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INEQUALITIES FOR GENERALIZED RATIONAL FUNCTIONS

Abstract

In this paper, we obtain two inequalities for generalized rational functions of one variable in L^p spaces when a partition of the domain with a suitable number of measurable subsets is considered.

1 Introduction.

Let μ be the Lebesgue measure on $\mathbb{R}$, and let $n, m \in \mathbb{N}$. Set $\{\phi_1, \phi_2, ..., \phi_n\}$ is a set of linearly independent continuous functions on $[a, b]$, and let $\{\psi_1, \psi_2, ..., \psi_m\}$ be a linearly independent continuous function set on the interval $[a, b]$ satisfying a Haar condition [2]; i.e., 0 is the only function of the form $\sum_{i=1}^{m} c_i\psi_i(x)$ which has m or more roots on $[a, b]$. We denote by V and W, respectively, the subspaces generated by them. We consider the set of generalized rational functions $\mathcal{R} := \{P/Q : P \in V, Q \in W, Q \neq 0 \text{ in } [a, b]\}$. Clearly, all elements in $\mathcal{R}$ can be written as P/Q, with $\|Q\|_1 = 1$, where $\| \sum_{j=1}^{m} a_j\psi_j(x)\|_1 := \sum_{j=1}^{m} |a_j|$. Henceforth, we assume that $\|Q\|_1 = 1$ for all $Q \in W$. If D is a measurable set and g is a measurable function on D, we consider the p-norm

$$\|g\|_{p,D} := \Big(\int_D |g(x)|^p \, d\mu\Big)^{1/p}, \quad 0 < p < \infty,$$

and $\|g\|_{\infty,D} = \sup \text{ess}_{x\in D}|g(x)|$. If $D_j \subset [a, b], 1 \le j \le m$, are pairwise disjoint closed sets of positive measure, $D = \cup_{i=1}^{m} D_i$, and f is a measurable function

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defined on $[a,b]$ which satisfies a suitable condition, we obtain in Section 2 an inequality of the following type:

$$\min_{1\leq j\leq m}\left\|f-\frac{P}{Q}\right\|_{\infty,D_j}\leq K\left\|f-\frac{P}{Q}\right\|_{p,D}, \tag{1.1}$$

for all $P/Q\in\mathcal{R}$. The function $f\equiv 0$ satisfies the required condition over f. Therefore, (1.1) is true in this case.

The Nikolskii type inequalities for algebraic polynomials ([3]); i.e., inequalities of the form (1.1) when $m=1,\psi_1(x)=1$, and $\phi_j(x)=x^{j-1}, 1\leq j\leq n$ does not hold for rational functions, as we shall show with an example. The theory of inequalities for univariate and multivariate algebraic polynomials has been developed extensively in the literature ([1], [3]). For certain classes of polynomials, Nikolskii type inequalities have been considered in [1]. In Section 3, we give an estimate of the constant K of the inequality (1.1) in terms of the $\mu(D_j)$, $1\leq j\leq m$, when $f=0$, $n=1$, $\phi_1(x)=1$, and $\psi_i(x)=x^{i-1}, 1\leq i\leq m$. Moreover, we prove that if g is a continuous function which oscillates r times; i.e., $|g|$ has r local maximum or minimum in the interval (a,b), then for any collection of measurable sets $D_j\subset[a,b]$, $\mu(D_j)>0$, $1\leq j\leq r+2$, with $\sup D_j\leq\inf D_{j+1}$, $1\leq j\leq r+1$, it has

$$\min_{1\leq j\leq r+2}\|g\|_{\infty,D_j}\leq\frac{1}{\min\limits_{1\leq i\leq r+2}\mu(D_i)^{1/p}}\|g\|_{p,\cup_{j=1}^{r+2}D_j}. \tag{1.2}$$

As an application of (1.2) we prove that for any partition of the interval $[a,b]$, say $a=a_0<a_1<\ldots<a_{r+2}=b$, there exists a finite set of points, $C\subset[a,b]$, such that

$$\|g\|_{p,[a,b]}\geq\min_{0\leq j\leq r+1}|a_j-a_{j+1}|^{1/p}\min_{y\in C}|g(y)| \tag{1.3}$$

for all continuous functions g who oscillate at most r times on $[a,b]$.

As example of a class whose members oscillate at the most r-times on $[a,b]$ for some $r\in\mathbb{N}$, we can mention $\mathcal{R}$ in the following cases:

- $\phi_i(x)=x^{i-1}, 1\leq i\leq n, \quad \psi_i(x)=x^{i-1}, 1\leq i\leq m$; i.e., algebraic rational functions;
- For n real numbers, $\lambda_1<\lambda_2<\ldots<\lambda_n$, let $\phi_i(x)=e^{\lambda_i x}, 1\leq i\leq n, \psi_1(x)=1, m=1$;
- Quotients of trigonometric polynomials.

2 Generalized Rational Functions.

We begin with a lemma, which is the key to prove the main theorem of this Section. It can be interesting itself. We denote $g|_D$ the restriction of a function g on the set D and $V|_D := \{g|_D : g \in V\}$.

Lemma 2.1. *Let $D_j \subset [a,b], 1 \le j \le m$, be pairwise disjoint closed subsets of positive measure. Let $D := \cup_{j=1}^m D_j$ and suppose that $f : D \to \mathbb{R}$ is an essentially bounded function such that $fW|_{D_j} \bigcap V|_{D_j} = \{0\}, 1 \le j \le m$. Then, for each $0 < s < 1$, there exists a constant $\alpha = \alpha(s) > 0$ that satisfies*

$$\mu\Big(\Big\{x \in D : \Big|f(x) - \frac{P(x)}{Q(x)}\Big| \ge s \min_{1\le j\le m}\Big\{\Big\|f - \frac{P}{Q}\Big\|_{\infty,D_j}\Big\}\Big\}\Big) \ge \alpha, \tag{2.1}$$

for all $P/Q \in \mathcal{R}$. The constant α depends only on W, V, f, D_j, and s.

PROOF. Clearly, (2.1) is equivalent to

$$\mu\Big(\Big\{x \in D : \Big|\lambda f(x) - \frac{P(x)}{Q(x)}\Big| \ge s \min_{1\le j\le m}\Big\{\Big\|\lambda f - \frac{P}{Q}\Big\|_{\infty,D_j}\Big\}\Big\}\Big) \ge \alpha, \tag{2.2}$$

for all $P/Q \in \mathcal{R}, \lambda \in \mathbb{R} - \{0\}$. Suppose that (2.2) is not true, then we can get $0 < s < 1$, $1 \le j_0 \le m$, a sequence $\lambda_k \in \mathbb{R} - \{0\}$, and a sequence $P_k/Q_k \in \mathcal{R}$ such that:

i) $0 < B_k := \|\lambda_k f - \frac{P_k}{Q_k}\|_{\infty,D_{j_0}} = \min\limits_{1\le j\le m}\{\|\lambda_k f - \frac{P_k}{Q_k}\|_{\infty,D_j}\}$, and

ii) the sets

$$A_k := \Big\{x \in D : \Big|\lambda_k f(x) - \frac{P_k(x)}{Q_k(x)}\Big| \ge s\Big\|\lambda_k f - \frac{P_k}{Q_k}\Big\|_{\infty,D_{j_0}}\Big\},$$

satisfy $\mu(A_k) \to 0$, for $k \to \infty$.

If we substitute $\frac{1}{B_k}(\lambda_k f - \frac{P_k}{Q_k})$ instead of $\lambda_k f - \frac{P_k}{Q_k}$ in i) and ii), we can assume without loss of generality that $\|\lambda_k f - \frac{P_k}{Q_k}\|_{\infty,D_{j_0}} = 1$. Only two cases can occur: a) $f \ne 0$ on a measure positive subset of D_{j_0}, and b) $f = 0$ on D_{j_0}(μ-a.e.). First, we suppose a). The condition $fW|_{D_{j_0}} \bigcap V|_{D_{j_0}} = \{0\}$ implies that all elements in $fW|_{D_{j_0}} + V|_{D_{j_0}}$ can only be written as $(Qf - P)|_{D_{j_0}}, Q \in W, P \in V$. We consider the norms over the linear space $fW|_{D_{j_0}} \bigoplus V|_{D_{j_0}}$ defined by $\rho_1(Qf - P) := \|Qf - P\|_{\infty,D_{j_0}}$ and $\rho_2(Qf - P) := \|Q\|_{\infty,D_{j_0}}\|f\|_{\infty,D_{j_0}} + \|P\|_{\infty,D_{j_0}}$. On the other hand, we have

$$\|Q_k\lambda_k f - P_k\|_{\infty,D_{j_0}} \le \|Q_k\|_{\infty,D_{j_0}}\Big\|\lambda_k f - \frac{P_k}{Q_k}\Big\|_{\infty,D_{j_0}} \le K, \tag{2.3}$$

for some constant K. Since $\|f\|_{\infty,D_{j_0}} > 0$, by the equivalence of the norms ρ_1 and ρ_2, we get that λ_k and $\|P_k\|_{\infty,D_{j_0}}$ are bounded sequences. Therefore, there are subsequences, denoted with the same index, such that $Q_k \to Q_0 \in W$, $P_k \to P_0 \in V$, and $\lambda_k \to \lambda_0 \in \mathbb{R}$. Since W satisfies a Haar condition, there exists $1 \le i \le m$ such that $|Q_0(x)| > 0$ for all $x \in D_i$. In addition, D_i is closed, thus, we have

$$\lambda_k f - \frac{P_k}{Q_k} \to \lambda_0 f - \frac{P_0}{Q_0} \tag{2.4}$$

uniformly on D_i. As $\|\lambda_k f - \frac{P_k}{Q_k}\|_{\infty,D_i} \ge \|\lambda_k f - \frac{P_k}{Q_k}\|_{\infty,D_{j_0}} = 1$, we obtain

$$\Big\|\lambda_0 f - \frac{P_0}{Q_0}\Big\|_{\infty,D_i} \ge 1. \tag{2.5}$$

Let $t \in (s,1)$. From (2.5) it follows that there is a μ-measurable set $B \subset D_i$, $\mu(B) > 0$ such that $|\lambda_0 f(x) - \frac{P_0(x)}{Q_0(x)}| \ge t$ for all $x \in B$. Then, there exists N such that

$$\Big|\lambda_k f(x) - \frac{P_k(x)}{Q_k(x)}\Big| \ge s,\ \forall k \ge N,\ \forall x \in B. \tag{2.6}$$

It follows from (2.6) that $B \subset A_k$ for all $k \ge N$. As consequence of ii), we obtain $\mu(B) = 0$, which is a contradiction. Now we assume b). As in item a), we obtain subsequences P_k and Q_k, converging to P_0 and Q_0, respectively, and $1 \le i \le m$ such that $|Q_0(x)| > 0$ for all $x \in D_i$. If $f = 0$ on D_i (μ-a.e.), in a similar way to a), we get a contradiction. On the contrary, there is a set $T \subset D_i$, $\mu(T) > 0$ such that $|f|$ has a positive lower bound on T. In the case that the sequence λ_k is bounded, it has a convergent subsequence, and again we get a contradiction. If the sequence λ_k is not bounded, there exists $N_1 > 0$ such that $|\lambda_k f(x) - \frac{P_0}{Q_0}| \ge 1$, for all $k \ge N_1$, $x \in T$. Since $\frac{P_k}{Q_k}$ uniformly converges to $\frac{P_0}{Q_0}$ on T, there is $N_2 > 0$ such that $|\lambda_k f(x) - \frac{P_k}{Q_k}| \ge |\lambda_k f(x) - \frac{P_0}{Q_0}| - \frac{1-s}{2}$ for all $k \ge N_2$, $x \in T$. Finally, we obtain $|\lambda_k f(x) - \frac{P_k}{Q_k}| \ge s$, for all $k \ge \max\{N_1, N_2\}$, $x \in T$ which implies $T \subset A_k$ for all $k \ge \max\{N_1, N_2\}$. Therefore, $\mu(T) = 0$, which is a contradiction. □

Now, we prove the main result of this Section.

Theorem 2.2. *Let $D_j \subset [a,b], 1 \le j \le m$, be pairwise disjoint closed subsets of positive measure, and let $0 < p < \infty$. Let $D := \cup_{j=1}^m D_j$, and suppose that $f : D \to \mathbb{R}$ is an essentially bounded function such that $fW|_{D_j} \bigcap V|_{D_j} = \{0\}$, $1 \le j \le m$. Then there exists a constant $K > 0$ such that*

$$\min_{1\le j\le m}\Big\|f - \frac{P}{Q}\Big\|_{\infty,D_j} \le K\Big\|f - \frac{P}{Q}\Big\|_{p,D}$$

for all $P/Q \in \mathcal{R}$. The constant K depends only on p, D_j, W, V, and f.

In particular, when $f = 0$, there exists a constant $K > 0$ such that

$$\min_{1\le j\le m} \left\|\frac{P}{Q}\right\|_{\infty,D_j} \le K \left\|\frac{P}{Q}\right\|_{p,D} \tag{2.7}$$

for all $P/Q \in \mathcal{R}$.

PROOF. Fix $0 < s < 1$. Let $0 < \alpha = \alpha(s)$ be as in Lemma 2.1. For $\frac{P}{Q} \in \mathcal{R}$, we consider the set

$$A = A(P/Q) := \Big\{x \in D : \Big|f(x) - \frac{P(x)}{Q(x)}\Big| \ge s \min_{1\le j\le m}\Big\|f - \frac{P}{Q}\Big\|_{\infty,D_j}\Big\}.$$

By the Lemma 2.1, we have $\mu(A) \ge \alpha$. Then for all $P/Q \in \mathcal{R}$,

$$\begin{aligned}\Big\|f - \frac{P}{Q}\Big\|_{p,D}^p = & \int_D \Big|f(x) - \frac{P(x)}{Q(x)}\Big|^p d\mu \\ & \ge \int_A \Big|f(x) - \frac{P(x)}{Q(x)}\Big|^p d\mu \\ & \ge \alpha\Big(s \min_{1\le j\le m}\Big\|f - \frac{P}{Q}\Big\|_{\infty,D_j}\Big)^p.\end{aligned} \tag{2.8}$$

The result follows with $K = 1/s\alpha^{1/p}$. □

We observe with a simple example that the inequality (2.7) is not true, in general, if we consider only r sets D_j with $r < m$. In fact, we can consider $m = 2$, $p = 1$, and the sequence

$$\frac{P_k(x)}{Q_k(x)} = \frac{1/k}{1/k + (1-1/k)x} \quad \text{and} \quad D_1 = [0,1].$$

It is easy to see that there is not a constant M such that $\|P_k/Q_k\|_{\infty,D_1} \le M\|P_k/Q_k\|_{1,D_1}$, for all $k \in \mathbb{N}$. Next, we see that the condition that the D_j are closed sets cannot be removed in Theorem 2.2. We take $f = 0$, $n = 1$, $m = 2$, $\phi_1(x) = 1$, and $\psi_i(x) = x^{i-1}$, $i = 1, 2$. Let

$$D_1 := \bigcup_{n=1}^{\infty}\Big[\frac{1}{2n}, \frac{1}{2n-1}\Big] \text{ and } D_2 := \bigcup_{n=1}^{\infty}\Big(\frac{1}{2n+1}, \frac{1}{2n}\Big).$$

Then, for $0 < \alpha < 1$,

$$\Big\|\frac{1}{\alpha + (1-\alpha)x}\Big\|_{\infty,D_j} = \frac{1}{\alpha},\ \ j = 1,2 \ \text{ and } \ \Big\|\frac{1}{\alpha + (1-\alpha)x}\Big\|_{1,[0,1]} = \frac{-\ln\alpha}{1-\alpha}.$$

On the other hand,

$$\lim_{\alpha\to 0^+} \frac{1/\alpha}{-\ln\alpha/(1-\alpha)} = \infty.$$

So, the inequality (2.7) is not true.

3 Finite Oscillation Functions.

In this Section, we give an estimate for the constant K in (2.7), as a function of the numbers $\mu(D_j)$. Inequalities of this type for multivariate algebraic polynomials have been given by Ganzburg and other authors (see [3]).

Definition 3.1. We shall say that a continuous function $f : [a,b] \to \mathbb{R}$ oscillates r-times on $[a,b]$ if $|f|$ has exactly r local maximums or minimums in (a,b).

We begin with an auxiliary lemma.

Lemma 3.2. *Let f be a continuous function which oscillates r-times on $[a,b]$. Let $D_j \subset [a,b], 1 \le j \le r+2$, be a family of measurable sets, $\mu(D_j) > 0$, $\sup D_j < \inf D_{j+1}$. Let $\alpha := \min\limits_{1\le j\le r+2} \|f\|_{\infty,D_j}$. If $A := \{1 \le j \le r+2 : \|f\|_{\infty,D_j} = \alpha\}$, then $\sharp A \le r+1$.*

Proof. Let $s := \sharp A$. Then there are s sets D_j, w.l.o.g. say D_j, $1 \le j \le s$, such that $\|f\|_{\infty,D_j} = \alpha$. Let $x_j \in \overline{D_j}$, $1 \le j \le s$, for $i \ne k$ such that $|f(x_j)| = \alpha$. Clearly, we must have a local maximum or minimum in each interval (x_i, x_{i+1}), $1 \le i \le s-1$. As the sets $\overline{D_j}$ are pairwise disjoint, it follows that $s - 1 \le r$. □

Theorem 3.3. *Let f be a continuous function which oscillates r-times on $[a,b]$, and let $0 < p \le \infty$. Let $D_j \subset [a,b], 1 \le j \le r+2$, be a family of measurable sets, $\mu(D_j) > 0$, $\sup D_j \le \inf D_{j+1}$, $1 \le j \le r+1$. Then*

$$\min_{1\le j\le r+2} \|f\|_{\infty,D_j} \le \frac{1}{\min\limits_{1\le i\le r+2} \mu(D_i)^{1/p}} \|f\|_{p,\cup_{j=1}^{r+2} D_j}. \tag{3.1}$$

Proof. If $p = \infty$, it is trivial. Suppose that $0 < p < \infty$. We write $\alpha := \min\limits_{1\le j\le r+2} \|f\|_{\infty,D_j}$, and $A := \{1 \le j \le r+2 : \|f\|_{\infty,D_j} = \alpha\}$. First, we suppose that $\sup D_j < \inf D_{j+1}, 1 \le j \le r+1$. Lemma 3.2 implies that $B := \{1 \le j \le$

$r+2 : j \notin A\} \neq \emptyset$. Suppose that for all $i \in B$, $\inf_{x\in D_i}|f(x)| < \alpha$. Then for each $i \in B$, there are $x_i, y_i \in D_i$ such that $|f(x_i)| < \alpha < |f(y_i)|$. Therefore, for each $i \in B$, there exists z_i belonging to the interval of extremes x_i and y_i such that $|f(z_i)| = \alpha$. Since $\sup D_j < \inf D_{j+1}, 1 \leq j \leq r+1$, then $z_i \neq z_k$, $i \neq k$. On the other hand, for each $j \in A$, there exists $t_j \in \overline{D_j}$ satisfying $|f(t_j)| = \alpha$. As the points t_j are different from the points z_i, we get $r+2$ points x with $|f(x)| = \alpha$. Thus, the function f does not oscillate r-times on $[a,b]$. It is a contradiction. In consequence, there is $j_0 \in B$ such that $\inf_{x\in D_{j_0}}|f(x)| \geq \alpha$. Therefore, we obtain

$$\|f\|_{p,\cup_{j=1}^{r+2}D_j} \geq \|f\|_{p,D_{j_0}} \geq \inf_{x\in D_{j_0}}|f(x)|\ \mu(D_{j_0})^{1/p} \geq \alpha\mu(D_{j_0})^{1/p},$$

as we want to show. Now, suppose $\sup D_j \leq \inf D_{j+1}, 1 \leq j \leq r+1$. Let $E_j(\epsilon) := [\inf D_j + \epsilon, \sup D_j - \epsilon] \cap D_j$, $1 \leq j \leq r+2$. For ϵ sufficiently small, the sets $E_j(\epsilon)$ satisfy that $\mu(E_j(\epsilon)) > 0$ and $\sup E_j(\epsilon) < \inf E_{j+1}(\epsilon)$, $1 \leq j \leq r+1$. We have proved, for the first part, that

$$\min_{1\leq j\leq r+2}\|f\|_{\infty,E_j(\epsilon)} \leq \frac{1}{\min\limits_{1\leq i\leq r+2}\mu(E_i(\epsilon))^{1/p}}\|f\|_{p,\cup_{j=1}^{r+2}E_j(\epsilon)}. \tag{3.2}$$

Finally, the Theorem follows by a limit process in (3.2) for ϵ tending to 0. □

Remark 3.4. We note that the Theorem 3.3 gives an estimate of the constant K in the inequality (2.7) for the particular case of algebraic rational functions mentioned in Section 1, with $n = 1$. In fact, here $r+2 = m$.

The next example shows that the amount $r+2$ of sets D_j in Theorem 3.3 is essential. Let $f(x) = x$, $p = 2$, $D_1 = [0,a]$, and $D_2 = [-a,0]$ with $a > 0$. Then

$$\|f\|_{2,D_1\cup D_2} = \sqrt{\frac{2}{3}a^3} < a^{3/2} = \min\{\mu(D_1)^{1/2}, \mu(D_2)^{1/2}\}\min\|f\|_{\infty,D_1\cup D_2}.$$

Now, we give another example which shows that the condition $\sup D_j \leq \inf D_{j+1}, 1 \leq j \leq r+1$, in Theorem 3.3 is also essential. Let $f(x) = \frac{1}{x^2+1}$ and $p = 1$. Let $a_i > 0, 1 \leq i \leq 3$, and $b_1 = 1 - a_1$. Consider the following sets

$$D_1 = [b_1, 1] \cup [a_2, b_1 + a_2],\ D_2 = -D_1, \text{and } D_3 = [0, a_1] \cup [a_3, b_1 + a_3].$$

Then $\mu(D_i) = 1$, $1 \leq i \leq 3$. It is easy to see that $\|f\|_{1,\cup_{i=1}^3 D_i} \to 0$ as $a_1 \to 0$, $a_2 \to \infty$, and $a_3 \to \infty$. However, $\min\limits_{1\leq i\leq 3}\|f\|_{\infty,D_i} = 1/2$. So, (3.1) is not true.

An immediate consequence of Theorem 3.3 is the following Corollary.

Corollary 3.5. *Let $B := \{a_j \in [a,b] : 0 \leq j \leq r+2\}$, $a = a_0 < a_1 < ... < a_{r+2} = b$, be a partition of the interval $[a,b]$. If*

$$\mathcal{H} := \{C \subset B : C \cap [a_j, a_{j+1}] \neq \emptyset, 0 \leq j \leq r+1\}.$$

Then

$$\|f\|_{p,[a,b]} \geq \min_{0 \leq j \leq r+1} |a_j - a_{j+1}|^{1/p} \max_{C \in \mathcal{H}} \min_{y \in C} |f(y)|, \tag{3.3}$$

for all continuous functions f oscillating at the most r times on $[a,b]$ and for all $0 < p \leq \infty$.

Proof. If $p = \infty$, it is trivial. Suppose that $0 < p < \infty$. Let f be a continuous function which oscillates at the most r times. We write $D_j = [a_j, a_{j+1}], 0 \leq j \leq r+1$. Given $C \in \mathcal{H}$, we have

$$\min_{y \in C} |f(y)| \leq \min_{0 \leq j \leq r+1} \|f\|_{\infty, D_j}.$$

Therefore, Theorem 3.3 implies the inequality (3.3). $\square$

The inequality (3.3) gives a lower bound for the p-norm of a continuous function f, oscillating r times, in terms of the values of f on a finite set of $[(r+3)/2]$ points, where [ ] denotes the integer part. In fact, we can take in the Corollary 3.5, $a_j = a + j\frac{b-a}{r+2}$, $0 \leq j \leq r+2$, and $C = \{a + (2j-1)\frac{b-a}{r+2} : 1 \leq j \leq [(r+3)/2]\}$.

Remark 3.6. Let $\mathcal{F}$ be the class of algebraic polynomials of degree less than or equal to n. Let $a_j = a + j\frac{b-a}{2n+1}$, $0 \leq j \leq 2n+1$, be with $a, b \in \mathbb{R}$. By Corollary 3.5, we get

$$\min_{1 \leq j \leq 2n+1} \|P\|_{\infty,[a_j,a_{j+1}]} \leq \frac{(2n+1)^{1/p}}{(b-a)^{1/p}} \|P\|_{p,[a,b]}, \ \forall P \in \mathcal{F}. \tag{3.4}$$

A comparison of (3.4) with the well known Nikolskii inequality (see [3], p.298),

$$\|P\|_{\infty,[a,b]} \leq \frac{8(n+1)^{2/p}}{(b-a)^{1/p}} \|P\|_{p,[a,b]}, \ \forall P \in \mathcal{F}, \tag{3.5}$$

shows that as the constant for a suitable partition of the domain decreases, the constant in (3.4) is of order $n^{1/p}$, while that in (3.5) is of order $n^{2/p}$.

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