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ON CONVERGENCE OF THE GAP-INTEGRAL

Abstract

The concept of the GAP-integral was introduced by the authors [5]. In this paper some convergence theorems for the GAP-integral are presented.

1 Introduction.

The Approximately Continuous Perron integral was introduced by Burkill [1] and its Riemann-type definition was given by Bullen [2]. Schwabik [6] presented a generalized version of the Perron integral leading to the new approach to a generalized ordinary differential equation. The authors introduced the concept of the Generalized Approximately Continuous Perron integral together with some important properties of the integral in [5]. In the present paper we obtain some convergence theorems of the GAP-integral. First we obtain the uniform convergence theorem. Then we prove the monotone convergence theorem and the basic convergence theorem for the GAP-integral. As an application of the basic convergence theorem, we obtain the mean convergence theorem for the GAP-integral.

2 Preliminaries.

Definition 2.1. A collection Δ of closed subintervals of $[a, b]$ is called an approximate full cover (AFC) if for every $x \in [a, b]$ there exists a measurable set $D_x \subset [a, b]$ such that $x \in D_x$ and D_x has density 1 at x, with $[u, v] \in \Delta$ whenever $u, v \in D_x$ and $u \leq x \leq v$.

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For all approximate full covers that occur in this paper the sets $D_x \subset [a,b]$ are the same. Then the relations $\Delta_1 \subseteq \Delta_2$ or $\Delta_1 \cap \Delta_2$ for approximate full covers Δ_1, Δ_2 are clear.

A division of $[a,b]$ obtained by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is called a Δ-division if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, if we have $x_{i-1} \leq \xi_i \leq x_i$ and $x_{i-1}, x_i \in D_{\xi_i}$ for all i. We call ξ_i the associated point of $[x_{i-1}, x_i]$ and x_i $(i = 0, 1, \ldots, n)$ the division points.

A division of $[a,b]$ given by $a \leq y_1 \leq \zeta_1 \leq z_1 \leq y_2 \leq \zeta_2 \leq z_2 \leq \cdots \leq y_m \leq \zeta_m \leq z_m \leq b$ is called a Δ-partial division if Δ is an approximate full cover with $([y_i, z_i], \zeta_i) \in \Delta$, for $i = 1, 2, \ldots, m$.

In [5], the GAP-integral is defined as follows :

Definition 2.2. A function $U : [a,b] \times [a,b] \to R$ is said to be generalized AP (GAP)-integrable to a real number A if for every $\epsilon > 0$ there is an AFC Δ of $[a,b]$ such that for every Δ-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$ we have

$$|(D)\sum\{U(\tau,\beta) - U(\tau,\alpha)\} - A| < \epsilon$$

and we write $A = (GAP)\int_a^b U$.

The set of all functions U which are Generalized Approximate Perron integrable on $[a,b]$ is denoted by GAP$[a,b]$. We use the notation

$$S(U,D) = (D)\sum\{U(\tau,\beta) - U(\tau,\alpha)\}$$

for the Riemann-type sum corresponding to the function U and the Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$. Note that the integral is uniquely determined.

Remark 2.3. If the AFC Δ in Definition 2.2 is replaced by an *ordinary full cover*, that is, the family of all $([\alpha,\beta],\tau)$ which are δ-fine for some $\delta(\tau) > 0$, i.e., $\tau \in [\alpha,\beta]$, $[\alpha,\beta] \subset [\tau - \delta(\tau), \tau + \delta(\tau)]$, then we have a general definition of Henstock integral [4] .

Setting $U(\tau,t) = f(\tau)t$ and $U(\tau,t) = f(\tau)g(t)$ where $f, g : [a,b] \to \mathbb{R}$ and $\tau, t \in [a,b]$, we obtain Riemann-type and Riemann-Stieltjes type integrals respectively for the functions f, g and a given Δ-division D of $[a,b]$.

Considering $U(\tau,t) = f(\tau)t$ in Definition 2.2, we obtain the classical approximately continuous Perron integral.

This definition is given in a more general form because of the general form of the function U.

For a given function $U : [a,b] \times [a,b] \to \mathbb{R}$ and a tagged interval (τ, J) with $\tau \in J = [\alpha,\beta] \subset [a,b]$ we will use the notation

$$U(\tau,J) = U(\tau,\beta) - U(\tau,\alpha) \tag{2.1}$$

for the point-interval function which corresponds to U.

Setting $U(\tau,t)=f(\tau)t$, $t\in[a,b]$, (2.1) becomes $U(\tau,J)=f(\tau)(\beta-\alpha)=f(\tau)|J|$ ($|J|$ denotes the length of the interval $J=[\alpha,\beta]$).

Let $U:[a,b]\times[a,b]\to\mathbb{R}$ and let δ be a positive function on $[a,b]$. Let D be an ordinary full cover of an interval $I\subset[a,b]$, that is, a δ-fine division $D=(J,\tau)$ of the interval $I\subset[a,b]$. We define the following interval functions, if they exist.

$$V(I)=\sup\,(D)\sum U(\tau,J)$$

and

$$W(I)=\inf\,(D)\sum U(\tau,J),$$

where the supremum and the infimum are over all δ-fine divisions $D=(J,\tau)$ of $I\subset[a,b]$.

The functions V and W serve as major and minor functions for U in a particular form.

We remark that if f has the Locally Small Riemann Sum ($LSRS$) property, then in view of Theorem 17.3 from [4], there exists a positive function δ such that both V and W exist for $I\subset[a,b]$.

Let

$$\underline{D}V(t)=\sup_{t\in I\subset[a,b]}\ \inf_{\delta>0}\ \frac{V(I)}{|I|}$$

and

$$\overline{D}W(t)=\inf_{\delta>0}\ \sup_{t\in I\subset[a,b]}\ \frac{W(I)}{|I|},$$

where $\underline{D}$ and $\overline{D}$ denote respectively the lower and the upper derivative of V and W at $t\in[a,b]$, respectively.

With the notion of a partial division we have proved in [5] the following theorem.

Theorem 2.4. **(Saks-Henstock Lemma)** *Let $U:[a,b]\times[a,b]\to\mathbb{R}$ be GAP-integrable over $[a,b]$. Then, given $\epsilon>0$, there is an approximate full cover Δ of $[a,b]$ such that for every Δ-division $D=\{([\alpha_{j-1},\alpha_j],\tau_j);j=1,2,\dots,q\}$ of $[a,b]$, we have*

$$\Big|\sum_{j=1}^{q}\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}-(GAP)\int_a^b U\Big|<\epsilon.$$

Then, if $\{([\beta_j,\gamma_j],\zeta_j); j=1,2,\ldots,m\}$ *represents a* Δ*-partial division of* $[a,b]$*, we have*

$$\Big|\sum_{j=1}^{m}[\{U(\zeta_j,\gamma_j)-U(\zeta_j,\beta_j)\}-(GAP)\int_{\beta_j}^{\gamma_j}U]\Big|<\epsilon.$$

The above theorem has an important use in the theory of generalized Perron integral.

3 Some Convergence Results.

We now give some convergence theorems for the GAP-integral.

Theorem 3.1. (Uniform Integrability Theorem) *Let*

(i) $U,U_n:[a,b]\times[a,b]\to\mathbb{R}$*,* $n=1,2,\ldots$ *be such that* $U_n\in GAP[a,b]$ *for all* $n=1,2,\ldots,$

(ii) there be an approximate full cover Δ_0 *of* $[a,b]$ *such that*

$$\lim_{n\longrightarrow\infty}[U_n(\tau,t_2)-U_n(\tau,t_1)]=U(\tau,t_2)-U(\tau,t_1)$$

for each $\tau\in[a,b]$*, and for every interval-point pair* $([t_1,t_2],\tau)\in\Delta_0$*,*

(iii) for every $\eta>0$ *there be an approximate full cover* Δ *of* $[a,b]$ *such that*

$$|S(U_n,D)-(GAP)\int_a^b U_n|<\eta$$

for every Δ*-division* D *of* $[a,b]$ *and every* $n=1,2,\ldots.$

Then $(GAP)\int_a^b U$ *exists, and*

$$\lim_{n\longrightarrow\infty}(GAP)\int_a^b U_n=(GAP)\int_a^b U.$$

Proof. Let $\epsilon>0$ be given and $A_n=(GAP)\int_a^b U_n$. By (iii), there is an approximate full cover $\Delta\subseteq\Delta_0$ of $[a,b]$ such that for every Δ-division $D=([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\big|S(U_n,D)-A_n\big|<\epsilon/2 \text{ for } n=1,2,\ldots.$$

By (ii), for every fixed Δ-division D of $[a,b]$ there exists a positive integer m_1 such that for $n > m_1$, we get $|S(U_n, D) - S(U, D)| =$

$$\Big| \sum \{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \sum \{U(\tau,\beta) - U(\tau,\alpha)\} \Big| < \epsilon/2$$

That is, $\lim_{n\to\infty} S(U_n, D) = S(U, D)$. Therefore, for any Δ-division D of $[a,b]$ there is a positive integer m_1 such that for $n > m_1$ we have

$$\begin{aligned} &|S(U,D) - A_n| \\ \le &|S(U,D) - S(U_n,D)| + |S(U_n,D) - A_n| < \epsilon/2 + \epsilon/2 \ = \ \epsilon \end{aligned} \tag{3.1}$$

First, we get from (3.1) that for all positive integers $n, p > m_1$

$$|A_n - A_p| \le |A_n - S(U,D)| + |S(U,D) - A_p| < \epsilon + \epsilon = 2\epsilon.$$

Thus, $\{A_n\}$ is a Cauchy sequence in $\mathbb{R}$ and let $A = \lim_{n\to\infty} A_n$. Then, given $\epsilon > 0$, there exists a positive integer m_2 such that

$$|A_n - A| < \epsilon \text{ for all } n > m_2. \tag{3.2}$$

Let $m = \max(m_1, m_2)$. Then we get from (3.1) and (3.2) for $n > m$, that

$$|S(U,D) - A| \le |S(U,D) - A_n| + |A_n - A| < \epsilon + \epsilon \ = \ 2\epsilon.$$

Hence, $U \in GAP[a,b]$ with $(GAP)\int_a^b U$, and

$$\lim_{n\to\infty} (GAP)\int_a^b U_n = (GAP)\int_a^b U. \qquad \square$$

Lemma 3.2. *Let $U, V : [a,b] \times [a,b] \to \mathbb{R}$ be such that $U, V \in GAP[a,b]$ and if there be an approximate full cover Δ_0 of $[a,b]$ such that*

$$U(\tau,t) - U(\tau,\tau) \le V(\tau,t) - V(\tau,\tau)$$

for every interval-point pair $([\tau,t],\tau) \in \Delta_0$ where $\tau < t$ and

$$U(\tau,\tau) - U(\tau,t) \le V(\tau,\tau) - V(\tau,t)$$

for every interval-point pair $([t,\tau],\tau) \in \Delta_0$ where $t < \tau$, then

$$(GAP)\int_a^b U \le (GAP)\int_a^b V.$$

Proof. Let $\epsilon > 0$ be arbitrary. Since $U, V \in GAP[a,b]$ given $\epsilon > 0$, there exists an approximate full cover Δ of $[a,b]$ with $\Delta \subseteq \Delta_0$ such that for every Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\Big|\sum\{U(\tau,\beta) - U(\tau,\alpha)\} - (GAP)\int_a^b U\Big| < \epsilon/2,$$

$$\Big|\sum\{V(\tau,\beta) - V(\tau,\alpha)\} - (GAP)\int_a^b V\Big| < \epsilon/2.$$

These give

$$\begin{aligned}(GAP)\int_a^b U - \epsilon/2 \ &< \sum\{U(\tau,\beta) - U(\tau,\alpha)\}\\ &= \sum[\{U(\tau,\beta) - U(\tau,\tau)\} + \{U(\tau,\tau) - U(\tau,\alpha)\}]\\ &\leq \sum[\{V(\tau,\beta) - V(\tau,\tau)\} + \{V(\tau,\tau) - V(\tau,\alpha)\}]\\ &= \sum\{V(\tau,\beta) - V(\tau,\alpha)\} \ < (GAP)\int_a^b V + \epsilon/2.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$(GAP)\int_a^b U \leq (GAP)\int_a^b V. \qquad \square$$

Theorem 3.3. (Monotone Convergence Theorem) *Let*

(i) $U, U_n : [a,b] \times [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ *be such that* $U_n \in GAP[a,b]$ *for all* $n = 1, 2, \ldots$ *with* $\sup\ (GAP)\int_a^b U_n < \infty$,

(ii) there be an approximate full cover Δ_0 *of* $[a,b]$ *such that*

$$U_n(\tau,t) - U_n(\tau,\tau) \leq U_{n+1}(\tau,t) - U_{n+1}(\tau,\tau)$$

for every interval-point pair $([\tau,t],\tau) \in \Delta_0$ *where* $\tau < t$ *and*

$$U_n(\tau,\tau) - U_n(\tau,t) \leq U_{n+1}(\tau,\tau) - U_{n+1}(\tau,t)$$

for every interval-point pair $([t,\tau],\tau) \in \Delta_0$ *where* $t < \tau$, $(n = 1, 2, \ldots)$,

(iii) there be an approximate full cover Δ' *of* $[a,b]$ *such that*

$$\lim_{n\to\infty}[U_n(\tau,t_2) - U_n(\tau,t_1)] = U(\tau,t_2) - U(\tau,t_1)$$

for each $\tau \in [a,b]$ *and every interval-point pair* $([t_1,t_2],\tau) \in \Delta'$.

Then, $U \in GAP[a,b]$ *and*

$$\lim_{n\to\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U.$$

Proof. Let $\epsilon > 0$ be given. Since each $U_n \in GAP[a,b]$ for each positive integer n, there is an approximate full cover Δ_n of $[a,b]$ such that for any Δ_n-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\sum |\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - (GAP)\int_\alpha^\beta U_n| < \epsilon/2^n.$$

By (iii), given $\epsilon > 0$, for every fixed Δ'-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, there exists an integer $M(\tau)$ such that whenever $m(\tau)$ is an integer with $m(\tau) \geq M(\tau)$ we have

$$|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \epsilon/2^{m(\tau)}$$

for every $\tau \in [a,b]$. Since $\{(GAP)\int_a^b U_n\}$ is non-decreasing by Lemma 3.2 and bounded above, $\lim_{n\to\infty}(GAP)\int_a^b U_n$ exists. Let $\lim_{n\to\infty}(GAP)\int_a^b U_n = A$. For each $\tau \in [a,b]$, we choose any integer $m(\tau) \geq M(\tau)$ and we take $\Delta = \Delta' \cap \Delta_0 \cap \Delta_{m(\tau)}$. Then, for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, we have

$$\begin{aligned}
&\Big|\sum\{U(\tau,\beta) - U(\tau,\alpha)\} - A\Big| \\
\leq &\Big|\sum[\{U(\tau,\beta) - U(\tau,\alpha)\} - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}]\Big| \\
&+ \sum\Big|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - (GAP)\int_\alpha^\beta U_{m(\tau)}\Big| \\
&+ \Big|\sum(GAP)\int_\alpha^\beta U_{m(\tau)} - A\Big| \\
< &\sum \epsilon/2^{m(\tau)} + \sum \epsilon/2^{m(\tau)} + \Big|\sum(GAP)\int_\alpha^\beta U_{m(\tau)} - A\Big|,
\end{aligned} \tag{3.3}$$

where all the sums involved run over all elements of the division D ($\sum = (D)\sum$). Therefore, if we can show that the last term $|(D)\sum(GAP)\int_\alpha^\beta U_{m(\tau)} - A| < \epsilon$, then the proof will be complete.

The number of associated points τ in the division D is finite and so is the number of those different $m(\tau)$ in the above sum over D. Let p denote the minimum of those $m(\tau)$ and q be the maximum. Then we have

$$\begin{aligned}(GAP)\int_a^b U_p &= (D)\sum(GAP)\int_\alpha^\beta U_p \le (D)\sum(GAP)\int_\alpha^\beta U_{m(\tau)}\\ &\le (D)\sum(GAP)\int_\alpha^\beta U_q = (GAP)\int_a^b U_q \le A.\end{aligned}$$

We can also find a positive integer m_0 such that

$$0 \le A - (GAP)\int_a^b U_m < \epsilon \text{ for all } m \ge m_0,$$

while defining $m(\tau)$ we always take $m(\tau) \ge m_0$ and so $p \ge m_0$. Hence

$$\begin{aligned}\Big|\sum(GAP)\int_\alpha^\beta U_{m(\tau)} - A\Big| &= A - \sum(GAP)\int_\alpha^\beta U_{m(\tau)}\\ &\le A - \sum(GAP)\int_\alpha^\beta U_p = A - (GAP)\int_a^b U_p < \epsilon.\end{aligned}$$

Therefore $U \in GAP[a,b]$ by (3.3) and

$$\lim_{n\to\infty}(GAP)\int_a^b U_n = A = (GAP)\int_a^b U. \qquad \square$$

In [5] the indefinite GAP-integral is defined as follows.

Definition 3.4. Let $U \in GAP[a,b]$. The function $\phi : [a,b] \to \mathbb{R}$ defined by

$$\phi(s) = (GAP)\int_a^s U,\ a < s \le b,\ \phi(a) = 0$$

is called the indefinite GAP-integral of U.

For $[\alpha,\beta] \subset [a,b]$ put $\phi(\alpha,\beta) = \phi(\beta) - \phi(\alpha) = (GAP)\int_\alpha^\beta U$.

Theorem 3.5. (Basic Convergence Theorem) *Let*

(i) $U_n : [a,b]\times[a,b] \to \mathbb{R}$ *be GAP-integrable on* $[a,b]$ *with the primitives* ϕ_n, $n = 1, 2, \dots$,

(ii) there be an approximate full cover Δ' *of* $[a,b]$ *such that*

$$\lim_{n\to\infty}[U_n(\tau,t_2) - U_n(\tau,t_1)] = U(\tau,t_2) - U(\tau,t_1)$$

for each $\tau \in [a,b]$ *and every interval-point pair* $([t_1,t_2],\tau) \in \Delta'$, *(iii)* ϕ_n *converge point-wise to a limit function* ϕ.

Then $U \in GAP[a,b]$ *with primitive* ϕ *if and only if for every* $\epsilon > 0$ *there is a function* $M(\tau)$ *defined on* $[a,b]$ *taking integer values such that for infinitely many* $m(\tau) \geq M(\tau)$ *there is an approximate full cover* Δ *such that for any* Δ*-division* $D = ([\alpha,\beta],\tau)$ *of* $[a,b]$ *we have*

$$\Big|\sum \{\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)\}\Big| < \epsilon.$$

Proof. Suppose $U \in GAP[a,b]$ with the primitive ϕ. Then there is an approximate full cover Δ_0 of $[a,b]$ such that for any Δ_0-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\Big|\sum [\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)]\Big| < \epsilon.$$

Again, since $U_n \in GAP[a,b]$ with primitive ϕ_n, $n = 1,2,\ldots,$ there is an approximate full cover Δ_n of $[a,b]$ such that for any Δ_n-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\Big|\sum [\{U_n(\tau,\beta) - U_n(\tau,\alpha)\} - \phi_n(\alpha,\beta)]\Big| < \epsilon/2^n.$$

Given $\epsilon > 0$, for every fixed Δ'-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, there exists an integer $M(\tau)$ such that whenever $m(\tau) \geq M(\tau)$ we have

$$|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \epsilon/2^{m(\tau)}$$

for every $\tau \in [a,b]$. Without any loss of generality, we may assume that $\Delta' = \Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_{m(\tau)}$. For each $\tau \in [a,b]$, we choose any integer $m(\tau) \geq M(\tau)$ and we take $\Delta = \Delta' \cap \Delta_0$. Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, we have

$$\begin{aligned}
&\Big|\sum \{\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)\}\Big| \\
\leq &\Big|\sum [\phi_{m(\tau)}(\alpha,\beta) - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}]\Big| \\
&+ \Big|\sum [\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}]\Big| \\
&+ \Big|\sum [\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)]\Big| \\
< &\epsilon + \sum \epsilon/2^{m(\tau)} + \epsilon < \epsilon + \epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

Conversely, suppose that the condition is satisfied. Then for every $\epsilon > 0$ there is a function $M(\tau)$ defined on $[a,b]$ taking integer values such that for infinitely many $m(\tau) \geq M(\tau)$ there is an approximate full cover Δ such that for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\Big|\sum\{\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)\}\Big| < \epsilon.$$

Also, for every fixed Δ'-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we can find $m(\tau) \geq M(\tau)$ such that

$$|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \epsilon/2^{m(\tau)}$$

for every $\tau \in [a,b]$. Using the same notation as in the first part, we choose $\Delta = \Delta' \cap \Delta_0$, $\tau \in [a,b]$. Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, we have

$$\begin{aligned}
&\Big|\sum[\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)]\Big| \\
\leq &\Big|\sum[\{U(\tau,\beta) - U(\tau,\alpha)\} - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}]\Big| \\
&+ \Big|\sum[\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \phi_{m(\tau)}(\alpha,\beta)]\Big| \\
&+ \Big|\sum\{\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)\}\Big| < \epsilon + \epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

Hence U is GAP-integrable on $[a,b]$. □

Theorem 3.6. **(Mean Convergence Theorem)** *Let*

(i) $U_n : [a,b] \times [a,b] \to \mathbb{R}$ *be GAP-integrable on* $[a,b]$ *with the primitives* ϕ_n, $n = 1,2,\dots$,

(ii) there be an approximate full cover Δ' *of* $[a,b]$ *such that*

$$\lim_{n\to\infty}[U_n(\tau,t_2) - U_n(\tau,t_1)] = U(\tau,t_2) - U(\tau,t_1)$$

for each $\tau \in [a,b]$ *and every interval-point pair* $([t_1,t_2],\tau) \in \Delta'$,

(iii) $[a,b]$ *be the union of a sequence of closed sets* X_i, $i = 1,2,\dots,$*and for every* i *and* $\epsilon > 0$ *there exist an integer* N *and an approximate full cover* Δ *of* $[a,b]$ *such that for any* Δ*-division* $D = ([\alpha,\beta],\tau)$ *of* $[a,b]$ *tagged in* X_i,*for each* i *we have* $\big|\sum\{\phi_n(\alpha,\beta) - \phi(\alpha,\beta)\}\big| < \epsilon$ *for some function* ϕ, *whenever* $n \geq N$,

(iv) the primitives ϕ_n *converge uniformly to* ϕ *on* $[a,b]$.

Then $U \in GAP[a,b]$ *with the primitive* ϕ *and*

$$\lim_{n\to\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U.$$

Proof. Let $\epsilon > 0$. By (*iii*) above, for every i and j there exists an integer N_{ij} and an approximate full cover Δ_{ij} of $[a,b]$ such that for any Δ_{ij}-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ with $\tau \in X_i$ we have

$$\Big|\sum\{\phi_n(\alpha,\beta) - \phi(\alpha,\beta)\}\Big| < \epsilon/2^{i+j} \text{ for all } n \geq N_{ij}.$$

Take $n = n(i,j)$ so that the above inequality holds. We may assume that for each i, $\{\phi_{n(i,j)}\}$ is a subsequence of $\{\phi_{n(i-1,j)}\}$. Now consider $\phi_{n(j)} = \phi_{n(j,j)}$ in place of ϕ_n and write $Y_1 = X_1$ and

$$Y_i = X_i - (X_1 \cup X_2 \cup \cdots \cup X_{i-1}) \text{ for } i = 2,\, 3, \ldots .$$

Put $M(\tau) = n(i)$ when $\tau \in Y_i$.

We note that there are infinitely many $m(\tau) \geq M(\tau)$, namely all $n(i) \geq n(j)$. If $m(\tau)$ takes values in $\{n(j) : j \geq i\}$ when $m(\tau) \geq M(\tau) = n(i)$, we put $\Delta = \Delta_{m(\tau)}$. Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ with $\tau \in Y_i$, for some i, we have

$$\Big|\sum\{\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)\}\Big| \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\epsilon/2^{i+j} = \epsilon.$$

This means that the condition of the basic convergence theorem is satisfied. Hence $U \in GAP[a,b]$ with the primitive ϕ and

$$\lim_{n\to\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U. \qquad \square$$

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