyset \neq A^{(\alpha)}$. (See, e.g., [8, Exercise 2.5.15].) Let f be the characteristic function of the set

$$\bigcup_{\beta \text{ odd},\ \beta<\omega_1} \big(A^{(\beta)} \setminus A^{(\beta+1)}\big).$$

One can easily verify that $\mathcal{U}_\beta(f) = \mathbb{R} \setminus A^{(\beta+1)}$ for each ordinal $\beta < \omega_1$. So, $f \in \mathcal{S}_\alpha \setminus \bigcup_{\beta<\alpha} \mathcal{S}_\beta$. □

Now we will investigate the sums of functions from the defined classes. We will need the following theorem [5].

Theorem 5. *Let $1 \leq \alpha < \omega_1$. Then α can be uniquely written in the form*

$$\alpha = \omega^{\eta_0} r_0 + \cdots + \omega^{\eta_n} r_n,$$

where $r_0, \ldots, r_n$ are finite nonzero ordinals, and $\langle \eta_0, \ldots, \eta_n \rangle$ is a decreasing sequence of countable ordinals.

The notion of the natural addition was defined in 1906 by G. Hessenberg [2]. We define the natural addition for countable ordinals in the following way. If

$$\alpha = \omega^{\xi_0} p_0 + \cdots + \omega^{\xi_k} p_k, \quad \beta = \omega^{\xi_0} q_0 + \cdots + \omega^{\xi_k} q_k,$$

where $\xi_0 > \cdots > \xi_k$ and $p_0, \ldots, p_k, q_0, \ldots, q_k$ are finite (we allow zeros here), then we define

$$\alpha \,(+)\, \beta \stackrel{\text{df}}{=} \omega^{\xi_0}(p_0 + q_0) + \cdots + \omega^{\xi_k}(p_k + q_k).$$

Clearly, the natural addition is commutative.

The following lemma is quite trivial.

Lemma 6. *Let $\alpha < \alpha'$ and $\beta \leq \beta'$. Then $\alpha \,(+)\, \beta < \alpha' \,(+)\, \beta'$.*

Now we can prove the next main result.

Theorem 7. *Let $\alpha, \beta < \omega_1$, $f \in \mathcal{S}_\alpha$, and $g \in \mathcal{S}_\beta$. Then $f + g \in \mathcal{S}_{\alpha(+)\beta}$.*

Proof. For brevity, for each $\gamma < \omega_1$, we denote

$$U_\gamma \overset{\text{df}}{=} \mathcal{U}_\gamma(f), \qquad V_\gamma \overset{\text{df}}{=} \mathcal{U}_\gamma(g),$$
$$\widetilde{U}_\gamma \overset{\text{df}}{=} U_\gamma \setminus \bigcup_{\sigma<\gamma} U_\sigma, \qquad \widetilde{V}_\gamma \overset{\text{df}}{=} V_\gamma \setminus \bigcup_{\sigma<\gamma} V_\sigma,$$

and

$$W_\gamma \overset{\text{df}}{=} \bigcup_{\delta(+)\varepsilon=\gamma} (U_\delta \cap V_\varepsilon) = \bigcup_{\delta(+)\varepsilon=\gamma} \bigcup_{\mu\le\delta,\nu\le\varepsilon} (\widetilde{U}_\mu \cap \widetilde{V}_\nu).$$

Notice that each set W_γ is open.

We will show by transfinite induction that for each $\gamma < \omega_1$,

$$W_\gamma \subset \mathcal{U}_\gamma(f+g). \tag{1}$$

Let $\gamma < \omega_1$ and assume that $W_\sigma \subset \mathcal{U}_\sigma(f+g)$ for each $\sigma < \gamma$. Clearly, it suffices to show that $U_\delta \cap V_\varepsilon \subset \mathcal{U}_\gamma(f+g)$ whenever $\delta\,(+)\,\varepsilon = \gamma$. So, fix $\delta, \varepsilon < \omega_1$ with $\delta\,(+)\,\varepsilon = \gamma$. First, we will show that

$$U_\delta \cap V_\varepsilon \setminus \bigcup_{\sigma<\gamma} W_\sigma = \widetilde{U}_\delta \cap \widetilde{V}_\varepsilon. \tag{2}$$

Let $x \in \widetilde{U}_\delta \cap \widetilde{V}_\varepsilon$. Then clearly

$$x \in U_\delta \cap V_\varepsilon = \bigcup_{\mu\le\delta,\nu\le\varepsilon} \big(\widetilde{U}_\mu \cap \widetilde{V}_\nu\big).$$

Suppose that $x \in W_\sigma$ for some $\sigma < \gamma$. There exist $\mu' \le \delta'$ and $\nu' \le \varepsilon'$ such that $\mu'\,(+)\,\nu' \le \delta'\,(+)\,\varepsilon' = \sigma$ and $x \in \widetilde{U}_{\mu'} \cap \widetilde{V}_{\nu'}$. Hence, $\widetilde{U}_{\mu'} \cap \widetilde{U}_\delta \neq \emptyset$ and $\widetilde{V}_{\nu'} \cap \widetilde{V}_\varepsilon \neq \emptyset$. Notice that the sequences $\big\langle \widetilde{U}_\xi; \xi < \omega_1 \big\rangle$ and $\big\langle \widetilde{V}_\xi; \xi < \omega_1 \big\rangle$ consist of pairwise disjoint sets. Thus, $\mu' = \delta$ and $\nu' = \varepsilon$, and consequently,

$$\gamma = \delta\,(+)\,\varepsilon = \mu'\,(+)\,\nu' \le \sigma < \gamma,$$

which is impossible. It follows that $x \in U_\delta \cap V_\varepsilon \setminus \bigcup_{\sigma<\gamma} W_\sigma$.

Now let $x \in U_\delta \cap V_\varepsilon \setminus \bigcup_{\sigma<\gamma} W_\sigma$. Since $x \in U_\delta \cap V_\varepsilon$, there exist $\mu \le \delta$ and $\nu \le \varepsilon$ such that $x \in \widetilde{U}_\mu \cap \widetilde{V}_\nu$. If $\mu\,(+)\,\nu < \gamma$, then $x \in W_{\mu(+)\nu} \subset \bigcup_{\sigma<\gamma} W_\sigma$, which is a contradiction. Thus, $\mu\,(+)\,\nu \ge \gamma$. By Lemma 6, we have $\mu = \delta$ and $\nu = \varepsilon$. It follows that $x \in \widetilde{U}_\delta \cap \widetilde{V}_\varepsilon$.

Now observe that by (2) and the induction assumption,

$$U_\delta \cap V_\varepsilon \setminus \bigcup_{\sigma<\gamma} \mathcal{U}_\sigma(f+g) \subset U_\delta \cap V_\varepsilon \setminus \bigcup_{\sigma<\gamma} W_\sigma = \widetilde{U}_\delta \cap \widetilde{V}_\varepsilon.$$

Since $f{\restriction}\widetilde{U}_\delta$ and $g{\restriction}\widetilde{V}_\varepsilon$ are continuous, the functions

$$(f+g){\restriction}\widetilde{U}_\delta \cap \widetilde{V}_\varepsilon \quad \text{and} \quad (f+g){\restriction}U_\delta \cap V_\varepsilon \setminus \bigcup_{\sigma<\gamma} \mathcal{U}_\sigma(f+g)$$

are continuous as well. Thus, $U_\delta \cap V_\varepsilon \subset \mathcal{U}_\gamma(f+g)$ (cf. Lemma 2). It completes the proof of (1).

By (1), we have, in particular,

$$W_{\alpha(+)\beta} = \mathbb{R} \subset \mathcal{U}_{\alpha(+)\beta}(f+g);$$

i.e., $f+g \in \mathcal{S}_{\alpha(+)\beta}$. □

Proposition 8. *Let $\alpha, \beta < \omega_1$ and $h \in \mathcal{S}_{\alpha+\beta}$. There are $f \in \mathcal{S}_\alpha$ and $g \in \mathcal{S}_\beta$ such that $h = f+g$.*

Proof. By definition, the restriction $\varphi \stackrel{\text{df}}{=} h{\restriction}\mathcal{U}_\alpha(h) \setminus \bigcup_{\delta<\alpha} \mathcal{U}_\delta(h)$ is continuous. Since $\mathcal{U}_\alpha(h) \setminus \bigcup_{\delta<\alpha} \mathcal{U}_\delta(h)$ is a closed subspace of $\mathcal{U}_\alpha(h)$, by Tietze Extension Theorem, we can extend φ to a continuous function $\widetilde{\varphi}\colon \mathcal{U}_\alpha(h) \to \mathbb{R}$. (See, e.g., [1].) Define

$$g(x) = \begin{cases} h(x) & \text{if } x \in \mathbb{R} \setminus \mathcal{U}_\alpha(h), \\ \widetilde{\varphi}(x) & \text{if } x \in \mathcal{U}_\alpha(h). \end{cases}$$

To prove $g \in \mathcal{S}_\beta$, it suffices to show that

$$\mathcal{U}_{\alpha+\sigma}(h) \subset \mathcal{U}_\sigma(g) \quad \text{for each } \sigma \le \beta.$$

(Recall that $\mathcal{U}_{\alpha+\beta}(h) = \mathbb{R}$.) We proceed by transfinite induction.

Clearly, $\mathcal{U}_{\alpha+0}(h) = \mathcal{U}_\alpha(h) \subset \mathcal{U}_0(g)$. So, let $0 < \sigma \le \beta$ and assume that $\mathcal{U}_{\alpha+\nu}(h) \subset \mathcal{U}_\nu(g)$ for each $\nu < \sigma$. Recall that if $\alpha \le \xi < \alpha+\sigma$, then $\xi = \alpha+\nu$ for some $\nu < \sigma$. So, since $\sigma > 0$, by induction assumption, we obtain

$$\begin{aligned} \mathcal{U}_{\alpha+\sigma}(h) \setminus \bigcup_{\nu<\sigma} \mathcal{U}_\nu(g) &\subset \mathcal{U}_{\alpha+\sigma}(h) \setminus \bigcup_{\nu<\sigma} \mathcal{U}_{\alpha+\nu}(h) \\ &\subset \mathcal{U}_{\alpha+\sigma}(h) \setminus \bigcup_{\xi<\alpha+\sigma} \mathcal{U}_\xi(h) \subset \mathbb{R} \setminus \mathcal{U}_\alpha(h). \end{aligned}$$

Thus, the restriction

$$g{\restriction}\mathcal{U}_{\alpha+\sigma}(h) \setminus \bigcup_{\nu<\sigma} \mathcal{U}_\nu(g) = h{\restriction}\mathcal{U}_{\alpha+\sigma}(h) \setminus \bigcup_{\nu<\sigma} \mathcal{U}_\nu(g)$$

is continuous. By Lemma 2, we obtain $\mathcal{U}_{\alpha+\sigma}(h) \subset \mathcal{U}_\sigma(g)$. It follows that $g \in \mathcal{S}_\beta$.

Now define the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) \stackrel{\mathrm{df}}{=} h(x) - g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \bigcup_{\delta<\alpha} \mathcal{U}_\delta(h), \\ (h - \widetilde{\varphi})(x) & \text{otherwise.} \end{cases}$$

We will first show that

$$\mathcal{U}_\delta(h) \subset \mathcal{U}_\delta(f) \quad \text{for each } \delta < \alpha. \tag{3}$$

Let $\delta < \alpha$ and assume that $\mathcal{U}_\mu(h) \subset \mathcal{U}_\mu(f)$ for each $\mu < \delta$. Then by induction assumption,

$$\mathcal{U}_\delta(h) \setminus \bigcup_{\mu<\delta} \mathcal{U}_\mu(f) \subset \mathcal{U}_\delta(h) \setminus \bigcup_{\mu<\delta} \mathcal{U}_\mu(h) \subset \mathcal{U}_\alpha(h).$$

So, the function

$$f{\restriction}\mathcal{U}_\delta(h) \setminus \bigcup_{\mu<\delta} \mathcal{U}_\mu(f) = (h - \widetilde{\varphi}){\restriction}\mathcal{U}_\delta(h) \setminus \bigcup_{\mu<\delta} \mathcal{U}_\mu(f)$$

is continuous, and by Lemma 2, we obtain $\mathcal{U}_\delta(h) \subset \mathcal{U}_\delta(f)$.

Now observe that by (3), since the restriction $f{\restriction}\mathbb{R}\setminus\bigcup_{\delta<\alpha} \mathcal{U}_\delta(f)$ is constant. Hence, $\mathcal{U}_\alpha(f) = \mathbb{R}$ and $f \in \mathcal{S}_\alpha$. This completes the proof. □

Using Proposition 8 several times, we obtain the following corollary.

Corollary 9. *Let $\gamma = \omega^{\eta_0} r_0 + \dots + \omega^{\eta_k} r_k$, where $r_0, \dots, r_k$ are finite nonzero ordinals, and $\langle \eta_0, \dots, \eta_k \rangle$ is a decreasing sequence of countable ordinals. Then for each $h \in \mathcal{S}_\gamma$, there are functions $f_{i,j} \in \mathcal{S}_{\omega^{\eta_i}}$, where $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, r_i\}$, such that $h = \sum_{i=0}^k \sum_{j=1}^{r_i} f_{i,j}$.*

Corollary 10. *Let $\alpha, \beta < \omega_1$ and $h\colon \mathbb{R} \to \mathbb{R}$. The following are equivalent:*

(i) *$h \in \mathcal{S}_{\alpha(+)\beta}$,*

(ii) *there are functions $f \in \mathcal{S}_\alpha$ and $g \in \mathcal{S}_\beta$ such that $h = f + g$.*

PROOF. (i) ⇒ (ii). Let $h \in \mathcal{S}_{\alpha(+)\beta}$. Write ordinals α and β in the form

$$\alpha = \omega^{\xi_0} p_0 + \dots + \omega^{\xi_k} p_k, \quad \beta = \omega^{\xi_0} q_0 + \dots + \omega^{\xi_k} q_k,$$

where $\xi_0 > \dots > \xi_k$ and $p_0, \dots, p_k, q_0, \dots, q_k$ are finite. By Corollary 9, there are functions $f_{i,j} \in \mathcal{S}_{\omega^{\eta_i}}$, where $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, p_i + q_i\}$ such that $h = \sum_{i=0}^k \sum_{j=1}^{p_i+q_i} f_{i,j}$. Put

$$f = \sum_{i=0}^{k} \sum_{j=1}^{p_i} f_{i,j}, \quad g = \sum_{i=0}^{k} \sum_{j=1}^{q_i} f_{i,p_i+j}.$$

By Theorem 7, $f \in \mathcal{S}_\alpha$ and $g \in \mathcal{S}_\beta$. Clearly, $h = f + g$.

The implication (ii) $\Rightarrow$ (i) follows by Theorem 7. $\square$

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