RESEARCH

Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2005/2006, pp. 97–118

S. D'Asero, Dipartimento di Matematica e Informatica, Università di Catania, I-95125 Catania, Italy. email: dasero@dmi.unict.it

V. Pata, Dipartimento di Matematica "F.Brioschi", Politecnico di Milano, I-20133 Milano, Italy. email: pata@mate.polimi.it

P. Ursino,* Dipartimento di Matematica e Informatica, Università di Catania, I-95125 Catania, Italy. email: ursino@dmi.unict.it

ON A GENERALIZED NOTION OF DIFFERENTIABILITY

Abstract

We discuss and compare some generalized types of continuity and differentiability. In particular, we focus on a notion of differentiability based on an integral average, and we establish a link with an approximation procedure by polynomials.

1 Introduction.

The purpose of a limiting procedure is to provide an approximation of a function in a neighborhood of a given point. In the present article, we consider three different generalized notions of limit for a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$, denoted by C-limit, M-limit (or approximate limit) and L-limit, respectively (see §2). In fact, all of these notions apply, rather than to f, to the entire equivalence class of f (with respect to the equivalence relation of equality almost everywhere), since they are immediately seen to be *independent* of the choice of the particular representative.

The kind of approximation provided by these limits is quite different. For the C-limit (see [9]), the portion of the neighborhood of x where f is far away from the desired limiting value $\ell_0 \in \mathbb{R}$ has zero measure, provided that the neighborhood is small enough, while for the M-limit (widely studied in the literature, see [4, 5, 8]), the measure of this set may not be zero, but it tends

Key Words: L-derivative, generalized Taylor sums, best approximating polynomials, Legendre polynomials
Mathematical Reviews subject classification: 26A24, 41A10, 42C10
Received by the editors December 1, 2005
Communicated by: Udayan B. Darji
*Partially supported by CNRS of France

to become irrelevant as the neighborhood shrinks to x. In either case, f is considered close to the limit ℓ_0 when the measure of the "bad" set is small. On the contrary, f has L-limit (see [3]) equal to ℓ_0 if its *integral average* about x is close to ℓ_0. Hence, from the point of view of the measure, f might be *always* away from ℓ_0, and still behave as a constant function (equal to ℓ_0) in a neighborhood of x, with respect to the integral. An enlightening example is provided by the function $f(t) = \sin(1/t)$, which, although is very often far away from zero, satisfies

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \sin\frac{1}{t}\,\mathrm{d}t = 0.$$

Here, the cancellations play a fundamental role, since $t = 0$ is not a Lebesgue point for f, and it is quite easy to verify that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \Big|\sin\frac{1}{t}\Big|\,\mathrm{d}t = \frac{2}{\pi}.$$

As we will see, some implications occur among the above-mentioned limits. In particular, the existence of the C-limit ensures the existence of the other ones. Besides, if f is essentially bounded in a neighborhood of x, then the existence of the M-limit at x implies the existence of the L-limit at x. Nonetheless, these two notions are not, in general, an extension of each other. Indeed, Example 2.3 below shows that a function may possess *different* M- and L-limits at x. Therefore, it is somehow a philosophical choice to decide which of the two limits better reflects the behavior of f in a neighborhood of x. For instance, from the physical viewpoint the notion of L-limit is certainly more interesting. Indeed, if f represents, say, a mass density, then the effective way to measure the mass is to use the integral. In this sense, the local behavior of f is not so important, whereas what one really measures is its average. Hence, the L-limit is more suitable to detect small oscillations, which are hidden using different approximation procedures, such as limits in L^1 (see [7], §B.43).

If we agree that the L-limit is an effective way to measure the behavior of f at a given point x, we can deepen our analysis. A natural development (see §3 and §4) is to look not just for a limit value ℓ_0, but rather for a polynomial of order n in t of the form

$$\mathcal{T}_n^f(x;t) = \sum_{k=0}^{n} \frac{\ell_k}{k!}(t-x)^k \tag{1.1}$$

such that the integral average of $f - \mathcal{T}_n^f(x)$ about x is a zero of order n, with respect to the measure of the interval of integration. This leads quite naturally to the concept of L-derivative of order n at the point x. In this respect, $\mathcal{T}_n^f(x)$ can be viewed as a sort of Taylor sum of order n for the function f at the point x.

Quite surprisingly, the existence of such a $\mathcal{T}_n^f(x)$; that is, the L-differentiability of f of order n at x, is related to a least square approximation procedure. Indeed, in §5, we construct on each interval $[x, y]$ a polynomial $P_n^{x,y}f$ of order n, which, if f is locally square summable, realizes the minimum L^2-distance between f and the subspace of polynomials of order n on $[x, y]$. Then, setting for $x \in \mathbb{R}$

$$\mathcal{P}_n^f(x;t) = \begin{cases} P_n^{x,t}f(t) & \text{if } t > x, \\ P_n^{t,x}f(t) & \text{if } t < x, \end{cases}$$

we prove the following result (see §6).

Theorem 1.1. *Let f be n-times L-differentiable at $x \in \mathbb{R}$. Then $\mathcal{P}_n^f(x;t)$ is defined by continuity at $t = x$, and*

$$\mathcal{T}_n^f(x;t) = \mathcal{P}_n^f(x;t) + o(|t-x|^n),$$

for every $t \in \mathbb{R}$.

Finally, using a suitable basis $\{G_n^{x,y}\}$ of orthogonal polynomials in $L^2(x, y)$, we write

$$P_n^{x,y}f(t) = \sum_{k=0}^{n} \beta_k^f(x,y) G_k^{x,y}(t), \ \ t \in [x, y].$$

Then, the limiting values of the coefficients $\beta_k^f(x, y)$ happen to be related to the successive L-derivatives of f. Precisely, we prove the following theorem (see §7).

Theorem 1.2. *The following are equivalent:*

(i) f is n-times L-differentiable at $x \in \mathbb{R}$.

(ii) There exist $\ell_0, \dots, \ell_n \in \mathbb{R}$ such that

$$\lim_{h \to 0^+} \beta_k^f(x, x+h) = \lim_{h \to 0^+} \beta_k^f(x-h, x) = \frac{\ell_k}{k!},$$

for all $k = 0, \dots, n$.

In that case, (1.1) *holds.*

The proofs of the above theorems lean on some combinatorial identities. In particular, the second theorem is based on a quite general Tauberian result (see Theorem 7.1 below).

2 Some Generalized Notions of Continuity.

We begin to establish and compare some different notions of limit at a given point $x \in \mathbb{R}$ for a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ (cf. [3, 4, 5, 8, 9]).

Definition 2.1. Let $\ell_0 \in \mathbb{R}$.

(i) f has *C-limit* equal to ℓ_0 at x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\lambda\big([x-\delta, x+\delta] \cap \big\{t : |f(t) - \ell_0| > \varepsilon\big\}\big) = 0,$$
λ being Lebesgue measure.

(ii) f has *M-limit* (or *approximate limit*) equal to ℓ_0 at x if for every $\varepsilon > 0$ the set $\big\{t : |f(t) - \ell_0| > \varepsilon\big\}$ has zero Lebesgue density at x.

(iii) f has *L-limit* equal to ℓ_0 at x if $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t = \ell_0$.

It is understood that (iii) requires that f is Lebesgue integrable in a neighborhood of x. Analogous definitions can be given for limits from the right and from the left. The function f is (C-, M- and L-, respectively) continuous at x if $f(x)$ is equal to the (appropriate) limit as $t \to x$.

Remark 2.2. It is apparent that the above limits are independent of the the choice of the particular representative f in the same equivalence class (the equivalence relation is obviously equality almost everywhere with respect to λ).

If we denote by Cf (Mf, Lf) the function obtained replacing $f(t)$ with its C-limit (M-limit, L-limit) whenever it exists, we obtain a function in the same equivalence class which is C-continuous (M-continuous, L-continuous) at all points where the limit exists (see [9]). Hence, in the sequel we will not make distinctions between continuity and having a limit at a point. In this respect, our approach differs from the one usually adopted in the literature (see e.g. [3, 4, 5, 8]), where the notions of M-continuity and L-continuity are given just for a single function (and indeed, ℓ_0 is replaced by $f(x)$), rather than for the equivalence class. On the other hand, in light of Remark 2.2, we think that our point of view is more effective and better reflects the nature of the limits under consideration.

We now dwell on the relationships among the three definitions. Clearly, notion (i) is the strongest, and implies (ii) and (iii). It is not hard to see that Cf might not be continuous in the usual sense at a point of C-continuity. Nonetheless, as shown in [9], there exists a function $B_c f$ (the "best continuous function"), equal to f almost everywhere, whose continuity points are exactly the points of C-continuity, whereas it is clear that no function in the same equivalence class of f can be continuous at a point where the f is not C-continuous. The choice of function $B_c f$ may have some degrees of arbitrariness (see the following Example 3.4).

Let us examine in more detail the mutual implications between (ii) and (iii). The following two facts are well known (see [4, 5, 8]).

- If f is essentially bounded in a neighborhood of x, then (ii) implies that x is a Lebesgue point for f; i.e.,

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}|f(t)-\ell_0|\,\mathrm{d}t=0,$$

  so that, in particular, (iii) holds.

- Conversely, if x is a Lebesgue point for f, then f is M-continuous at x.

However, the implication (ii) $\Rightarrow$ (iii) is in general false without the boundedness of f. An example in such sense is provided by the function (see [8])

$$f(t)=\begin{cases}2^n & \text{if } \frac{1}{2^n}-\frac{1}{2^{2n+1}}\le t\le\frac{1}{2^n},\ n\in\mathbb{N}=\{0,1,2,\ldots\},\\ 0 & \text{otherwise},\end{cases}$$

which is M-continuous at 0, with $Mf(0)=0$, but not L-continuous. The situation can be even worse, since it is possible to construct measurable functions which are nowhere locally summable, and thus nowhere L-continuous. On the other hand, a classical result (see e.g. [4], Theorem 2.9.13) ensures that any measurable function is M-continuous almost everywhere. The reverse implication (iii) $\Rightarrow$ (ii) may also fail, even if f is essentially bounded in a neighborhood of x. For instance, let A be a set which has Lebesgue density 1/2 at x, and let

$$f(t)=\chi_A(t)-\chi_{A^C}(t).$$

Then $L\text{-}\lim_{t\to x}f(t)=0$, but f has no M-limit at x (nor x is a Lebesgue point). A more familiar example, recalled in the introduction, is given by the function $f(t)=\sin(1/t)$, which is easily verified to be everywhere L-continuous, but has no M-limit at $t=0$. But perhaps the most peculiar fact is that (ii) and (iii) may hold simultaneously for *different* values ℓ_0. Indeed, in the next example we exhibit a function that possesses both a right M-limit and a right L-limit at $t=0$, but the values of the limits do not coincide. Obviously, the same can be done with (left) limits. Just notice that this singular feature would have been somehow hidden if we restricted ourselves to consider the continuity of a single function rather than the one of the whole equivalence class.

Example 2.3. For every integer $n \geq 1$, call $I_n = \left(\frac{1}{n}, \frac{1}{n} + \frac{1}{2^n}\right]$ and define

$$f(t) = \begin{cases} \frac{2^n}{n(n+1)} & \text{if } t \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for every $n \geq 1$,

$$\int_0^{\frac{1}{n}} f(t)\,\mathrm{d}t = \sum_{j=n+1}^{\infty} \int_{I_j} f(t)\,\mathrm{d}t = \sum_{j=n+1}^{\infty} \frac{1}{j(j+1)} = \frac{1}{n+1},$$

and

$$\lambda\big(\left[0, \tfrac{1}{n}\right] \cap \{t : f(t) > 0\}\big) = \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^n}.$$

Fix $h \in (0,1]$. Then $h \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for some $n \geq 1$. Hence,

$$\frac{n}{n+2} = n \int_0^{\frac{1}{n+1}} f(t)\,\mathrm{d}t \leq \frac{1}{h} \int_0^h f(t)\,\mathrm{d}t \leq (n+1) \int_0^{\frac{1}{n}} f(t)\,\mathrm{d}t = 1,$$

and

$$\frac{\lambda\big([0,h] \cap \{t : f(t) > 0\}\big)}{h} \leq (n+1)\,\lambda\big([0, \tfrac{1}{n}] \cap \{t : f(t) > 0\}\big) = \frac{n+1}{2^n}.$$

Since $n \to \infty$ as $h \to 0^+$, from the last two inequalities we conclude that, at zero, f possesses right L-limit equal to 1 and right M-limit equal to 0.

3 Generalized Derivatives.

In the same spirit of the previous section, we provide different generalized notions of differentiability.

Definition 3.1. Let $\ell_1 \in \mathbb{R}$.

(i) f has *C-derivative* equal to ℓ_1 at x if, for some $\ell_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\lambda\left([x-\delta, x+\delta] \cap \left\{t : \left|\frac{f(t) - \ell_0}{t - x} - \ell_1\right| > \varepsilon\right\}\right) = 0.$$

(ii) f has *M-derivative* (or *approximate derivative*) equal to ℓ_1 at x if, for some $\ell_0 \in \mathbb{R}$ and every $\varepsilon > 0$, the set

$$\left\{t : \left|\frac{f(t) - \ell_0}{t - x} - \ell_1\right| > \varepsilon\right\}$$

has zero Lebesgue density at x.

(iii) f has *L-derivative* equal to ℓ_1 at x if, for some $\ell_0 \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{1}{h^2} \int_x^{x+h} \big[f(t) - \ell_0 - \ell_1(t-x)\big]\,\mathrm{d}t = 0.$$

The number ℓ_1 is the (C-, M- and L-, respectively) derivative of f at x, and we write $\ell_1 = C'f(x)$, $\ell_1 = M'f(x)$ and $\ell_1 = L'f(x)$ in the three cases.

Analogous definitions hold for right and left derivatives. It is straightforward to prove the following corollary.

Corollary 3.2. *Assume that f has C- (M-, L-) right (left) derivative ℓ_1 at x. Then f has C- (M-, L-) right (left) limit ℓ_0 at x, where ℓ_0 is the value occurring in the above formulae.*

Remark 3.3. It is worth observing that, assuming that the ratio

$$r(t) = \frac{f(t) - \ell_0}{t - x}$$

is summable, f has L-derivative ℓ_1 at x if and only if r has L-limit ℓ_1 at x. Indeed, it is immediate to see that if r has L-limit at x then f has L-limit equal to ℓ_0 at x. Then, upon setting

$$\psi(h) = \frac{1}{h^2} \int_x^{x+h} \big[f(t) - \ell_0 - \ell_1(t-x)\big]\,\mathrm{d}t \text{ and } \varphi(h) = \frac{1}{h} \int_x^{x+h} \big[r(t) - \ell_1\big]\,\mathrm{d}t,$$

integration by parts yields the relations

$$\psi(h) = \varphi(h) - \frac{1}{h^2} \int_0^h t\varphi(t)\,\mathrm{d}t \text{ and } \varphi(h) = \psi(h) + \frac{1}{h} \int_0^h \psi(t)\,\mathrm{d}t.$$

Hence, $\psi(h) \to 0$ if and only if $\varphi(h) \to 0$, as $h \to 0$.

The picture here is similar to the one we have seen just before. The stronger condition (i) implies both (ii) and (iii), and it amounts to saying that there is a function equivalent to f which is differentiable at x in the usual sense. As in the case of continuity, it is possible to construct the "best differentiable function" $B_d f$. Again, $B_d f$ is not uniquely defined, and it may differ from a particular choice of $B_c f$.

Example 3.4. Write $\mathbb{R} = A \cup B \cup N$, where A and B are sets of Lebesgue density $1/2$ at 0, and N is a countable set with 0 as cluster point. Define

$$f(t) = \begin{cases} t^2 & \text{if } t \in A, \\ 0 & \text{if } t \in B, \\ \sqrt{|t|} & \text{if } t \in N. \end{cases}$$

Then, a possible choice of $B_c f$ is just f, which is not differentiable at 0. To obtain $B_d f$ one must alter f on N, say to $B_d f(t) = 0$ for $t \in N$, for instance.

As should be expected after Remark 3.3, neither the implication (iii) $\Rightarrow$ (ii) nor its converse remain valid in general as illustrated by the following two examples.

Example 3.5. Let

$$f(t) = \begin{cases} 1 & \text{if } \frac{1}{2^n} - \frac{1}{2^{2n+1}} \leq t \leq \frac{1}{2^n}, \quad n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $M'f(0) = 0$, but f is not L-differentiable at 0.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be constructed in the following way. For all $n \in \mathbb{N}$, divide each interval $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ into 4^n subintervals I_{nj} of equal length 8^{-n}, and define

$$f(t) = \begin{cases} 1 & \text{if } t \in I_{nj} \text{ with } j \text{ even}, \\ -1 & \text{if } t \in I_{nj} \text{ with } j \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L'f(0) = 0$, but f is not M-differentiable at 0.

On the other hand, as a consequence of Remark 3.3 and the implications between the M-limit and the L-limit discussed in the previous section, we have the following proposition.

Proposition 3.7. *Let f be M-differentiable at x. If the ratio*

$$r(t) = \frac{f(t) - Mf(x)}{t - x}$$

is essentially bounded in a neighborhood of x, then f is L-differentiable at x, and we have the equality $M'f(x) = L'f(x)$. The analogous statements hold for right and left derivatives.

Remark 3.8. In fact, in the hypotheses of the above proposition, a stronger conclusion holds true. Namely, f is L^1-differentiable at x in the sense of Calderón-Zygmund [1]; that is,

$$\lim_{h \to 0} \frac{1}{h^2} \int_x^{x+h} |f(t) - Mf(x) - M'f(x)(t - x)| \, \mathrm{d}t = 0.$$

Let us prove this fact for the right derivative. Set $M_+ f(x) = \ell_0$ and $M'_+ f(x) = \ell_1$, and assume there exists $k > 0$ such that the function r is essentially bounded by a positive constant K in the interval $[x, x+k]$. Select $\varepsilon > 0$, and put $E = \{t : |r(t) - \ell_1| > \varepsilon\}$. Then, introduce the sets $A_h = [x, x+h] \cap E$ and $B_h = [x, x+h] \cap E^C$. Hence, for $0 < h \leq k$, we have

$$\begin{aligned}\frac{1}{h^2} \int_x^{x+h} |f(t) - \ell_0 - \ell_1(t-x)| \, \mathrm{d}t &\leq \frac{1}{h} \int_{A_h} |r(t) - \ell_1| \, \mathrm{d}t + \frac{1}{h} \int_{B_h} |r(t) - \ell_1| \, \mathrm{d}t \\ &\leq (K + |\ell_1|) \frac{\lambda(A_h)}{h} + \varepsilon.\end{aligned}$$

Since we know that $\lim_{h \to 0^+} \lambda(A_h)/h = 0$ and $\varepsilon > 0$ is arbitrary, taking the limit $h \to 0^+$ we get the claim.

4 Successive L-Derivatives.

We concentrate hereafter on the concept of L-derivative. Firstly, we define the L-derivative of order n.

Definition 4.1. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is n-times L-differentiable at $x \in \mathbb{R}$ if there exist $\ell_0, \ell_1, \ldots, \ell_n \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{1}{h^{n+1}} \int_x^{x+h} \left[f(t) - \sum_{k=0}^{n} \frac{\ell_k}{k!} (t-x)^k \right] \mathrm{d}t = 0.$$

The number ℓ_k is the k^{th} L-derivative of f at x, and we write $\ell_k = L^{(k)} f(x)$. It is readily seen that if such numbers ℓ_k exist, then they are unique.

Indeed, it is clear from the definition that if f is n-times L-differentiable at x, then it is, in particular, k-times L-differentiable at x for all $k \leq n$. Note that the construction of the n^{th} L-derivative of f at x does not require the knowledge (nor the existence) of the L-derivatives of f of order k (for $k < n$) in a neighborhood of x.

Analogous definitions hold for the right derivative $L^{(k)}_+ f(x)$ and the left derivative $L^{(k)}_- f(x)$.

Definition 4.2. The *generalized Taylor sum* of order n relative to f centered at x is the polynomial of order n in the variable t given by

$$\mathcal{T}^f_n(x; t) = \sum_{k=0}^{n} \frac{L^{(k)} f(x)}{k!} (t-x)^k$$

Directly from the definition, the n^{th} L-derivative of f at x can be computed (once the preceding L-derivatives at x are known) by the simple formula

$$L^{(n)}f(x) = \lim_{h\to 0} \frac{(n+1)!}{h^{n+1}} \int_x^{x+h} \big[f(t) - \mathcal{T}^f_{n-1}(x;t)\big]\,\mathrm{d}t.$$

Notation.

We agree to denote by $\mathbb{L}^n[x]$ ($\mathbb{L}^n_+[x]$ and $\mathbb{L}^n_-[x]$, respectively) the subset of $L^1_{\text{loc}}(\mathbb{R})$ of functions which have L-derivatives (right L-derivative and left L-derivative, respectively) at x up to order n.

5 Best Approximating Polynomials.

We assume for the moment that $f \in L^2_{\text{loc}}(\mathbb{R})$ and we construct a family of approximating polynomials on the finite intervals of the real line. Precisely, given $x < y$, we consider the projection operator $P^{x,y}_n : L^2(x,y) \to L^2(x,y)$ onto the subspace of $L^2(x,y)$ consisting of polynomials of order n. Thus, $P^{x,y}_n f$ is uniquely defined by the relation

$$\|f - P^{x,y}_n f\|_{L^2(x,y)} = \min_p \|f - p\|_{L^2(x,y)},$$

where the minimum is taken over all the polynomials p of degree less than or equal to n.

Definition 5.1. We agree to call $P^{x,y}_n f$ the *best approximating polynomial* of order n on the interval $[x,y]$.

In order to compute $P^{x,y}_n f$, it is convenient to consider the orthogonal basis in $L^2(x,y)$ formed by the *generalized Legendre polynomials* (cf. the classical treatise [10], or see [11] for a more concise exposition)

$$G^{x,y}_n(t) = \frac{(n!)^2}{(2n)!} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{(2n-2i)!}{i!(n-i)!(n-2i)!} \Big(\frac{y-x}{2}\Big)^{2i} \Big(t - \frac{y+x}{2}\Big)^{n-2i}, \tag{5.1}$$

for $n \in \mathbb{N}$, where $\lfloor \cdot \rfloor$ denotes the *floor function*. This normalization choice entails the equality

$$G^{x,y}_n(y) = \frac{(n!)^2}{(2n)!}\,(y-x)^n. \tag{5.2}$$

The $G_n^{x,y}$ are obtained by rescaling the Legendre polynomials (which are defined on $[-1,1]$) on the interval $[x,y]$, and they satisfy the normalization condition

$$\|G_n^{x,y}\|^2_{L^2(x,y)} = \frac{(n!)^4}{(2n+1)!(2n)!}(y-x)^{2n+1}. \tag{5.3}$$

Therefore,

$$P_n^{x,y} f(t) = \sum_{k=0}^{n} \beta_k^f(x,y) G_k^{x,y}(t),$$

where

$$\beta_k^f(x,y) = \frac{\langle G_k^{x,y}, f\rangle_{L^2(x,y)}}{\|G_k^{x,y}\|^2_{L^2(x,y)}}. \tag{5.4}$$

We now note that (5.4) makes sense even if we assume $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, provided that we correctly interpret $\langle G_k^{x,y}, f\rangle_{L^2(x,y)}$ as $\int_x^y G_k^{x,y}(t) f(t)\,\mathrm{d}t$. Hence, we can construct the polynomial $P_n^{x,y} f$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ as well, and, with abuse of notation, we keep referring to it as the best approximating polynomial.

The next step is to derive an explicit representation of the coefficient $\beta_k^f(x,y)$. Setting

$$\Gamma_l^{x,y}[f] = \begin{cases} \displaystyle\int_x^y f(t)\,\mathrm{d}t & \text{if } l = 0, \\ \displaystyle\int_x^y \int_x^{t_0} \cdots \int_x^{t_{l-1}} f(t_l)\,\mathrm{d}t_l \cdots \mathrm{d}t_0 & \text{if } l > 0, \end{cases}$$

we have the following proposition.

Proposition 5.2. *For every $k \in \mathbb{N}$,*

$$\beta_k^f(x,y) = \frac{(2k+1)!}{(k!)^2} \sum_{l=0}^{k} (-1)^l \frac{(k+l)!}{l!(k-l)!} \frac{\Gamma_l^{x,y}[f]}{(y-x)^{k+l+1}}. \tag{5.5}$$

The proof of the proposition is based on the following combinatorial identity

$$\sum_{i=0}^{\lfloor \frac{k-l}{2} \rfloor} (-1)^i \frac{(2k-2i)!}{i!(k-i)!(k-2i-l)!} = 2^{k-l} \frac{(k+l)!}{l!(k-l)!}, \quad \forall k \in \mathbb{N},\ \forall l = 0, \dots, k. \tag{5.6}$$

Indeed, the *shifted Legendre polynomials*, defined as $P_k^*(t) = \frac{(2k)!}{(k!)^2} G_k^{0,1}(t)$, have the closed form (see [10, 11])

$$P_k^*(t) = \sum_{l=0}^{k} (-1)^{k+l} \frac{(k+l)!}{(l!)^2 (k-l)!}\, t^l.$$

On the other hand, exploiting (5.1), we have also the equivalent expression

$$\begin{aligned}
P_k^*(t) &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{(2k-2i)!}{i!(k-i)!(k-2i)!4^i} \Big(t - \frac{1}{2}\Big)^{k-2i} \\
&= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{(2k-2i)!}{i!(k-i)!(k-2i)!} \sum_{l=0}^{k-2i} (-1)^{k+l} \frac{(k-2i)!}{l!(k-2i-l)!2^{k-l}}\, t^l \\
&= \sum_{l=0}^{k} (-1)^{k+l} \frac{1}{l!2^{k-l}} \sum_{i=0}^{\lfloor \frac{k-l}{2} \rfloor} (-1)^i \frac{(2k-2i)!}{i!(k-i)!(k-2i-l)!}\, t^l,
\end{aligned}$$

after a change of the summation order. Hence, (5.6) follows comparing the coefficients of each term t^l appearing in P_k^*.

Remark 5.3. Arguing in a similar fashion, $G_n^{0,y}$ (with $y > 0$) can be equivalently written as

$$G_n^{0,y}(t) = \frac{(n!)^2}{(2n)!} \sum_{l=0}^{n} (-y)^{n-l} \frac{(n+l)!}{(l!)^2 (n-l)!}\, t^l.$$

We can now proceed to the proof of Proposition 5.2.

Proof of Proposition 5.2. Exploiting (5.1) and (5.3)-(5.4), we are led to the equality

$$\begin{aligned}
\beta_k^f(x,y) =& \left(\frac{(2k+1)!}{(k!)^2} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^i\,(2k-2i)!}{4^i\, i!(k-i)!(k-2i)!} (y-x)^{2i-2k-1} \right) \\
&\times \left(\int_x^y \Big(t - \frac{y+x}{2}\Big)^{k-2i} f(t)\, \mathrm{d}t. \right)
\end{aligned}$$

Integration by parts $(k-2i)$-times yields

$$\int_x^y \Big(t - \frac{y+x}{2}\Big)^{k-2i} f(t)\, \mathrm{d}t = \sum_{l=0}^{k-2i} (-1)^l \frac{(k-2i)!}{(k-2i-l)!} \Big(\frac{y-x}{2}\Big)^{k-2i-l} \Gamma_l^{x,y}[f],$$

so that

$$\beta_k^f(x,y) = \frac{(2k+1)!}{2^k\,(k!)^2}\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}(-1)^i\frac{(2k-2i)!}{i!(k-i)!}\sum_{l=0}^{k-2i}(-1)^l\frac{2^l\,\Gamma_l^{x,y}[f]}{(k-2i-l)!(y-x)^{k+l+1}}.$$

Changing the summation order, this is equivalent to

$$\beta_k^f(x,y) = \frac{(2k+1)!}{2^k\,(k!)^2}\sum_{l=0}^{k}(-1)^l\frac{2^l\,\Gamma_l^{x,y}[f]}{(y-x)^{k+l+1}}\sum_{i=0}^{\lfloor \frac{k-l}{2}\rfloor}(-1)^i\frac{(2k-2i)!}{i!(k-i)!(k-2i-l)!},$$

and the conclusion follows from (5.6). □

6 Approximating Polynomials and L-Derivatives.

Let us investigate the relationships between the best approximating polynomials and the L-derivatives of a given function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. We begin with the following result on the limiting behavior of the coefficients $\beta_k^f(x,y)$.

Theorem 6.1. *Let $f \in \mathbb{L}^n_+[x]$ for some $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, for every $k \in \{0,\dots,n\}$ and every $h > 0$,*

$$\beta_k^f(x,x+h) = \frac{(2k+1)!}{(k!)^2}\sum_{m=0}^{n-k}\frac{(m+k)!}{m!(m+2k+1)!}L_+^{(m+k)}f(x)h^m + o(h^{n-k}).$$

Analogously, if $f \in \mathbb{L}^n_-[x]$,

$$\beta_k^f(x-h,x) = \frac{(2k+1)!}{(k!)^2}\sum_{m=0}^{n-k}\frac{(m+k)!}{m!(m+2k+1)!}L_-^{(m+k)}f(x)(-h)^m + o(h^{n-k}).$$

A particular instance of the above theorem follows.

Corollary 6.2. *If $f \in \mathbb{L}^n[x]$ for some $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then*

$$\lim_{h\to 0^+}\beta_n^f(x,x+h) = \lim_{h\to 0^+}\beta_n^f(x-h,x) = \frac{L^{(n)}f(x)}{n!}.$$

Remark 6.3. If $f \in L^p_{\mathrm{loc}}(\mathbb{R})$, for some $p \geq 1$, from the straightforward inequality

$$\left|\frac{1}{h}\int_x^{x+h}\left[f(t)-\mathcal{T}_n^f(x;t)\right]\mathrm{d}t\right| \leq \left(\frac{1}{h}\int_x^{x+h}\left|f(t)-\mathcal{T}_n^f(x;t)\right|^p\mathrm{d}t\right)^{1/p}$$

we see that if f has L^p-derivative of order n in the sense of Calderón-Zygmund [1] (see also [2]), then f has L-derivative of order n. Thus, the existence of the left and right limits in Corollary 6.2 is a necessary condition in order for f to have L^p-derivative of order n. In that case, the value of the limit is the coefficient of the Taylor sum relative to the L^p-derivative.

The proof of Theorem 6.1 requires the following lemma.

Lemma 6.4. *For every $k, j \in \mathbb{N}$,*

$$\sum_{l=0}^{k}(-1)^l \frac{(k+l)!}{l!(k-l)!(j+l+1)!} = \begin{cases} 0 & \text{if } j < k, \\ \frac{j!}{(j-k)!(k+j+1)!} & \text{if } j \geq k. \end{cases} \tag{6.1}$$

Proof. Note that (6.1) can be equivalently written as the binomial identity

$$\sum_{l=0}^{k}(-1)^l \binom{k+l}{l}\binom{k+j+1}{k-l} = \binom{j}{k},$$

which is just a particular instance of the more general identity

$$\sum_{l\in\mathbb{Z}}(-1)^l \binom{p+l}{n+l}\binom{q}{k-l} = (-1)^n \binom{q+n-p-1}{n+k}, \ \forall n,k\in\mathbb{N}, \ \forall p,q\in\mathbb{R}.$$

To prove that, we use the relation

$$(-1)^l \binom{p+l}{n+l} = (-1)^n \binom{n-p-1}{n+l},$$

obtained by negating the upper index. Hence, the above sum turns into

$$(-1)^n \sum_{l\in\mathbb{Z}} \binom{n-p-1}{n+l}\binom{q}{k-l}.$$

This is a *Vandermonde's convolution* (see e.g. §5 of [6]), whose result is exactly the right-hand side of the identity to be proved. □

Remark 6.5. Here is an interesting application of Lemma 6.4. If we consider the shifted Legendre polynomials $P_k^*(t)$, collecting (5.5) and (6.1) we have

$$t^j = \sum_{k=0}^{j} \frac{(2k+1)(j!)^2}{(j-k)!(k+j+1)!} P_k^*(t),$$

for every $t \in [0,1]$ and every $j \in \mathbb{N}$.

Proof of Theorem 6.1. Let $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $h > 0$. We will limit ourselves to give the proof of the asymptotic expansion of $\beta_k^f(x, x+h)$ (the other case is analogous and left to the reader). For a fixed $k \in \{0, \dots, n\}$, using (5.5), we derive the equality

$$\beta_k^f(x, x+h) = \frac{(2k+1)!}{(k!)^2} \sum_{l=0}^{k} (-1)^l \frac{(k+l)!}{l!(k-l)!} \frac{\Gamma_l^{x,y}[f]}{h^{k+l+1}} = \Lambda_1 + \Lambda_2,$$

where

$$\Lambda_1 = \frac{(2k+1)!}{(k!)^2} \sum_{j=0}^{n} L_+^{(j)} f(x)\, h^{j-k} \sum_{l=0}^{k} (-1)^l \frac{(k+l)!}{l!(k-l)!(j+l+1)!}$$

and

$$\Lambda_2 = \frac{(2k+1)!}{(k!)^2} \sum_{l=0}^{k} (-1)^l \frac{(k+l)!}{l!(k-l)!} \frac{\Gamma_l^{x,x+h}[f - \mathcal{T}_n^f(x)]}{h^{k+l+1}}.$$

Here, we used the facts that

$$\Gamma_l^{x,x+h}[f] = \Gamma_l^{x,x+h}[\mathcal{T}_n^f(x)] + \Gamma_l^{x,x+h}[f - \mathcal{T}_n^f(x)]$$

and

$$\Gamma_l^{x,x+h}[\mathcal{T}_n^f(x)] = \sum_{j=0}^{n} \frac{L_+^{(j)} f(x)}{(j+l+1)!} h^{j+l+1}.$$

Exploiting (6.1), we obtain

$$\begin{aligned}
\Lambda_1 &= \frac{(2k+1)!}{(k!)^2} \sum_{j=k}^{n} \frac{j!}{(j-k)!(k+j+1)!} L_+^{(j)} f(x)\, h^{j-k} \\
&= \frac{(2k+1)!}{(k!)^2} \sum_{m=0}^{n-k} \frac{(m+k)!}{m!(m+2k+1)!} L_+^{(k+m)} f(x)\, h^m.
\end{aligned}$$

Hence, we are left to show that $\Lambda_2 = o(h^{n-k})$. Indeed, for every $l \in \{0, \dots, k\}$, using the L'Hospital rule l-times, and recalling that $f \in \mathbb{L}_+^n[x]$, we have

$$\begin{aligned}
&\lim_{h \to 0^+} \frac{1}{h^{n-k}} \frac{\Gamma_l^{x,x+h}[f - \mathcal{T}_n^f(x)]}{h^{k+l+1}} \\
&= \cdots \\
&= \frac{(n+1)!}{(n+l+1)!} \lim_{h \to 0^+} \frac{1}{h^{n+1}} \int_x^{x+h} \left[f(t) - \mathcal{T}_n^f(x;t) \right] \mathrm{d}t = 0.
\end{aligned}$$

The proof is completed. □

We are now ready to prove the main result of this section, namely, to show how the best approximating polynomial of order n of a given function f is related to the generalized Taylor sum $\mathcal{T}_n^f$. To this end, for $x \in \mathbb{R}$, we introduce the function

$$\mathcal{P}_n^f(x;t) = \begin{cases} P_n^{x,t} f(t) & \text{if } t > x, \\ P_n^{t,x} f(t) & \text{if } t < x; \end{cases}$$

that is, the best approximating polynomial of f on the interval $[x,t]$ (or $[t,x]$ if $t < x$) calculated at the right (or left) endpoint. Then, we have the following theorem.

Theorem 6.6. *Let $f \in \mathbb{L}^n[x]$ for some $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $\mathcal{P}_n^f(x;t)$ is defined by continuity at $t = x$, and*

$$\mathcal{T}_n^f(x;t) = \mathcal{P}_n^f(x;t) + o(|t-x|^n),$$

for every $t \in \mathbb{R}$.

Proof. We assume that $t > x$, the argument for $t < x$ being the same. Hence, from (5.2),

$$\mathcal{P}_n^f(x;t) = \sum_{k=0}^{n} \beta_k^f(x,t) G_k^{x,t}(t) = \sum_{k=0}^{n} \frac{(k!)^2}{(2k)!} \beta_k^f(x,t)(t-x)^k.$$

On account of the explicit representation of the coefficients $\beta_k^f(x,t)$, provided by Theorem 6.1, we obtain

$$\mathcal{P}_n^f(x;t) = \sum_{k=0}^{n} (2k+1) \sum_{m=0}^{n-k} \frac{(m+k)!}{m!(m+2k+1)!} L^{(m+k)} f(x)(t-x)^{m+k} + o(t-x)^n.$$

Setting $r = m + k$, and grouping the terms corresponding to the same r, we get

$$\begin{aligned} \mathcal{P}_n^f(x;t) &= \sum_{r=0}^{n} r! \sum_{k=0}^{r} \frac{(2k+1)}{(r-k)!(r+k+1)!} L^{(r)} f(x)(t-x)^r + o(t-x)^n \\ &= \sum_{r=0}^{n} \frac{L^{(r)} f(x)}{r!} (t-x)^r + o(t-x)^n = \mathcal{T}_n^f(x;t) + o(t-x)^n. \end{aligned}$$

Indeed, setting $A_k = \frac{1}{(r-k)!(r+k)!}$, we have

$$\sum_{k=0}^{r} \frac{(2k+1)}{(r-k)!(r+k+1)!} = A_r + \sum_{k=0}^{r-1} \big[A_k - A_{k+1}\big] = A_0 = \frac{1}{(r!)^2},$$

which finishes the proof. □

Remark 6.7. The same result clearly holds if we only require $f \in \mathbb{L}^n_+[x]$ ($f \in \mathbb{L}^n_-[x]$, respectively), replacing $t \in \mathbb{R}$ with $t \geq x$ ($t \leq x$, respectively).

7 A Converse Result.

In this final section, we prove the converse of Corollary 6.2. This entails the following necessary and sufficient condition for L-differentiability of $L^1_{\mathrm{loc}}(\mathbb{R})$ functions.

Theorem 7.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then, the following are equivalent:*

(i) $f \in \mathbb{L}^n[x]$ for some $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(ii) There exist $\ell_0, \ldots, \ell_n \in \mathbb{R}$ such that

$$\lim_{h\to 0^+} \beta_k^f(x, x+h) = \lim_{h\to 0^+} \beta_k^f(x-h, x) = \frac{\ell_k}{k!},$$

for all $k = 0, \ldots, n$.

In that case, $L^{(k)}f(x) = \ell_k$, for $k = 0, \ldots, n$.

Remark 7.2. If in (ii) we ask the existence of the first (second) limit only, then the conclusion of the theorem holds replacing $f \in \mathbb{L}^n[x]$ with $f \in \mathbb{L}^n_+[x]$ ($f \in \mathbb{L}^n_-[x]$) in (i). In which case, the ℓ_k will be the right (left) successive derivatives.

The implication (i) $\Rightarrow$ (ii) is the content of Corollary 6.2. Thus, we have just to prove the reverse implication (ii) $\Rightarrow$ (i). To this end, the main ingredient needed is the following Tauberian result.

Theorem 7.3. *Let ψ be a function defined in a neighborhood of $x \in \mathbb{R}$, continuous except perhaps at x. Further, assume that*

$$\lim_{h\to 0} h\psi(x+h) = 0. \tag{7.1}$$

For $q \geq l \in \mathbb{N}$, let

$$J_{l,q}(h) = \begin{cases} h^{q+1}\psi(x+h) & \text{if } l = 0, \\ \displaystyle\int_x^{x+h} (t-x)^{q+1}\psi(t)\,\mathrm{d}t & \text{if } l = 1, \\ \displaystyle\int_x^{x+h}\int_x^{t_1}\cdots\int_x^{t_{l-1}} (t_l - x)^{q+1}\psi(t_l)\,\mathrm{d}t_l\cdots\mathrm{d}t_1 & \text{if } l > 1. \end{cases}$$

Assume that, for some given $q \geq n \in \mathbb{N}$,

$$\lim_{h\to 0} \sum_{l=0}^{n} (-1)^l \frac{(n+l)!}{l!(n-l)!} \frac{J_{l,q}(h)}{h^{q+l+1}} = 0. \tag{7.2}$$

Then, it follows that

$$\lim_{h\to 0} \psi(x+h) = 0.$$

The same statement holds replacing $h \to 0$ with $h \to 0^+$ (or $h \to 0^-$), whenever this limit occurs.

Proof. The proof is by induction on n. The case $n = 0$ is trivially satisfied. Thus, assume that the theorem holds for $n-1$, for some $n > 0$. Let then $q \geq n > 0$ be fixed. We set

$$B_{l,n} = \frac{(n+l)!}{l!(n-l)!}, \ l = 0, \dots, n$$

and

$$D_{l,n} = \frac{(n+l)!}{l!(n-l-1)!n}, \ l = 0, \dots, n-1.$$

By straightforward calculations,

$$B_{l,n} = \begin{cases} D_{0,n} & \text{if } l = 0, \\ (n+l)D_{l-1,n} + D_{l,n} & \text{if } l = 1, \dots, n-1, \\ 2nD_{n-1,n} & \text{if } l = n. \end{cases}$$

Observe now that

$$D_{l-1,n} \frac{\mathrm{d}}{\mathrm{d}h} \frac{J_{l,q}(h)}{h^{n+l}} = D_{l-1,n} \frac{J_{l-1,q}(h)}{h^{n+l}} - (n+l) D_{l-1,n} \frac{J_{l,q}(h)}{h^{n+l+1}},$$

which can be equivalently written as

$$D_{l-1,n} \frac{\mathrm{d}}{\mathrm{d}h} \frac{J_{l,q}(h)}{h^{n+l}} = \begin{cases} D_{l-1,n} \dfrac{J_{l-1,q}(h)}{h^{n+l}} + (D_{l,n} - B_{l,n}) \dfrac{J_{l,q}(h)}{h^{n+l+1}} & \text{if } l = 1, \dots, n-1, \\ D_{n-1,n} \dfrac{J_{n-1,q}(h)}{h^{2n}} - B_{n,n} \dfrac{J_{n,q}(h)}{h^{2n+1}} & \text{if } l = n. \end{cases}$$

Hence, recalling that $D_{0,n} = B_{0,n}$, we learn that

$$\frac{\mathrm{d}}{\mathrm{d}h} \sum_{l=1}^{n} (-1)^{l-1} D_{l-1,n} \frac{J_{l,q}(h)}{h^{n+l}} = \sum_{l=0}^{n} (-1)^l B_{l,n} \frac{J_{l,q}(h)}{h^{n+l+1}}.$$

Due to (7.1), we have that $J_{l,q}(h) = o(h^{q+l})$ as $h \to 0$. Since $q \ge n$, this implies that

$$\lim_{h\to 0} \sum_{l=1}^{n} (-1)^{l-1} D_{l-1,n} \frac{J_{l,q}(h)}{h^{n+l}} = 0.$$

Therefore, we are in a position to apply the L'Hospital rule to obtain

$$\begin{aligned}\lim_{h\to 0} \sum_{l=1}^{n} (-1)^{l-1} D_{l-1,n} \frac{J_{l,q}(h)}{h^{q+l+1}} &= \lim_{h\to 0} \frac{1}{h^{q-n+1}} \sum_{l=1}^{n} (-1)^{l-1} D_{l-1,n} \frac{J_{l,q}(h)}{h^{n+l}} \\ &= \lim_{h\to 0} \frac{1}{q-n+1} \sum_{l=0}^{n} (-1)^{l} B_{l,n} \frac{J_{l,q}(h)}{h^{q+l+1}} = 0,\end{aligned}$$

thanks to (7.2). Hence, using (7.2) again,

$$\lim_{h\to 0} \left[\sum_{l=0}^{n} (-1)^{l} B_{l,n} \frac{J_{l,q}(h)}{h^{q+l+1}} + 2n \sum_{l=1}^{n} (-1)^{l-1} D_{l-1,n} \frac{J_{l,q}(h)}{h^{q+l+1}} \right] = 0.$$

The expression between the brackets can be written more conveniently as

$$\psi(h) + \sum_{l=1}^{n} (-1)^{l} (B_{l,n} - 2nD_{l-1,n}) \frac{J_{l,q}(h)}{h^{q+l+1}} = \sum_{l=0}^{n-1} (-1)^{l} B_{l,n-1} \frac{J_{l,q}(h)}{h^{q+l+1}}.$$

Indeed, for $l > 0$,

$$B_{l,n} - 2nD_{l-1,n} = \frac{(n+l-1)!(n-l)}{l!(n-l)!} = \begin{cases} B_{l,n-1} & \text{if } l = 1, \dots, n-1, \\ 0 & \text{if } l = n. \end{cases}$$

In conclusion,

$$\lim_{h\to 0} \sum_{l=0}^{n-1} (-1)^{l} B_{l,n-1} \frac{J_{l,q}(h)}{h^{q+l+1}} = 0,$$

which, in light of the inductive hypotheses, yields the desired result. Obviously, the same argument applies if we consider right or left limits only. □

Proof of Theorem 7.1. We restrict ourselves to proving the right L-differentiability (the other case being completely analogous). We prove the implication (ii) $\Rightarrow$ (i) by induction on n. For $n = 0$, it is trivially true. Assume then that the result holds for $n - 1$, for some $n \ge 1$. Set

$$\widehat{\mathcal{T}}_n(x;t) = \sum_{j=0}^{n} \frac{\ell_j}{j!} (t-x)^j,$$

and introduce the function

$$\psi(\tau) = \frac{1}{(\tau - x)^{n+1}} \int_x^\tau \big[f(t) - \widehat{\mathcal{T}}_n(x;t)\big]\,\mathrm{d}t.$$

Note that, by the inductive hypothesis,

$$\lim_{h\to 0^+} h\psi(x+h) = 0. \tag{7.3}$$

Proving the theorem amounts to showing that

$$\lim_{h\to 0^+} \psi(x+h) = 0. \tag{7.4}$$

Arguing as in the proof of Theorem 6.1, we split the coefficient $\beta_n^f(x, x+h)$ into the sum

$$\beta_n^f(x, x+h) = \Lambda_1' + \Lambda_2',$$

where

$$\Lambda_1' = \frac{(2n+1)!}{(n!)^2} \sum_{j=0}^n \ell_j h^{j-n} \sum_{l=0}^n (-1)^l \frac{(n+l)!}{l!(n-l)!(j+l+1)!}$$

and

$$\Lambda_2' = \frac{(2n+1)!}{(n!)^2} \sum_{l=0}^n (-1)^l \frac{(n+l)!}{l!(n-l)!} \frac{\Gamma_l^{x,x+h}[f - \widehat{\mathcal{T}}_n(x)]}{h^{n+l+1}}.$$

By (6.1), it is readily seen that $\Lambda_1' = \frac{\ell_n}{n!}$, whereas, with the notation of Theorem 7.1, we have

$$\Gamma_l^{x,x+h}[f - \widehat{\mathcal{T}}_n(x)] = J_{l,n}(h),$$

so that

$$\Lambda_2' = \frac{(2n+1)!}{(n!)^2} \sum_{l=0}^n (-1)^l \frac{(n+l)!}{l!(n-l)!} \frac{J_{l,n}(h)}{h^{n+l+1}}$$

Therefore, the hypothesis

$$\lim_{h\to 0^+} \beta_n(x, x+h) = \frac{\ell_n}{n!}$$

is in fact equivalent to

$$\lim_{h\to 0^+} \sum_{l=0}^n (-1)^l \frac{(n+l)!}{l!(n-l)!} \frac{J_{l,n}(h)}{h^{n+l+1}} = 0. \tag{7.5}$$

In view of (7.3) and (7.5), we can apply Theorem 7.3 with $q = n$, thereby obtaining the desired conclusion (7.4). $\square$

Acknowledgment.

This work was conceived while the third author was visiting the University of Paris VII, which is gratefully acknowledged for its kind hospitality.

References

[1] A. P. Calderón, A. Zygmund, *Local Properties of Solutions of Elliptic Partial Differential Equations*, Studia Math., **20** (1961), 171–225.

[2] M. J. Evans, L_p *Derivatives and Approximate Peano Derivatives*, Trans. Amer. Math. Soc., **165** (1972), 381–388.

[3] M. J. Evans, P. D. Humke, *Approximate Continuity Points and L-points of Integrable Functions*, Real Anal. Exchange, **11** (1985/86), 390–410.

[4] H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin-Heidelberg, 1996.

[5] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, Providence, RI, 1994.

[6] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, 1990.

[7] M. Gromov, *Metric Structures for Riemannian and non-Riemannian Spaces. Based on the 1981 French Original*, Progress in Mathematics, 152, Birkhäuser Boston, Inc., Boston, MA, 1999.

[8] R. Kannan, C. K. Krueger, *Advanced Analysis on the Real Line*, Springer-Verlag, New York, 1996.

[9] V. Pata, A. Villani, *Some Measurability and Continuity Properties of Arbitrary Real Functions*, Matematiche (Catania), **57** (2004), 63–82.

[10] G. Szegö, *Orthogonal Polynomials, 4th edition*, American Mathematical Society, Providence, RI, 1975.

[11] `http://mathworld.wolfram.com/LegendrePolynomial.html`