RESEARCH

Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 79–86

Robert Menkyna, Department of Engineering Fundamentals, Faculty of Electrical Engineering, University of Žilina, Workplace Liptovský Mikuláš, Slovakia. email: menkyna@lm.utc.sk

ON A SPECIAL SUBCLASS OF THE SET OF DERIVATIVES

Abstract

We deal with the class of functions defined as a sum of a uniformly convergent series of functions continuous both on a closed set and on its complement. Such functions are mentioned in the literature, e.g., in [1], [2], [3], [4]. We investigate the particular class of derivatives.

We deal with classes of real functions defined on the interval $(0,1)$. As usual the symbols C, D, B_1, Δ, and A stand for the class of continuous, Darboux, Baire 1 functions, functions that are derivatives or approximately continuous functions, respectively.

Consider the following three properties of a function f on $(0,1)$.

(*) There exists a closed set $A \subset (0,1)$ such that $f\restriction_A$ and $f\restriction_{\sim A}$ are continuous;

(**) There exists a sequence of functions $f_n \in \mathcal{F}(C)$, $n = 1, 2, \ldots$, such that the series $\sum\limits_{n=1}^{\infty} f_n$ uniformly converges to f;

(***) There exists a closed set $A \subset (0,1)$ such that $f\restriction_A = 0$ and $f\restriction_{\sim A}$ is continuous.

Definition 1. Let $\mathcal{F}$ be a subclass of B_1. Let $\mathcal{F}(C) = \{f \in \mathcal{F}, f \text{ satisfies } (*)\}$.

Remark 2. In Definition 1, it suffices to consider nowhere dense sets A.

Remark 3. Evidently, $D(C) \subset DB_1$.

Definition 4. Let $\mathcal{F}$ be a subclass of B_1 such that $\mathcal{F} + \mathcal{F} \subset \mathcal{F}$ and $\mathcal{F}$ with the metric of uniform convergence is closed. Let $\sigma\mathcal{F}(C) = \{f \in \mathcal{F};\ f$ satisfies $(**)\}$.

Key Words: derivative, uniformly convergent series of functions
Mathematical Reviews subject classification: 26A15, 26A21
Received by the editors November 13, 2005
Communicated by: B. S. Thomson

Remark 5. Because $\Delta + \Delta \subset \Delta$ and $\Delta\,[unif]$ is closed, the definition of $\sigma\Delta(C) \subset \Delta$ is correct.

The main result of the present paper is the following theorem.

Theorem 6. *Consider Δ furnished with the metric of uniform convergence. Then, $\sigma\Delta(C)$ is a closed nowhere dense set in the space Δ.*

First of all we show that $\sigma\Delta(C) \subsetneqq \Delta$.

Lemma 7. *Let f_n, $n = 1, 2, \ldots$, be functions in $\mathcal{D}(C)$ such that the partial sums $s_k = \sum\limits_{n=1}^{k} f_n$, $k = 1, 2, \ldots$, belong to D and the series $\sum\limits_{n=1}^{\infty} f_n$ uniformly converges to the function f. Then, for each pair of real numbers α and β, $\alpha < \beta$, and for every open interval $I \subset (0,1)$, if $f^{-1}(\alpha, \beta) \cap I \neq \emptyset$, then there exists an interval $J \subset I$ such that $f(J) \subset (\alpha, \beta)$.*

Proof. Let $x_0 \in f^{-1}(\alpha, \beta) \cap I$. Without loss of generality, we may assume that $f(x_0) = 0$ and $x_0 \in f^{-1}(-\alpha, \alpha) \cap I$. The series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly to f. Hence for $\varepsilon = \frac{\alpha}{2}$ there is $k(\varepsilon) \in N$ such that $\left| f(x) - \sum\limits_{n=1}^{k} f_n \right| < \varepsilon$ for every $k > k(\varepsilon)$, $x \in I$. Take a fixed integer $k > k(\varepsilon)$. We show that there exists a point x_0^* of continuity of the function $s_k = \sum\limits_{n=1}^{k} f_n$ for which $|\, s_k(x_0^*)| < \varepsilon$. Let $A_n, n = 1, 2, \ldots$, be closed nowhere dense sets such that $f_n\restriction_{A_n}$, $f_n\restriction_{\sim A_n}$ are continuous functions and let

$$x_0 \in A_n \text{ for } n = 1, 2, \ldots, m_0,$$

$$x_0 \notin A_n \text{ for } n = m_0 + 1, \ldots, k.$$

Let

$$f^{10} = \sum_{n=1}^{m_0} f_n, \quad f^{20} = \sum_{n=m_0+1}^{k} f_n.$$

Since the functions $f_n\restriction_{A_n}, n = 1, 2, \ldots, m_0$, and f^{20} are continuous at x_0, for positive real numbers λ_0, ζ_0, $\lambda_0 + \zeta_0 + |\, s_k(x_0)| < \varepsilon$, there exists a neighborhood $O(x_0) \subset I$ of x_0 such that

$$|f_n(x) - f_n(x_0)| < \frac{\lambda_0}{m_0} \text{ for every } x \in A_n \cap O(x_0), n = 1, 2, \ldots, m_0,$$

$$\left|\, f^{20}(x) - \; f^{20}(x_0)\right| < \zeta_0 \text{ for every } x \in O(x_0)$$

and $\bigcup_{n=m_0+1}^{k} A_n \cap O(x_0) = \emptyset$. If $|\, s_k(x)| < \varepsilon$ for every $x \in O(x_0)$, then there exists $x_1 \in O(x_0) \setminus \bigcup_{n=1}^{k} A_n$. The function s_k is continuous at the point $x_0^* = x_1$ and $|s_k(x_0^*)| < \varepsilon$. Otherwise, $|s_k(x_1^*)| \geq \varepsilon$ for any $x_1^* \in O(x_0)$. The function s_k has the Darboux property. Hence for certain x_1 lying between x_0 and x_1^*, we have

$$\lambda_0 + \zeta_0 + |s_k(x_0)| < |s_k(x_1)| < \varepsilon \leq |s_k(x_1^*)|.$$

With a suitable change of subscripts, we get

$$\begin{aligned} &x_1 \in A_n \text{ for } n = 1, 2, \ldots, m_1, \\ &x_1 \notin A_n \text{ for } n = m_1 + 1, \ldots, k. \end{aligned}$$

Evidently, $m_1 \leq m_0$. Equality $m_1 = m_0$ leads to the contradiction of the selection of x_1, because in this case

$$\begin{aligned} |s_k(x_1)| &= \left|f^{10}(x_1) + f^{20}(x_1)\right| \\ &\leq \sum_{n=1}^{m_0} |f_n(x_1) - f_n(x_0)| + \left|f^{20}(x_1) - f^{20}(x_0)\right| + |s_k(x_0)| \\ &< \lambda_0 + \zeta_0 + |s_k(x_0)|. \end{aligned}$$

That is, $m_1 < m_0$. Now, we shall repeat the procedure. Let

$$f^{11} = \sum_{n=1}^{m_1} f_n, \quad f^{21} = \sum_{n=m_1+1}^{k} f_n$$

and let λ_1, ζ_1 be positive real numbers, $\lambda_1 + \zeta_1 + |\, s_k(x_1)| < \varepsilon$, and let $O(x_1)$ be a neighborhood of x_1, $O(x_1) \subset O(x_0)$, such that

$$|f_n(x) - f_n(x_1)| < \frac{\lambda_1}{m_1} \text{ for every } x \in A_n \cap O(x_1), \; n = 1, 2, \ldots, m_1,$$

$$\left|f^{21}(x) - f^{21}(x_1)\right| < \zeta_1 \text{ for every } x \in O(x_1)$$

and $\bigcup_{n=m_1+1}^{k} A_n \cap O(x_1) = \varnothing$. Again, if $|s_k(x)| < \varepsilon$ for all $x \in O(x_1)$, then there is $x_2 \in O(x_1) \setminus \bigcup_{n=1}^{k} A_n$. The function s_k is continuous at the point

$x_0^* = x_2$ and $|s_k(x_0^*)| < \varepsilon$. In the opposite case, we can analogously as above find x_2, such that $|s_k(x_2)| < \varepsilon$,

$$x_2 \in A_n \text{ for } n = 1, 2, \ldots, m_2,$$

$$x_2 \notin A_n \text{ for } n = m_2 + 1, \ldots, k$$

and moreover, $m_2 < m_1$. Continuing this way, after a finite number of steps, we shall find $x_0^* \in I$ such that $x_0^* \notin \bigcup_{n=1}^{k} A_n$ and $|s_k(x_0^*)| < \varepsilon$. From the continuity of the functions s_k at the point x_0^*, it follows that there exists an interval $J \subset I$ such that $x_0^* \in J$ and $|s_k(x)| < \varepsilon$ for every $x \in J$. From there,

$$|f(x)| \leq |f(x) - s_k(x)| + |s_k(x)| < \varepsilon + \varepsilon = \alpha \text{ for every } x \in J,$$

and then $J \subset f^{-1}(-\alpha, \alpha)$. □

Example 8. *Let K be a perfect, nowhere dense subset of the interval $(0, 1)$ of positive Lebesgue measure, $\lambda(K) > 0$, and let E be a subset of K such that E is of type F_σ and the density $d(x, E) = 1$ for all $x \in E$. Then, from [1] Theorem 6.5. we get the existence of a function $f \in bA$ such that*

$$\begin{aligned} 0 < &f(x) \leq 1 \quad \text{for all } x \in E \\ &f(x) = 0 \quad \text{for all } x \notin E. \end{aligned}$$

From the inclusion $bA \subset b\Delta$, it follows that $f \in \Delta$, but immediately from Lemma 7, $f \notin \sigma\Delta(C)$. Indeed, for any $0 < \alpha < \beta \leq 1$, the set $f^{-1}((\alpha, \beta)) \subset K$ is nonempty and nowhere dense.

Next, we prove that $\sigma\Delta(C)$ is closed in the space Δ.

Definition 9. Define $\Delta^0(C) = \{f \in \Delta;\ f \text{ satisfies } (***)\}$.

Lemma 10. *Let $f_i \in \Delta(C), i = 1, 2, \ldots, n$. If $\left|\sum_{i=1}^{n} f_i\right| < \varepsilon$, then for every $\delta > 0$ there exists an open set U and a sequence of functions $g_0 \in C$, $g_1, \ldots, g_n \in \Delta^0(C)$ such that $\lambda(U) < \delta$ and*

(a) $\sum_{i=1}^{n} f_i = \sum_{i=0}^{n} g_i$,

(b) $\left|\sum_{i=0}^{k} g_i\right| < \varepsilon$ *for every* $k = 0, 1, \ldots, n$,

(c) $g_i(x) = 0$ *for every* $x \notin U$, $\quad i = 1, \ldots, n$.

Proof. In the proof of the lemma, we use the induction principle. Let $f_1 \in \Delta(C)$, $|f_1| < \varepsilon$ and let the functions $f_1 \restriction_{A_1}$ and $f_1 \restriction_{\sim A_1}$ be continuous, where A_1 is a closed set. Choose an open set $V \supset A_1$ such that $\lambda(V \setminus A_1) < \delta$ and let $U = V \setminus A_1$. Since $f_1 \restriction_{\sim U}$ is continuous, according to Tietze's extension theorem there is a continuous function g_0 defined on $(0,1)$ such that $|g_0| < \varepsilon$, $g_0 \restriction_{\sim U} = f_1 \restriction_{\sim U}$. Then $g_1 = f_1 - g_0 \in \Delta$, $g_1 \restriction_{\sim U} = 0$ and $g_1 \restriction_U$ is continuous; that is, $g_1 \in \Delta^0(C)$ and conditions $(a), (b), (c)$ are true.

Now let the assertion of the lemma hold for an arbitrary sum of $n-1$ functions. We show the validity of the lemma for an arbitrary sum of n functions. Assume $f_i \in \Delta(C)$, $i = 1, 2, \ldots, n$, $\left|\sum_{i=1}^{n} f_i\right| < \varepsilon$, and let the closed set A_i correspond to the function f_i in the sense of the definition of $\Delta(C)$. Let $J_k = (a_k, b_k)$, $k = 1, 2, \ldots$, be the sequence of contiguous intervals of the closed set $A = \bigcap_{i=1}^{n} A_i$. On every interval J_k, we construct a decreasing sequence $x_k^j \searrow a_k$ and an increasing sequence $y_k^j \nearrow b_k$, $j = 1, 2, \ldots$, such that x_k^j, $y_k^j \notin \bigcup_{i=1}^{n} A_i$ and $x_k^1 < y_k^1$. We can require for the sequence of intervals I_k^j, $j = 1, 2, \ldots$, generated from intervals $\langle x_k^{j+1}, x_k^j \rangle$, $\langle x_k^1, y_k^1 \rangle$, $\langle y_k^j, y_k^{j+1} \rangle$ that for every $j = 1, 2, \ldots$, there exists at least one A_i such that $I_k^j \cap A_i = \emptyset$. That means that on every interval I_k^j, at least one function f_i is continuous. Therefore, the sum $\sum_{i=1}^{n} f_i$ can be expressed on every interval I_k^j as a sum of $n-1$ functions from $\Delta(C)$. According to (inductive hypothesis) the assumption, there exists an open set $V_k^j \subset I_k^j$ and a sequence of functions $h_1 \in C$, $h_2, h_3, \ldots, h_n \in \Delta^0(C)$ such that $\lambda\left(V_k^j\right)$ is sufficiently small and $\sum_{i=1}^{n} f_i = \sum_{i=1}^{n} h_i$, $\left|\sum_{i=1}^{k} h_i\right| < \varepsilon$ for every $k = 1, \ldots, n$, $h_i(x) = 0$ for every $x \notin V_k^j$, $i = 2, \ldots, n$. Then, on every interval J_k, we define an open set $V_k = \bigcup_{j=1}^{\infty} V_k^j$ and a sequence of functions $h_1 \in C$, $h_2, h_3, \ldots, h_n \in \Delta^0(C)$. We can demand that $\lambda(V_k) < \frac{\delta}{2^k} \lambda(J_k)$, and for the densities we have $d(a_k, V_k) = d(b_k, V_k) = 0$ to be valid. Define the functions $h_1, \ldots, h_n$ on the set A by

$$h_1(x) = \sum_{i=1}^{n} f_i(x), \text{ and } h_2(x) = \ldots = h_n(x) = 0, \ x \in A.$$

Evidently,

$$\left\{x, h_1(x) \neq \sum_{i=1}^{n} f_i(x)\right\} \subset V = \bigcup_{k=1}^{\infty} V_k, \quad \lambda(V) < \delta$$

and

$$h_i(x) = 0 \text{ for every } x \notin V, \quad i = 2, \ldots, n.$$

Moreover, $d(x, V) = 0$ for every $x \in A$. Because $\left|\sum_{i=1}^{k} h_i\right| < \varepsilon$ for every $k = 1, \ldots, n$, the functions $h_1, \ldots, h_n$ are bounded. To show that they belong to the class Δ, it suffices to verify that for every x_0,

$$h_i(x_0) = \lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m} h_i \, d\lambda \tag{1}$$

for each sequence E_m, $m = 1, 2, \ldots$, of intervals contracting to x_0 ([1] Theorem 8.4. p. 41). If $x_0 \notin A$, according to inductive hypothesis, the condition above yields (1). Now let $x_0 \in A$. Then, $\sum_{i=1}^{n} f_i \in b\Delta$ and

$$\begin{aligned}
h_1(x_0) = \sum_{i=1}^{n} f_i(x_0) &= \lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m} \sum_{i=1}^{n} f_i \, d\lambda \\
&= \lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m} h_1 \, d\lambda - \frac{1}{\lambda(E_m)} \int_{E_m} h_1 - \sum_{i=1}^{n} f_i \, d\lambda \\
&= \lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m} h_1 \, d\lambda - \frac{1}{\lambda(E_m)} \int_{E_m \cap V} h_1 - \sum_{i=1}^{n} f_i \, d\lambda \\
&= \lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m} h_1 \, d\lambda.
\end{aligned}$$

This follows from the boundedness of $h_1 - \sum_{i=1}^{n} f_i$ and from the fact that $d(x_0, V) = 0$. Thus, $h_1 \in \Delta$, the functions $h_1 \restriction_A$, $h_1 \restriction_{\sim A}$ are continuous, and hence $h_1 \in \Delta(C)$. Using the same arguments, we get

$$\lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m} h_i \, d\lambda = \lim_{E_m \to x_0} \frac{1}{\lambda(E_m)} \int_{E_m \cap V} h_i \, d\lambda = 0 = h_i(x_0)$$

for every $i = 2, \dots, n$, and hence, $h_i \in \Delta^0(C)$.

Since $h_1 \in \Delta(C)$ and $|h_1| < \varepsilon$, according to the first part of the proof, for every δ_1, $0 < \delta_1 < \delta - \lambda(V)$ there exists an open set W, $\lambda(W) < \delta_1$, and functions $g_0 \in C$ and $g_1 \in \Delta^0(C)$ such that $g_0 + g_1 = h_1$, $\left|\sum_{i=0}^{k} g_i\right| < \varepsilon$ for every $k = 0, 1$, and $g_1(x) = 0$ for every $x \notin W$. Let $U = W \cup V$, $g_i = h_i$ for $i = 2, \dots, n$. Because $\lambda(U) < \delta$ and $g_0 \in C$, $g_1, \dots, g_n \in \Delta^0(C)$ satisfy the conditions $(a), (b), (c)$, and the proof of Lemma 10 is complete. □

Next, we shall show that $\sigma\Delta(C)$ is closed in the space Δ with the metric of uniform convergence. If a sequence $f_n \in \sigma\Delta(C)$, $n = 1, 2, \dots$, uniformly converges to a function f, then

$$f = f_1 + \sum_{n=1}^{\infty}(f_{n+1} - f_n).$$

Since for each n, f_n is the sum of a uniformly convergent series, instead of the function f_n we can consider a partial sum s_n of functions from $\Delta(C)$ such that $s_n \rightrightarrows f$ and $|s_{n+p} - s_n| < \frac{1}{2^n}$ for every $p \in N$. Evidently,

$$f = s_1 + \sum_{n=1}^{\infty}(s_{n+1} - s_n).$$

According to Lemma 10 above, for every $n \in N$ there exists a sequence of functions $g_{n_1}, g_{n_2}, \dots, g_{n_{k_n}} \in \Delta(C)$ such that

$$s_{n+1} - s_n = \sum_{i=1}^{k_n} g_{n_i},$$

$$\left|\sum_{i=1}^{k} g_{n_i}\right| < \frac{1}{2^n} \text{ for every } k = 1, 2, \dots, k_n .$$

For $f = s_1 + \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} g_{n_i}$, we have $f \in \sigma\Delta(C)$, because the sequence of partial sums $s_n + \sum_{i=1}^{k} g_{n_i}$, $n = 1, 2, \dots,$ $k = 1, 2, \dots, k_n$, of the series $s_1 + \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} g_{n_i}$ is uniformly convergent, which follows from the inequality

$$\left|f - \left(s_n + \sum_{i=1}^{k} g_{n_i}\right)\right| \le |f - s_n| + \left|\sum_{i=1}^{k} g_{n_i}\right| < |f - s_n| + \frac{1}{2^n}$$

and from $s_n \rightrightarrows f$.

It remains to show that the set $\Delta \setminus \sigma\Delta(C)$ is dense in Δ. Let $f \in \Delta,\ \varepsilon > 0$. Since $f \in B_1 \supset \Delta$, we can choose points $x_1 < x_2$ of continuity of the function f, such that

$$|f(x) - f(x_1)| < \frac{\varepsilon}{3} \quad \text{for every } x \in \langle x_1, x_2 \rangle .$$

Define the function g by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin (x_1, x_2) \\ \text{linear} & \text{on } \langle x_1, x_2 \rangle \end{cases}$$

and the function h, by

$$h(x) = \begin{cases} 0 & \text{if } x \notin (x_1, x_2) \\ & \text{a copy of the function of Example 8 is on } (x_1, x_2). \end{cases}$$

Then $w = g + \frac{\varepsilon}{3} h \notin \sigma\Delta(C)$ but $|f - w| \le |f - g| + \frac{\varepsilon}{3} < 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, which means that $\sigma\Delta(C)$ is nowhere dense in Δ. □

References

[1] A. M. Bruckner, *Differentiation of Real Functions*, Lecture notes in Math. 659, Springer-Verlag, Berlin, (1978).

[2] J. G. Ceder, T. L. Pearson, *A Survey of Darboux Baire 1 Functions*, Real. Anal. Exch., **9** (1984), 179–194.

[3] Z. Grande, *On a Theorem of Menkyna*, Real. Anal. Exch., **18(2)** (1992-1993), 585–589.

[4] R. Menkyna, *Classifying the Set Where a Baire 1 Function is Approximately Continuous*, Real. Anal. Exch., **14(2)** (1988-1989), 413–419.