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Arnold W. Miller,* University of Wisconsin-Madison, Department of Mathematics, Van Vleck Hall, 480 Lincoln Drive, Madison, Wisconsin. 53706-1388. email: miller@math.wisc.edu
http://www.math.wisc.edu/$\sim$miller

HALF OF AN INSEPARABLE PAIR

Abstract

A classical theorem of Luzin is that the separation principle holds for the $\mathbf{\Pi}^0_\alpha$ sets but fails for the $\mathbf{\Sigma}^0_\alpha$ sets. We show that for every $\mathbf{\Sigma}^0_\alpha$ set A which is not $\mathbf{\Pi}^0_\alpha$ there exists a $\mathbf{\Sigma}^0_\alpha$ set B which is disjoint from A but cannot be separated from A by a $\mathbf{\Delta}^0_\alpha$ set C. Assuming $\mathbf{\Pi}^1_1$-determancy it follows from a theorem of Steel that a similar result holds for $\mathbf{\Pi}^1_1$ sets. On the other hand assuming V=L there is a proper $\mathbf{\Pi}^1_1$ set which is not half of a Borel inseparable pair. These results answer questions raised by F.Dashiell.

The separation principle is a classical property of point classes in descriptive set theory. For every countable ordinal α and every pair of disjoint sets $A, B \subseteq 2^\omega$ in the multiplicative class α ($\mathbf{\Pi}^0_\alpha$) there exists a set C in ambiguous class α ($\mathbf{\Delta}^0_\alpha$) which separates them; i.e., $A \subseteq C$ and $C \cap B = \emptyset$. It is also a classical result of Luzin that the separation principle must fail for the dual classes $\mathbf{\Sigma}^0_\alpha$. For proofs, see Kechris [15] §22.

For Γ a class of subsets of ω^ω, define the dual class $\widetilde{\Gamma} = \{\omega^\omega \setminus A : A \in \Gamma\}$, $\Delta = \Gamma \cap \widetilde{\Gamma}$, and

$\mathrm{Sep}(\Gamma) \equiv \forall A, B \in \Gamma \;\; A \cap B = \emptyset \to \exists C \in \Delta \;\; A \subseteq C$ and $C \cap B = \emptyset$.

Γ is continuously closed iff for all continuous $f : \omega^\omega \to \omega^\omega$ if $A \in \Gamma$, then $f^{-1}(A) \in \Gamma$. Γ is nonselfdual iff $\Gamma \neq \widetilde{\Gamma}$.

Van Wesep and Steel [34] [35] [32] proved that for continuously closed nonselfdual Γ in the Borel subsets of ω^ω either ($\neg\mathrm{Sep}(\Gamma)$ and $\mathrm{Sep}(\widetilde{\Gamma})$) or ($\neg\mathrm{Sep}(\widetilde{\Gamma})$ and $\mathrm{Sep}(\Gamma)$); i.e., separation holds on one side and fails on the other. This result is true for all continuously closed nonselfdual classes, if the Axiom of Determinacy holds.

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In Dashiell [8], Luzin's theorem on the failure of separation for $\mathbf{\Sigma}^0_\alpha$ is used to prove that the Banach space, $\mathcal{B}_\alpha$, of Baire class α-functions is not isomorphic to the space $\mathcal{B}_{\omega_1}$ of Baire functions.

The following Theorem settles a question raised by F. Dashiell. He already knew the result for $\mathbf{\Sigma}^0_1$ and $\mathbf{\Sigma}^0_2$. It was also asked by Luzin [19] in 1930, see the top of page 73, "Un autre problème ..." and the last paragraph on page 76. Henryk Torunczyk informs me that Theorem 1 follows from the results in the paper Louveau and Saint-Raymond [18].

Theorem 1. *Suppose X is a Polish space and $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$. Then there exists $A^* \subseteq X$ which is $\mathbf{\Sigma}^0_\alpha$ such that $A \cap A^* = \emptyset$ but there does not exist a $\mathbf{\Delta}^0_\alpha$ set C which separates A and A^*; i.e., $A \subseteq C$ and $C \cap A^* = \emptyset$.*

PROOF. For $\alpha = 1$, if A is any open set which is not closed, then it cannot be separated from the interior of $X \setminus A$. So we may assume $\alpha \geq 2$. By Theorem 4 of Kunen-Miller [16], there exists a set $P \subseteq X$ such that P is homeomorphic to a closed subset of 2^ω and $P \cap A$ is $\mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Delta}^0_\alpha$. So without loss of generality we may assume $A \subseteq 2^\omega$.

For subsets $B, C \subseteq 2^\omega$ define $B \leq_W C$ (Wadge reducible) iff there exists a continuous map $f : 2^\omega \to 2^\omega$ such that $f^{-1}(C) = B$. Associated with Wadge reducibility is the Wadge game whose payoff set is of roughly the same complexity as B and C. It follows from Borel determinacy (see Martin [23]) that for every pair of Borel sets B and C that either $B \leq_W C$ or $C \leq_W (2^\omega \setminus B)$, see for example Van Wesep [34]. It follows from this that for any $B \subseteq 2^\omega$ which is $\mathbf{\Sigma}^0_\alpha$ we have that $B \leq_W A$, since otherwise $A \leq_W (2^\omega \setminus B)$ would make A a $\mathbf{\Pi}^0_\alpha$ and hence $\mathbf{\Delta}^0_\alpha$, which is contrary to our assumption.

Now assume $\alpha = 2$. Let $D, D^* \subseteq 2^\omega$ be countable dense and disjoint. Note that they are $\mathbf{\Sigma}^0_2$ sets which cannot be separated, since dense $\mathbf{\Pi}^0_2$; i.e., G_δ, sets must intersect by the Baire Category Theorem. Since $D \leq_W A$, there exists a continuous map $f : 2^\omega \to 2^\omega$ with $f^{-1}(A) = D$. Let $A^* = f(D^*)$. Since it is countable, A^* is a $\mathbf{\Sigma}^0_2$ set. It cannot be separated from A, because if C is a $\mathbf{\Delta}^0_2$ with $A \subseteq C$ and $A^* \cap C = \emptyset$, then $D \subseteq f^{-1}(C)$ and $D^* \subseteq f^{-1}(2^\omega \setminus C)$ would separate D and D^*.

Now assume $\alpha > 2$. By a result of Harrington, see Steel [31] or Van Engelen, Miller, Steel [33], for any B which is $\mathbf{\Sigma}^0_\alpha$ there exists a one-to-one continuous map $f : 2^\omega \to 2^\omega$ such that $f^{-1}(A) = B$. By a classical theorem of descriptive set theory (see Kechris [15]) there exists disjoint $B, B^* \subseteq 2^\omega$ $\mathbf{\Sigma}^0_\alpha$ sets which cannot be separated by a $\mathbf{\Delta}^0_\alpha$ set. Let f be one-to-one and continuous with $f^{-1}(A) = B$. Let $A^* = f(B^*)$. Since f is one-to-one, it is a homeomorphism onto its range and hence A^* is a $\mathbf{\Sigma}^0_\alpha$ set disjoint from A. The set A^* cannot be separated from A because the preimage of a separating set would separate B and B^*. □

Dashiell's proof of Theorem 1 for $\alpha = 2$ is as follows. Suppose X is a Polish space and $A \subseteq X$ is some F_σ set which is not a G_δ. By Baire's theorem on functions of the first class, there exists a closed $F \subseteq X$ on which the characteristic function of A has no point of continuity relative to F. That is, both $A \cap F$ and $A \setminus F$ are dense in F. Let A^* be a countable dense set in $A \setminus F$ (hence an F_σ). Clearly now A and A^* can not be separated by disjoint G_δ sets of X, because intersecting with F would give two dense G_δ subsets of the complete metric space F, which must meet.

Dashiell pointed out that for a fixed countable ordinal α if we let X_α be the Stone space of the Boolean algebra of $\mathbf{\Delta}^0_\alpha$ subsets of the reals, then the cozero sets in X_α whose closures are not open (i.e., not clopen) correspond to the proper $\mathbf{\Sigma}^0_\alpha$ sets. (Recall that a zero set is a closed set which the preimage of singleton zero under a real-valued continuous map and a cozero set is the complement of a zero set.) Hence, by Theorem 1, we know that every cozero set A whose closure is not open has an inseparable disjoint sibling; i.e., a cozero set B disjoint from A but the closures of A and B must meet.

Dashiell tells us that the question from [8] of whether $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ can be isomorphic Banach spaces for some $1 < \alpha < \beta < \omega_1$ is still open.

Dashiell also raised the same question for the coanalytic sets, $\mathbf{\Pi}^1_1$. The classic result (see Kechris [15] §34,35) is that any pair of disjoint analytic sets ($\mathbf{\Sigma}^1_1$) can be separated by a Borel set ($\mathbf{\Delta}^1_1$), but separation fails for $\mathbf{\Pi}^1_1$. Luzin proved this by applying the reduction principle to a pair of doubly universal sets.

Theorem 2. *Suppose $\mathbf{\Pi}^1_1$-determinacy holds, then for any $\mathbf{\Pi}^1_1$ set A in a Polish space X, if A is not $\mathbf{\Sigma}^1_1$, then there exists $A^* \subseteq X$ a $\mathbf{\Pi}^1_1$ set disjoint from A which cannot be separated from A by a Borel set ($\mathbf{\Delta}^1_1$).*

Theorem 3. *Suppose $V = L$, then there exists a Π^1_1 set $A \subseteq 2^\omega$ which is not $\mathbf{\Sigma}^1_1$ with the property that for any $B \subseteq 2^\omega$ a $\mathbf{\Pi}^1_1$ set disjoint from A there exists a Borel set C with $A \subseteq C$ and $C \cap B = \emptyset$.*

PROOF. For Theorem 2 note that since there is a Borel bijection between X and 2^ω we may assume that $X = 2^\omega$. Theorem 2 is an immediate corollary of a Theorem of Steel [31], who showed that $\mathbf{\Pi}^1_1$-determinacy implies that for any two properly $\mathbf{\Pi}^1_1$ subsets A_1, A_2 of 2^ω there exists a Borel automorphism $f : 2^\omega \to 2^\omega$ such that $f(A_1) = A_2$. Hence if we take $C, C^* \subseteq 2^\omega$ to be any disjoint pair of $\mathbf{\Pi}^1_1$ sets which are not Borel separable and $f : 2^\omega \to 2^\omega$ a Borel automorphism with $f(A) = C$, then $f^{-1}(C^*) = A^*$ will be the required set.

For Theorem 3 we use for A the self-constructible reals studied by Guaspari, Kechris, and Sacks, see Kechris [14] §2, where the self-constructible reals A are denoted $\mathcal{C}_1$.

Define

$$A = \{x \in 2^\omega \ : \ x \in L_{\omega_1^x}\}$$

where ω_1^x is the least ordinal which is not the order type of a relation recursive in x. It is also the least ordinal α such that $L_\alpha[x]$ is an admissible set. Suppose that B is a $\mathbf{\Pi}^1_1$ set disjoint from A. Then we may assume that B is $\Pi^1_1(x_0)$ for some $x_0 \in A$ since by Kechris [14] 2A, every real in L is recursive in some $x_0 \in A$.

Let $\gamma < \omega_1^{x_0}$ be the least ordinal so that $x_0 \in L_\gamma$. For any $y \in 2^\omega$ define $\gamma^+(y)$ to be the least $\alpha > \gamma$ such that $L_\alpha[y]$ is an admissible set.

Lemma 4. *For any $C \subseteq 2^\omega$ a nonempty $\Pi^1_1(x_0)$ set there exists $y \in C$ such that $y \in L_{\gamma^+(y)}$.*

PROOF. The proof is a slight generalization of Sacks [27] III Lemma 9.3 p. 82.

Recall that a binary relation (X, R) is well-founded iff every nonempty subset of X has an R-minimal element. A map $f : X \to$ Ordinals is called a rank function iff

$$\forall s, t \in X \ \ sRt \to f(s) < f(t).$$

Then (X, R) is well-founded iff it has a rank function on it. For (X, R) well-founded the canonical rank function on X is defined inductively by

$$f(s) = \sup\{f(t) + 1 : tRs\}.$$

The range of the canonical rank function is called the rank of (X, R). Furthermore, if $(X, R) \in \mathbb{A}$ is a well-founded relation in an admissible set $\mathbb{A}$, then its rank and its canonical rank function are in $\mathbb{A}$. See Barwise [3] V.3.1 p.159.

Claim 4.1. Suppose δ_1 an aodinal and $T \subseteq \delta_1^{<\omega}$ is a subtree, $T \in L_{\delta_2}$ where $\delta_2 > \omega$ is a limit ordinal. For each $s \in T$ define $T_s = \{t \in T : s \subseteq t\}$. For each ordinal $\alpha < \delta_2$ if $\text{rank}(T_s) = \alpha$, then the canonical rank function, on T_s; i.e., $t \mapsto \text{rank}(T_t)$ is an element of $L_{\delta_2+\alpha+1}$.

PROOF. Note that $(T \times \alpha) \in L_{\delta_2}$ since α is small. Fix α and $s \in T$ with $\text{rank}(T_s) = \alpha$. For each $\delta < \delta_1$ if $s\delta \in T$ and $\text{rank}(T_{s\delta}) = \beta$, then the canonical rank function on $T_{s\delta}$ is in $L_{\delta_2+\beta+1} \subseteq L_{\delta_2+\alpha}$ and is uniformly definable from $T_{s\delta}$, hence the canonical rank function on T_s is in $L_{\delta_2+\alpha+1}$. □

Claim 4.2. Suppose T, δ_1 and δ_2 satisfy the hypothesis of Claim 1. For any ordinal α define

$$T(\alpha) = \{s \in T : \text{rank}(T_s) < \alpha\}.$$

Then $T(\alpha) \in L_{\delta_2+\alpha+1}$.

PROOF. This follows from the previous claim since the canonical rank functions are elements of $L_{\delta_2+\alpha}$. □

By the Addison-Kondo Theorem we may assume that C is a $\Pi^1_1(x_0)$ singleton, i.e. $C = \{y_0\}$.

Now by standard arguments there exists a tree $T \subseteq \cup_{n<\omega}(\omega^n \times 2^n)$ which is recursive in x_0 such that for every $y \in 2^\omega$ we have that

$$y = y_0 \text{ iff } T\langle y\rangle =^{def} \{s : (s, y \restriction |s|) \in T\} \subseteq \omega^{<\omega} \text{ is well-founded.}$$

Now since the tree $(T\langle y_0\rangle, \supset)$ is well-founded and it is an element of the admissible set $L_{\gamma^+(y)}[y]$, its rank δ_0 is strictly less than $\gamma^+(y)$ and its canonical rank function $R : T\langle y_0\rangle \to \delta_0$ is in $L_{\gamma^+(y)}[y]$.

Now define a tree

$$T^* \subseteq \cup_{n<\omega}(\delta_0^n \times 2^n)$$

which basically consists of attempts at a rank function into δ_0 for $T\langle y_0\rangle$. More formally, suppose $\{t_i : i < \omega\}$ is a reasonable recursive listing of $\omega^{<\omega}$; e.g., it should have the properties that $|s_i| \leq i$ and if $s_i \subset s_j$, then $i < j$.

Define $(r, s) \in T^* \cap (\delta^n \times 2^n)$ iff for each $i, j < n$ if $(t_i, s \restriction |t_i|), (t_j, s \restriction |t_j|) \in T$ and $t_i \subset t_j$, then $r(j) < r(i)$.
Let $R^* : \omega \to \delta_0$ be the corresponding map to R; i.e.,

$$R^*(i) = \begin{cases} R(t_i) & \text{if } t_i \in T\langle y_0\rangle \\ 0 & \text{otherwise.} \end{cases}$$

Note that T^* is an element of $L_{\gamma^+(y_0)}$ and (y_0, R^*) is an infinite branch thru it. We claim that (y_0, R^*) is the lexicographically least infinite branch through T^*. To see this, note that if (y, S) is an infinite branch in T^*, then $y = y_0$, since S will be a rank function for $T\langle y\rangle$; hence $T\langle y\rangle$ is well-founded and so $y = y_0$. On the other hand R assigns to any $s \in T\langle y_0\rangle$ the smallest possible ordinal for any rank function, and so R^* will be lexicographically less than S.

Let

$$LF = \{\sigma \in T^* : \sigma \text{ is lexicographically left of } (y_0, R^*)\}.$$

Then $(LF, \supset)$ is a well-founded relation and it is an element of the admissible set $L_{\gamma^+(y_0)}[y_0]$. Hence its rank δ_1 is strictly smaller than $\gamma^+(y_0)$. By identifying the tree T^* with a tree on $(\delta_0+\delta_0)^{<\omega}$; i.e., by mapping $(i, \alpha) \in 2\times\delta_0$ to $\delta_0\cdot i+\alpha$ we may apply Claim 2. Hence the tree $T^* \setminus T^*(\delta_1)$ and its leftmost branch (y_0, R^*) (which is Δ_1 in it) are elements of $L_{\gamma^+(y_0)}$.

Hence $y_0 \in L_{\gamma^+(y_0)}$ as was to be shown. This proves Lemma 4. □

Now we prove Theorem 3. The relation

$$\{(u,v) : u \in \Delta^1_1(v)\}$$

is Π^1_1. Hence the set

$$C = \{y \in B : x_0 \in \Delta^1_1(y)\}$$

is $\Pi^1_1(x_0)$. If it is nonempty, then there exists $y \in C$ with $y \in L_{\gamma^+(y)}$. But since $x_0 \in \Delta^1_1(y)$ we know that $\omega_1^y \geq \omega_1^{x_0} > \gamma$. Hence $y \in L_{\omega_1^y}$ which contradicts $A \cap B = \emptyset$. It follows that

$$B \subseteq \{y : x_0 \notin \Delta^1_1(y)\} \subseteq \{y : \omega_1^y < \gamma\}.$$

The second inclusion is true since every element of $L_{\omega_1^y}$ is in $\Delta^1_1(y)$. It is well known that for any countable γ the set $D = \{y \in 2^\omega : \omega_1^y < \gamma\}$ is Borel. For example, a $\mathbf{\Sigma}^1_1$ definition and $\mathbf{\Pi}^1_1$ definition are given by:

1. $y \in D$ iff there exists $\alpha < \gamma$ such that $\forall e \in \omega$ if $\{e\}^y$ is characteristic function of a well-ordering $(\omega, \leq_e^y)$, then order-type$(\omega, \leq_e^y) < \alpha$.
2. $y \in D$ iff there does not exist $e \in \omega$ and $f : (\omega, \leq_e^y) \to (\gamma, <)$ an isomorphism where $\{e\}^y$ is the characteristic function of the relation $(\omega, \leq_e^y)$.

But note that $D \cap A \subseteq L_\gamma$ is countable and $B \subseteq D$, so A and B can be separated by a Borel set. This proves Theorem 3. $\Box$

Martin and Solovay [22] have shown that assuming Martin's Axiom, not CH, and $\omega_1 = \omega_1^L$ that every set of reals of cardinality ω_1 is $\mathbf{\Pi}^1_1$. This result also appears in Fremlin [11] 23J. Henryk Torunczyk informs me that under these assumptions any set of reals of cardinality ω_1 cannot be half of an inseparable pair of $\mathbf{\Pi}^1_1$ sets.

Question 5. *If every non Borel $\mathbf{\Pi}^1_1$ set is half of an inseparable pair, then is $\mathbf{\Pi}^1_1$-determinacy true?*

See Harrington [12] for some properties of coanalytic sets which imply $\mathbf{\Pi}^1_1$-determinacy.

Clifford Weil raised the question of whether we can get a large number of examples in Theorem 3; e.g.,

Question 6. *Assuming V=L, does there exist continuum many coanalytic sets which are pairwise non Borel isomorphic and each of which is not half of an inseparable pair?*

In Cenzer and Mauldin [7] it is shown that assuming V=L there are continuum many coanalytic sets no two of which are Borel isomorphic.

1 Separation for Subsets of ω.

We could also consider the failure of separation for (lightface) classes of subsets of ω. Addison [1] shows that separation holds for the class of Π^0_n and fails for the class Σ^0_n subsets of ω. However, not every proper Σ^0_1 subset of ω is half of an inseparable pair. A set $A \subseteq \omega$ is simple iff it is recursively enumerable (equivalently Σ^0_1), coinfinite, but its complement does not contain an infinite recursively enumerable subset. Simple sets were first constructed by Post [26] (or see Soare [29]), and clearly a simple set cannot be half of an inseparable pair. We are not sure exactly which recursively enumerable sets are half of inseparable pair, perhaps just the complete ones.

Post also showed that a subset of ω is Σ^0_{n+1} iff it is $\Sigma^0_1(0^{(n)})$ (see Soare[29] IV 2.2). By relativizing his construction of a simple set to the oracle $0^{(n)}$ we get a properly Σ^0_{n+1} subset of ω which is not half of an inseparable pair.

Classically, separation holds for the class of Σ^1_1 subsets of ω and fails for Π^1_1. A proof analogous to the simple set type construction will give a proper Π^1_1 subset of ω which is not half of an inseparable pair (see the proof of Sacks [27] VI Theorem 2.1 or 2.4). Another "natural" example of such a Π^1_1-set can be given as follows. Let $(\omega, \preceq)$ be a recursive linear ordering whose well-ordered initial segment is isomorphic to ω_1^{CK}, the first non recursive ordinal. The existence of such a linear ordering is due to Feferman [10] or perhaps Harrison [13] see also Ash and Knight [2] 8.11. Now let A be the initial well-ordered segment of $\preceq$; i.e.,

$$A = \{n \in \omega : \{m : m \prec n\} \text{ is well-ordered by } \preceq\}.$$

Then A is a proper Π^1_1 set. It cannot be half of an inseparable pair because if $B \subseteq \omega$ is Π^1_1 and disjoint from A, then there must exists some $n_0 \notin A$ such that $k \succeq n_0$ for every $k \in B$. Otherwise

$$\omega \setminus A = \{m \in \omega \ : \ \exists k \in B \ \ k \preceq m\}$$

but A is not a Δ^1_1 set.

Another light-face question one might ask is the following. Suppose A and B are disjoint Π^1_1 subsets of ω^ω which cannot be separated by a Δ^1_1-set; then can they be separated by a $\boldsymbol{\Delta}^1_1$-set? Here is a counterexample. Let $A, B \subseteq \omega$ be disjoint Π^1_1 sets which cannot be separated by Δ^1_1 subset of ω. Define $A^* = \{f \in \omega^\omega : f(0) \in A\}$ and $B^* = \{f \in \omega^\omega : f(0) \in B\}$. Then A^* and B^* are disjoint Π^1_1 which are clopen and hence separable by clopen sets. But they cannot be separated by a Δ^1_1 subset of ω^ω. Suppose $C \subseteq \omega^\omega$ is Δ^1_1 and $A^* \subseteq C$ and $B^* \cap C = \emptyset$. For each $n < \omega$ let $x_n \in \omega^\omega$ be the constant function n. Then

$$C^* = \{n < \omega : x_n \in C\}$$

is a Δ^1_1 set separating A and B.

2 Natural Pairs of Inseparable Sets.

A number of authors have given natural examples of inseparable pairs of $\mathbf{\Pi}^1_1$ sets.

Luzin [20] p.263 gives the following example. Let

$$\phi : \omega^\omega \times \omega^\omega \to \omega^\omega$$

be a Borel function such that for every $f : \omega^\omega \to \omega^\omega$ continuous there exists x such that $\forall y \ \ \phi(x,y) = f(y)$. Let

$$E = \{(x,z) : \exists! y \ \phi(x,y) = z\}$$

$$E_0 = \{(x,z) \in E : \exists! y \ \phi(x,y) = z \text{ and } y(0) \text{ is even } \}$$

$$E_1 = \{(x,z) \in E : \exists! y \ \phi(x,y) = z \text{ and } y(0) \text{ is odd } \}$$

Then E_0 and E_1 are disjoint inseparable $\mathbf{\Pi}^1_1$ sets.

Sierpinski [28] gives the following pair of inseparable $\mathbf{\Pi}^1_1$ sets. Let $U \subseteq \mathbb{R}^3$ be a universal G_δ set for subsets of the plane; i.e., U is G_δ and for every G_δ set $V \subseteq \mathbb{R}^2$ there exists an $x \in \mathbb{R}$ with $U_x = V$. Then

$$S_1 = \{(x,y) : \neg\exists z \ (x,y,z) \in U\} \text{ and } S_2 = \{(x,y) : \exists! \ z \ (x,y,z) \in U\}$$

are a pair of inseparable $\mathbf{\Pi}^1_1$ subsets of the plane.

Dellacherie and Meyer [9] give the following pair of inseparable $\mathbf{\Pi}^1_1$ sets (or perhaps the analogous families of trees): Let LO be the space of linear orderings on ω which we can regard as a closed subspace of $P(\omega \times \omega) \equiv 2^{\omega\times\omega}$. Let $WO \subseteq LO$ be the well-orderings. For two linear orderings let $L_1 \not\hookrightarrow L_2$ mean that L_1 cannot be order embedded into L_2. The following two sets cannot be separated by a Borel set:

$D_1 = \{(L_1, L_2) \in LO^2 : L_1 \in WO \text{ and } L_2 \not\hookrightarrow L_1\}$

$D_2 = \{(L_1, L_2) \in LO^2 : L_2 \in WO \text{ and } L_1 \not\hookrightarrow L_2\}$

To see that these sets are not separable by a Borel set, first note that for any $\mathbf{\Pi}^1_1$ set $A \subseteq 2^\omega$ there exists a continuous map $f : 2^\omega \to LO$ such that $f^{-1}(WO) = A$. (Such a map can be obtained by using the Kleene-Brouwer ordering on a possible well-founded tree $T \subseteq \omega^{<\omega}$ and mapping $\omega^{<\omega} \setminus T$ to and ω sequence at the end.) Similar, for any $\mathbf{\Pi}^1_1$ set $B \subseteq 2^\omega$ there exists a continuous map $g : 2^\omega \to LO$ such that $g^{-1}(WO) = B$. Now if A and B

happen to be an inseparable disjoint pair, then the map $h(x) = (f(x), g(x))$ has the property that $h(A) \subseteq D_1$ and $h(B) \subseteq D_2$. Hence if C separated D_1 and D_2, then $h^{-1}(C)$ would separate A and B.

Maitra [21] uses an open game $G(x)$ on ω^ω due to Blackwell and shows that
$I = \{x \subseteq \omega^{<\omega} : G(x) \text{ is won by player I }\}$
$II = \{x \subseteq \omega^{<\omega} : G(x) \text{ is won by player II }\}$
are disjoint inseparable $\mathbf{\Pi}^1_1$ sets. They are not complementary sets because in the game considered there may be 'ties'.

Becker [4], [5] contains several examples of inseparable $\mathbf{\Pi}^1_1$ sets, for example,
$B_1 = \{f \in C([0,1]) : f \text{ is nowhere differentiable }\}$
$B_2 = \{f \in C([0,1]) : \exists! x\ f'(x) \text{ exists }\}$
are inseparable $\mathbf{\Pi}^1_1$ sets. He gives other examples in the compact subsets of the plane:
$C_1 = \{K \in \mathcal{K}(\mathbb{R}^2)\ :\ K \text{ is path-connected and simply connected}\}$
$C_2 = \{K \in \mathcal{K}(\mathbb{R}^2)\ :\ K \text{ is path-connected and has exactly one hole}\}$

Milewski [24] shows that the following pair of $\mathbf{\Pi}^1_1$ sets in the space of compact subsets of the Hilbert cube, $[0,1]^\omega$, are inseparable:
$M_1 = \{K \in \mathcal{K}([0,1]^\omega) : \text{all components of } K \text{ are finite dimensional }\}$
$M_2 = \{K \in \mathcal{K}([0,1]^\omega) : \text{exactly one component of } K \text{ is } \infty\text{-dim }\}$

Camerlo and Darji [6] give several families of pairwise inseparable coanalytic sets. For any compact set $K \subseteq \omega^\omega$ let
$CD(K) = \{T \subseteq \omega^{<\omega} : \{x \in \omega^\omega : \forall n\ x \restriction n \in T\} \text{ is homeomorphic to } K\}$
Then for any two nonhomeomorphic compact set K_1 and K_2 the sets $CD(K_1)$ and $CD(K_2)$ are inseparable $\mathbf{\Pi}^1_1$ sets.

One schema for obtaining natural disjoint inseparable pairs is to take a naturally defined filter F on ω and its dual ideal $F^* = \{\omega \setminus X : X \in F\}$. Note that F and F^* have the same complexity since there exists a recursive homeomorphism taking one to other; i.e., $X \mapsto \omega \setminus X$. The cofinite filter COF and its dual ideal FIN are naturally inseparable $\mathbf{\Sigma}^0_2$ sets in $P(\omega)$. Louveau's filter $G\mathcal{N}$ [17] is an example of a $\mathbf{\Pi}^1_1$ filter which cannot be separated from its dual ideal by a Borel set. This filter is on the subsets of $\omega^{<\omega}$ and is defined as follows:

$$A \in G\mathcal{N} \text{ iff Player I has a winning strategy in the game } J(A).$$

where $J(A)$ is the game:

Player I:	n_0		n_1		n_2		$\cdots$
Player II:		$m_0 \geq n_0$		$m_1 \geq n_1$		$m_2 \geq n_2$	$\cdots$

Player I wins iff for some k all $s \supseteq (m_i : i < k)$ are not in A. (We use $\supseteq$ to denote end extension of sequences.) This can also be described as follows: $A \in G\mathcal{N}$ iff $\exists \sigma : \omega^{<\omega} \to \omega \; \forall x \in \omega^\omega$ if $\forall n \; x(n) \geq \sigma(x \restriction n)$, then $\exists n \; \forall s \supseteq x \restriction n \;\; s \notin A$. Although superficially it seems as if $G\mathcal{N}$ is Σ^1_2, Louveau proves it is Π^1_1 by using the fact that open games are determined and noting that Player I has a winning strategy iff Player II does not.

Louveau proves that any Borel real valued function on a compact metric space is the $G\mathcal{N}$-limit of a sequence of continuous functions. Hence $G\mathcal{N}$ is a kind of ultimate generalization of the cofinite filter.

Proposition 7. *$G\mathcal{N}$ cannot be separated from its dual ideal $G\mathcal{N}^*$ by a Borel set.*

PROOF. This follows easily from Corollaire 8 (ii) in Louveau [17] which states that for any separable metric space X and disjoint $\mathbf{\Pi}^1_1$ sets C_1 and C_2, there exists a sequence, $(H_u)_{u \in \omega^{<\omega}}$ of closed subsets of X such that

$$C_1 \;\subseteq\; \liminf_{G\mathcal{N}} H_u \;\subseteq\; \limsup_{G\mathcal{N}} H_u \;\subseteq\; X \setminus C_2.$$

where

$$x \in \liminf_{G\mathcal{N}} H_u \;\text{ iff }\; \{u : x \in H_u\} \in G\mathcal{N}$$

and

$$x \in \limsup_{G\mathcal{N}} H_u \;\text{ iff }\; \{u : x \in H_u\} \notin G\mathcal{N}^*.$$

Now take $X = 2^\omega$ and let C_1 and C_2 be any two disjoint inseparable $\mathbf{\Pi}^1_1$ sets and take $H_u \subseteq 2^\omega$ to be the closed sets as in Louveau's Corollaire 8. Suppose for contradiction that $B \subseteq P(\omega^{<\omega})$ is a Borel set with $G\mathcal{N} \subseteq B$ and $G\mathcal{N}^* \cap B = \emptyset$. Define

$$Q = \{x \in 2^\omega : \{u : x \in H_u\} \in B\}.$$

Since B is Borel the set Q is Borel. Note that

$$\liminf_{G\mathcal{N}} H_u \subseteq Q \subseteq \limsup_{G\mathcal{N}} H_u$$

and so $C_1 \subseteq Q$ and $Q \subseteq 2^\omega \setminus C_2$ which contradicts that C_1 and C_2 cannot be separated. □

There are plenty of natural examples of proper $\mathbf{\Pi}^1_1$ filters which can be separated from their duals by Borel sets.

$W_1 = \{A \subseteq \omega^{<\omega} : \neg \exists f \in \omega^\omega \; \exists^\infty n \; f \restriction n \in A\}$

$W_2 = \{A \subseteq \omega^{<\omega} : \neg\exists f \in \omega^\omega \ \exists^\infty n \ \exists s \supseteq f \upharpoonright n \ \ s \in A\}$
W_1 is the ideal of well-founded subrelations, W_2 is the ideal generated by well-founded subtrees. However, note that $W_1 \subseteq W_2 \subseteq NWD$ where NWD is the Borel ideal of nowhere dense subsets of $\omega^{<\omega}$ defined by

$A \in NWD$ iff $\forall s \ \exists t \supseteq s \ \forall r \supseteq t \ \ r \notin A$.

Similarly,

$W_3 = \{A \subseteq \mathbb{Q} : A \text{ is well-ordered } \}$

$W_4 = \{A \subseteq \mathbb{Q} : cl(A) \subseteq \mathbb{Q} \text{ is compact } \}$

we have that $W_3 \subseteq W_4 \subseteq NWDQ$ where $NWDQ$ is the Borel ideal of nowhere dense subsets of the rationals $\mathbb{Q}$. Hence, it is the case that each of W_1, W_2, W_3, W_4 can be separated from their duals by a Borel set.

In Solecki [30] it is shown that for any $\mathbf{\Pi}^0_3$ filter F there exists a $\mathbf{\Sigma}^0_2$ set B with $F \subseteq B$ and $F^* \cap B = \emptyset$. He leaves open whether the analogous result holds for $\mathbf{\Pi}^0_4$ filters. Let F be the cofinite $\times$ cofinite filter on $\omega \times \omega$; i.e., for each $A \subseteq \omega \times \omega$ we have that

$$A \in F \ \text{ iff } \ \forall^\infty n \ \forall^\infty m \ (n,m) \in A$$

Then F is a proper $\mathbf{\Sigma}^0_4$ set (see Kechris [15] §23) and so is its dual ideal F^*. In Solecki [30] Example 1.7, it is shown that F cannot be separated from F^* by a $\mathbf{\Sigma}^0_2$ set. Also according to [30] Corollary 1.5, they cannot be separated by a $\mathbf{\Delta}^0_3$ sets. They can however be separated by a $\mathbf{\Sigma}^0_3$ set. Let

$$Q = \{A \subseteq \omega \times \omega : \forall^\infty n \ \exists^\infty m \ (n,m) \in A\}$$

Then Q is Σ^0_3 and $F \subseteq Q$ and $F^* \cap Q = \emptyset$.

Question 8. *Is there a $\mathbf{\Sigma}^0_3$ filter F which cannot be separated from its dual ideal F^* by a $\mathbf{\Delta}^0_3$ set? In fact, is there a $\mathbf{\Sigma}^0_3$ filter F which is not $\mathbf{\Sigma}^0_2$?*

Question 9. *For F the cofinite $\times$ cofinite filter does there exist a natural $\mathbf{\Sigma}^0_4$ set G such that F and G are a disjoint inseparable pair. (How would you prove there isn't a natural one?)*

There is an easy way to generate examples of inseparable $\mathbf{\Sigma}^0_n$ sets.

Proposition 10. *Suppose that $Q \subseteq 2^\omega$ is a complete $\mathbf{\Pi}^0_n$ set. Let*

$Q_0 = \{(x_n : n < \omega) : \exists n \text{ even } x_n \in Q \text{ and } \forall m < n \ \ x_m \notin Q\}$

$Q_1 = \{(x_n : n < \omega) : \exists n \text{ odd } x_n \in Q \text{ and } \forall m < n \ \ x_m \notin Q\}$

Then Q_0 and Q_1 are $\mathbf{\Sigma}^0_{n+1}$ sets which cannot be separated by a $\mathbf{\Delta}^0_{n+1}$ set.

PROOF. Let $A, B \subseteq 2^\omega$ be a disjoint inseparable pair of $\mathbf{\Sigma}^0_{n+1}$ sets. Write them as unions of $\mathbf{\Pi}^0_n$ sets, $A = \cup_{n<\omega} U^0_n$ and $B = \cup_{n<\omega} U^1_n$. Since Q is complete, there are continuous maps $f_{2n+i} : 2^\omega \to 2^\omega$ with $f^{-1}_{2n+i}(Q) = U^i_n$. Then the map $x \mapsto (f_m(x) : m < \omega)$ shows that Q_0 and Q_1 are inseparable. $\square$

Similarly there is a natural pair of inseparable $\mathbf{\Sigma}^0_3$ sets.

Proposition 11. *Let*
$E = \{x \in \omega^\omega : \liminf_n \ x(n) \text{ is even } \}$
$O = \{x \in \omega^\omega : \liminf_n \ x(n) \text{ is odd } \}$
Then E and O are disjoint inseparable $\mathbf{\Sigma}^0_3$ sets.

PROOF. The set $A = \{x \in \omega^\omega : \liminf_n \ x(n) < \infty\}$ is known to be a complete $\mathbf{\Sigma}^0_3$, see Kechris [15] p.180. This means the given any $\mathbf{\Sigma}^0_3$ set $B \subseteq 2^\omega$ there exists a continuous map $f : 2^\omega \to \omega^\omega$ with $f(A) = B$. Now suppose that B_1 and B_2 are a disjoint inseparable pair of $\mathbf{\Sigma}^0_3$ sets and f_i continuous with $f_i^{-1}(A) = B_i$. Define $h : 2^\omega \to \omega^\omega$ by $h(x)(n) = 2f_1(x(n)$ if $f_1(x)(n) \leq f_2(x)(n)$ and $h(x)(n) = 2f_2(x(n) + 1$ otherwise. Then h is continuous and $h(B_1) \subseteq E$ and $h(B_2) \subseteq O$ and so E and O cannot be separated. $\square$

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