RESEARCH

Real Analysis Exchange
ISSN:0147-1937
Vol. 31(2), 2005/2006, pp. 315–354

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ANALYSIS OF MINKOWSKI CONTENTS OF FRACTAL SETS AND APPLICATIONS

Abstract

Using fractal sets and Minkowski contents we extend the repertoire of Lebesgue integrable functions to those with large singular sets. A new method of constructing fractal sets is proposed, using a class of absolutely continuous functions, called swarming functions. We obtain bounds on Minkowski contents of fractals in terms of two natural parameters contained in $[-\infty, \infty]$, called the upper and lower dispersions of the fractal. Assuming that upper and lower box dimensions of a fractal are equal, we show that if the difference of dispersions is sufficiently large, then the set is not Minkowski measurable. Fractals with nondegenerate d-dimensional Minkowski contents (i.e., contained in $(0, \infty)$) are characterized as those with nondegenerate dispersions (i.e., contained in $(-\infty, \infty)$). The Weierstrass function, a class of affine fractal functions and a class of McMullen's sets have nondegenerate Minkowski contents. Also some classes of spirals of focus and limit cycle type in the plane are shown to be Minkowski measurable. Using swarming functions we can easily construct fractal sets with maximally separated lower and upper box dimensions, and a pair of fractal sets with maximal instability of lower box dimension with respect to union. We also study gauge functions associated with fractals having degenerate Minkowski contents, and obtain new integrability criteria for a class of singular integrals.

Key Words: fractals, Minkowski contents, swarming functions, box dimensions, gauge functions, Lebesgue's integral, singular integral
Mathematical Reviews subject classification: 26A30, 26B15, 28A12, 28A80
Received by the editors February 28, 2004
Communicated by: R. Daniel Mauldin

1 Introduction.

Our motivation for this work is twofold. First, it has been noticed that estimates of Minkowski contents of fractal sets (more precisely, nondegeneracy of Minkowski contents) are related to understanding the problem of asymptotics of eigenfrequencies of fractal strings and fractal drums (Weyl–Berry conjecture), see Lapidus [12], [13], Lapidus and Pomerance [15], [16], Lapidus and Maier [14], Falconer [5], He and Lapidus [10], Lapidus and van Frankenhuysen [17], and the references therein. We say that a Minkowski content is nondegenerate if it is different from 0 and ∞. Second, Lebesgue integrable functions with large singular sets on prescribed fractal sets can be generated assuming precisely the same type of condition on Minkowski contents, see Žubrinić [26]. We also have in mind applications to the study of singular sets of Sobolev functions (see [24] and the references therein), and the fact that Lebesgue and Sobolev spaces possess maximally singular functions in the sense described in [27]. For analogous problems in Besov and Lizorkin-Triebel spaces see [25]. Minkowski contents of graphs of rapidly oscillating solutions of the one-dimensional p-Laplace equation are studied in Pašić and Županović [21]. Applications to analysis of spiral trajectories of some planar vector fields (Hopf-Takens bifurcation) can be seen in Žubrinić and Županović [28], see also [29], and a review article [30] about fractal dimensions in dynamics.

Using fractal sets and Minkowski contents, in Section 3 we extend the repertoire of Lebesgue integrable functions to those with large singular sets, (see Theorems 4.2 and 4.4). We use functions of the form $u(x) = d(x, A)^{-\alpha}$ with $\alpha > 0$, where $d(\cdot, A)$ is the Euclidean distance function from the set $A \subset \mathbb{R}^N$, and u is defined on the Minkowski sausage A_r of the set A; that is, on the open r-neighborhood of A. Assuming that the upper and lower box dimensions of A are equal, the problem of integrability of u turns out to be related to nondegeneracy condition on Minkowski contents of A.

Nondegeneracy of Minkowski contents is studied in Section 2, as a continuation of our previous work in [26] and [27]. The main result is stated in Theorem 3.1, in which dispersion parameters $\underline{D}$ and $\overline{D}$ associated with A are introduced, that enable us partial control over the Minkowski contents of A. Assuming that the upper and lower box dimensions coincide, we show that the corresponding Minkowski contents of A are nondegenerate if and only if both dispersion parameters of A are nondegenerate; that is, different from $\pm\infty$, see Theorem 3.2(c). As a consequence we obtain that the Weierstrass function, the Knopp (or Takagi) function, a class of affine fractal functions, a class of McMullen sets, and some other fractals, all have nondegenerate Minkowski contents, see Theorem 3.5.

The difference of upper and lower dispersions of a fractal set, called the amplitude of the fractal, is shown to have effect on d-dimensional Minkowski contents; if the amplitude of the fractal set is sufficiently large, then the Minkowski contents are separated, (see Theorem 3.2(b)). Moreover, the quotient of the upper and lower Minkowski contents of a fractal has at least exponential growth with respect to its amplitude, see (22). We also find a class of fractals enabling partial control over their Minkowski contents, (see Corollaries 3.3 and 3.4).

Motivated by the problem of integrability of the above function u (and by He, Lapidus [10]), in Section 4 we study natural gauge functions associated with fractals having degenerate Minkowski contents, (see Theorem 5.1). In Theorem 5.2 we also show that the Minkowski content condition is indeed essential in order to have the natural characterization of integrability as stated in Theorem 4.3. In Section 5 we study two classes of spirals in the plane, and show that they are Minkowski measurable, see Theorems 6.1 and 6.2. In the proof we use excision property of the upper Minkowski content, see Lemma 6.6(a).

In introductory Section 1 we propose a new method of defining fractal sets, using appropriate absolutely continuous functions related to the logarithmic scale of the box counting function of A, called swarming functions, (see Lemma 2.1). It enables us to view fractals almost like their swarming functions. Using zigzagging swarming functions we construct a class of fractal sets with maximally separated lower and upper box dimensions, (see Theorem 2.2 simplifying known examples, see Tricot [23, example on p. 29], Mattila [19, examples on p. 77]), and a pair of fractal sets with maximal instability of lower box dimension with respect to union, (see Theorem 2.4 improving and simplifying Tricot [23, examples on pp. 30 and 123]).

The paper is organized as follows:

2. Generating fractal sets using swarming functions;
3. Bounds and separation of Minkowski contents;
4. Minkowski contents and the Lebesgue integral;
5. Gauge functions for Minkowski contents;
6. Minkowski measurable spirals in the plane.

2 Generating Fractal Sets Using Swarming Functions.

Let A be a bounded set in $\mathbb{R}^N$, $N \geq 1$. Let us fix the base $b = 2$ and for any integer $n \geq 1$ we consider the natural 2^{-n}-grid in $\mathbb{R}^N$, which divides the space into closed cubes of sides 2^{-n} with at most overlapping faces. By $\omega_n(A)$ we denote the number of cubes containing at least one point of A. Note that a point of A can be contained in at most 2^N cubes. Let us define a sequence

$$s_n := \log_2 \omega_n(A), \tag{1}$$

that we call the swarming sequence of the set A. Here and in the sequel we use the notation $\log_a b := \log b / \log a$. It is well known that we can express the upper and lower box dimensions of A as

$$\overline{\dim}_B A = \limsup_{n\to\infty} \frac{s_n}{n}, \quad \underline{\dim}_B A = \liminf_{n\to\infty} \frac{s_n}{n}, \tag{2}$$

(see Falconer [4, p. 41] or Tricot [23, p. 24]). The sequence s_n associated with the set A will be of greatest importance in this paper. We will use it for defining fractal sets (in a nonunique way) and for studying their properties. More precisely, we will try to define a class of swarming sequences σ_n in advance, for which one can generate the corresponding collections of fractal sets A such that $\sigma_n \sim \log_2 \omega_n(A)$ as $n \to \infty$. (We say that $a_n \sim b_n$ if $a_n/b_n \to 1$.) Loosely speaking, this will enable us to identify fractal sets with their swarming sequences. See Lemma 2.1 for precise formulation.

Let A be a given bounded, infinite subset of $[0,1]^N \subset \mathbb{R}^N$. The overall number of cubes of 2^{-n}-grid in $[0,1]^N$ is equal to 2^{Nn}. We have $\omega_{n+1}(A) \le 2^N \omega_n(A)$ since each cube of the 2^{-n}-grid, containing a point of A, is the union of 2^N subcubes of the $2^{-(n+1)}$-grid. Hence, the corresponding swarming sequence s_n defined by (1) has the following basic properties:

$$\begin{cases} s_n \text{ is nondecreasing, } s_n \to \infty \text{ as } n \to \infty, \\ 0 \le s_1 \le (\log_2 3)N, \\ s_{n+1} \le s_n + N. \end{cases} \tag{3}$$

Remark 2.1. If we require the set A to be bounded only, then we can change initial condition on s_1 in (3) to $s_1 \ge 0$.

Conversely, compact fractal sets in $[0,1]^N$ can be generated (in a nonunique way) using a sequence σ_n given in advance and satisfying the properties listed in (3), so that $\sigma_n \sim \log_2 \omega_n(A)$. Before stating the general result, we illustrate the idea in the case of $N = 1$.

Example 2.1. Let a sequence (σ_n) with elements contained in $\mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ be given such that the following conditions are satisfied:

$$\begin{cases} \sigma_n \text{ is nondecreasing, } \sigma_n \to \infty \text{ as } n \to \infty, \\ \sigma_1 \in \{0, 1\}, \\ \sigma_{n+1} \le \sigma_n + 1. \end{cases} \tag{4}$$

Let us construct a set $A \subseteq [0,1]$ such that $\log_2 \omega_n(A) \sim \sigma_n$ as $n \to \infty$. First, it is easy to verify that there exists a unique set $S \subseteq N$ such that

$$\#(S \cap \{1,2,\dots,n\}) = \sigma_n, \ \forall n \in \mathbb{N}, \tag{5}$$

where # denotes the cardinal number of a set. We can construct the set S inductively as follows. Assume that $n = 1$. If $\sigma_1 = 0$, then we let $1 \notin S$, while for $\sigma_1 = 1$ we let $1 \in S$. Assume that $n \geq 2$. If $\sigma_n = \sigma_{n-1}$, then we let $n \notin S$, while for $\sigma_n = \sigma_{n-1} + 1$ we let $n \in S$. It is easy to see that (5) holds.

Now using the sequence σ_n we construct a subset A of $[0,1]$ by representing its elements in base $b = 2$. It consists of all numbers $x = 0.x_1x_2x_3 \cdots := \sum_{i=1}^{\infty} x_i 2^{-i}$ such that $x_i \in \{0,1\}$ if $i \in S$, while $x_i = 0$ if $i \notin S$ (or more generally, for $i \notin S$ we can let x_i have any *fixed*, prescribed value, 0 or 1). It is easy to see that

$$2^{\sigma_n} \leq \omega_n(A) \leq 3 \cdot 2^{\sigma_n}. \tag{6}$$

Indeed, let us define a closed interval

$$I_{x_1 \dots x_n} := \{x = 0.x_1 \dots x_n x_{n+1} \dots : x_i \in \{0,1\} \text{ for } i \geq n+1\},$$

where we require that $x_1, \dots, x_n$ satisfy conditions in the definition of A. Clearly, there are 2^{σ_n} such intervals, see (6), each of them belongs to 2^{-n}-grid, and their union A_n contains A. Note that the family of sets A_n is decreasing and $A = \cap_n A_n$. Each interval $I_{x_1 \dots x_n}$ meets A, therefore $\omega_n(A) \geq 2^{\sigma_n}$. On the other hand, each point of A is in at least of one of intervals $I_{x_1 \dots x_n}$, which meets 2^{-n}-grid in at most three intervals, and we conclude that $\omega_n(A) \leq 3 \cdot 2^{\sigma_n}$.

From (6) we see that $\log_2 \omega_n(A) \sim \sigma_n$, and therefore, $\underline{\dim}_B A = \liminf_n \frac{\sigma_n}{n}$ and $\overline{\dim}_B A = \limsup_n \frac{\sigma_n}{n}$, see (2). Compare with Bishop [1, Chapter 1, Example 2.2].

In the following lemma we say that a compact set A in $\mathbb{R}^N$ is constructive if it is representable as the intersection of a decreasing family of sets A_n such that each A_n can be constructed by effective procedure as a finite union of closed cubes of 2^{-n}-grid. For given $x \in \mathbb{R}$ by $\lfloor x \rfloor$ we denote the greatest integer part of x.

Lemma 2.1. (Swarming functions) *Let $f : [1,\infty) \to [0,\infty)$ be an absolutely continuous function such that*

$$f(1) \in [0, N+1), \ f'(t) \in [0,N] \textit{ for a.e. } t. \tag{7}$$

Then the sequence $\sigma_n := \lfloor f(n) \rfloor$ is such that σ_n is nondecreasing, $0 \leq \sigma_1 \leq N$, and $\sigma_{n+1} \leq \sigma_n + N$. There exists a constructive subset A of $[0,1]^N$ such that

$$\sigma_n \leq \log_2 \omega_n(A) \leq \sigma_n + N \log_2 3. \tag{8}$$

In particular, $\sigma_n \sim \log_2 \omega_n(A)$ *as* $n \to \infty$ *and*

$$\overline{\dim}_B A = \limsup_{n\to\infty} \frac{f(n)}{n}, \ \underline{\dim}_B A = \liminf_{n\to\infty} \frac{f(n)}{n}. \tag{9}$$

PROOF. Monotonicity of σ_n is clear since f is nondecreasing. We have $f(n+1) - f(n) = \int_n^{n+1} f'(t)\,dt \le N$, so that

$$\sigma_{n+1} = \lfloor f(n+1) \rfloor \le \lfloor f(n) + N \rfloor = \lfloor f(n) \rfloor + N = \sigma_n + N. \tag{10}$$

We construct inductively a decreasing sequence of compact subsets A_n of $[0,1]^N$ as follows. Each A_n will be a union of 2^{σ_n} closed cubes with disjoint interiors, belonging to the 2^{-n}-grid. For $n = 1$ define A_1 as the union of 2^{σ_1} cubes in $[0,1]^N$ with disjoint interiors, belonging to the 2^{-1}-grid.

Assume that A_{n-1} has been constructed as a union of $2^{\sigma_{n-1}}$ cubes with disjoint interiors, belonging to the $2^{-(n-1)}$-grid. Define A_n as a subset of A_{n-1} obtained in two steps.

(a) Change each $2^{-(n-1)}$-cube of A_{n-1} by an arbitrary subcube belonging to the 2^{-n}-grid. Their union A'_n makes $2^{\sigma_{n-1}}$ cubes of the 2^{-n}-grid with disjoint interiors.

(b) Note that A_{n-1} contains precisely $2^N 2^{\sigma_{n-1}}$ subcubes of the 2^{-n}-grid. Since $2^{\sigma_n} \le 2^N 2^{\sigma_{n-1}}$, (see (10)), it is possible to define the union A''_n of $2^{\sigma_n} - 2^{\sigma_{n-1}}$ arbitrarily chosen subcubes with disjoint interiors, contained in $A_{n-1} \setminus \operatorname{int} A'$ and belonging to the 2^{-n}-grid.

Define $A_n = A'_n \cup A''_n$. The set $A = \cap_n A_n$ has desired properties since $2^{\sigma_n} \le \omega_n(A) \le 3^N 2^{\sigma_n}$. □

Remark 2.2. The construction of approximating sets A_n described in the proof of Lemma 2.1 is to be interpreted as the process of "swarming" of the fractal A, thinking of n as time. The analogue of Lemma 2.1 can be stated for any base $b > 2$ instead of base $b = 2$.

Remark 2.3. It is easy to see that for any infinite set A in $[0,1]^N$ there exists an absolutely continuous function $f(t)$, $t \ge 1$, having properties (7), such that $\omega_n(A) = 2^{f(n)}$. Indeed, the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) := \log_2 \omega_n(A)$ can be extended to a piecewise linear and continuous function $f : [1,\infty) \to \mathbb{R}$ satisfying properties (7). The function $f(t)$ in Lemma 2.1, satisfying conditions (7), will be called a swarming function of the set A. A swarming function defines the set A up to the choice of $2^{f(n)}$ among 2^{nN} cubes of the 2^{-n}-grid in $[0,1]^N$ for each n.

Remark 2.4. If in Lemma 2.1 we want A to be only a bounded set in $\mathbb{R}^N$ (not necessarily contained in $[0,1]^N$), then we can relax the condition on $f(1)$ in (7) to $f(1) \ge 0$. The initial set A_1 containing A will be the union of $2^{\lfloor f(1) \rfloor}$ closed cubes of 2^{-1}-grid in $\mathbb{R}^N$.

Remark 2.5. It is easy to see that (9) holds also in the form

$$\overline{\dim}_B A = \limsup_{t\to\infty} \frac{f(t)}{t}, \quad \underline{\dim}_B A = \liminf_{t\to\infty} \frac{f(t)}{t}. \tag{11}$$

To show this, it suffices to note that for any $t \ge 1$ there exists a unique $n \in \mathbb{N}$ such that $n \le t < n+1$; hence, $\frac{f(n)}{n+1} \le \frac{f(t)}{t} \le \frac{f(n+1)}{n}$, and the claim follows from the fact that $\limsup_n(a_n b_n) = \limsup_n a_n \lim_n b_n$, provided b_n is convergent (take $a_n := f(n)/(n+1))$, $b_n := (n+1)/n$, and then $a_n := f(n+1)/n$, $b_n := n/(n+1))$, and analogously for $\liminf$.

The following result is known, (see Mattila [19, p. 77]). The novelty is in the construction of fractal sets with separated box dimensions, based on the use of (zigzagging) swarming functions. We hope this construction is more transparent than those based on the use of modified Cantor sets. Here we do not study Hausdorff's dimension of fractal sets.

Theorem 2.2. (Separation of box dimensions) *For any prescribed pair of values $\underline{d} \le \overline{d}$ in $[0,N]$ there exists a constructive subset A of $[0,1]^N$, generated by an explicit swarming function, such that*

$$\underline{\dim}_B A = \underline{d} \text{ and } \overline{\dim}_B A = \overline{d}.$$

In particular, there exists a set A with maximally separated box dimensions $\underline{\dim}_B A = 0$ and $\underline{\dim}_B A = N$.

Proof. (a) Let us first consider the case when $\underline{d}, \overline{d} \in (0,1)$. We use Lemma 2.1. Let a set A be generated by a swarming function $f(t)$ defined in the following way. If $\underline{d} = \overline{d} =: d$ we let $f(t) = d \cdot t$. If $\underline{d} < \overline{d}$ we first define two auxiliary functions $f_1(t) := \underline{d} \cdot t$ and $f_2(t) := \overline{d} \cdot t$, $t \ge 1$. Now it is easy to construct a piecewise linear, continuous function $f(t)$ such that $f'(t) = 0$ or N a.e., $f_1(t) \le f(t) \le f_2(t)$ for all $t \ge t_0$ where t_0 is sufficiently large, and there exist two monotone, divergent sequences t'_n and t''_n in $[t_0, \infty)$ such that for all n,

$$f(t'_n) = f_1(t'_n) \text{ and } f(t''_n) = f_2(t''_n).$$

From this we get $\overline{\dim}_B A = \limsup_n f(t)/t = \limsup_n f(t''_n)/t''_n = \overline{d}$, and analogously for $\underline{\dim}_B A$. The graph of the function f can be constructed by drawing a line from the origin O to arbitrarily chosen point P_1' on the graph of f_1, then we take the line with the slope N from P_1' to the point P_1'' on the graph of f_2, then continue with horizontal line to the point P_2' on the graph of f_1, then take again the line with the slope N to the point P_2'' on the graph of f_2, and so on. The zigzagging function f constructed in this way satisfies the assumptions of Lemma 2.1, and the claim follows from (9).

(b) Consider the case $\underline{d} = 0$ and $\overline{d} = N$. Now we define auxiliary functions $f_1(t) = \sqrt{t}$ and $f_2(t) = Nt - \sqrt{t}$. We construct a piecewise linear, absolutely continuous function $f(t)$ analogously as in (a). Using Lemma 1 again, the corresponding set A in $[0,1]^N$ satisfies the desired properties.

(c) The remaining cases where either $\underline{d} = 0$ and $\overline{d} \in [0, N)$, or $\overline{d} = N$ and $\underline{d} \in (0, N]$, can be treated similarly. $\square$

It is well known that Hausdorff's dimension is countably stable, (see Falconer [4, p 29]), while the upper box dimension is only finitely stable. There are known examples showing that lower box dimension is unstable with respect to finite unions; that is, there are subsets A and B of $[0,1]$ such that

$$\underline{\dim}_B(A \cup B) > \max\{\underline{\dim}_B A, \underline{\dim}_B B\},$$

(see e.g. Tricot [23, pp. 30 and 123]). Here we provide a simple construction of subsets achieving maximal instability of lower box dimension with respect to union. We shall first state a simple result about box dimensions of unions and Cartesian products of sets.

Lemma 2.3. *Let A and B be two bounded subsets of $\mathbb{R}^N$. We have*

$$\underline{\dim}_B(A \cup B) = \liminf_{n\to\infty} \frac{\max\{s_n(A), s_n(B)\}}{n},$$
$$\underline{\dim}_B(A \times B) = \liminf_{n\to\infty} \frac{s_n(A) + s_n(B)}{n},$$

and analogously for the upper box dimension, where $s_n(\cdot)$ is defined by (1). In particular, if $s_n(A) \sim \lfloor f_A(n) \rfloor$ and $s_n(B) \sim \lfloor f_B(n) \rfloor$ as $n \to \infty$, where f_A and f_B are swarming functions of A and B, see Lemma 2.1, then

$$\underline{\dim}_B(A \cup B) = \liminf_{n\to\infty} \frac{\max\{f_A(n), f_B(n)\}}{n},$$
$$\underline{\dim}_B(A \times B) = \liminf_{n\to\infty} \frac{f_A(n) + f_B(n)}{n},$$

and analogously for the upper box dimension.

PROOF. Using

$$2^{\max\{s_n(A),s_n(B)\}} = \max\{\omega_n(A),\omega_n(B)\} \le \omega_n(A\cup B) \le \omega_n(A)+\omega_n(B)$$
$$= 2^{s_n(A)} + 2^{s_n(B)} \le 2\cdot 2^{\max\{s_n(A),s_n(B)\}}.$$

we get the following estimate for $s_n(A\cup B) := \log_2 \omega_n(A\cup B)$:

$$\max\{s_n(A),s_n(B)\} \le s_n(A\cup B) \le 1+\max\{s_n(A),s_n(B)\}.$$

The claim about box dimension of union follows immediately. The claim for Cartesian products follows easily from $\omega_n(A\times B) = \omega_n(A)\cdot\omega_n(B)$. □

Theorem 2.4. (Instability of lower box dimension) *There exist two constructive subsets A and B in $[0,1]^N$ such that $\underline{\dim}_B A = \underline{\dim}_B B = 0$, while $\dim_B(A\cup B) = N$. Furthermore, there exist constructive subsets A and B of $[0,1]^N$ such that $\underline{\dim}_B A = \underline{\dim}_B B = 0$, while $\underline{\dim}_B(A\times B) = N$ and $\overline{\dim}_B A = \overline{\dim}_B B = N$, $\overline{\dim}_B(A\times B) = N$.*

PROOF. (a) Let $f_1(t) = \sqrt{t}$ and $f_2(t) = Nt-\sqrt{t}$, $t\ge 1$. The idea is to construct sets A and B with swarming functions f_a and f_b such that $\max\{f_a(t), f_b(t)\} = f_2(t)$, and such that $f_a(t_n') = f_1(t_n')$ for odd n, and $f_b(t_n') = f_1(t_n')$ for even n, where $t_n' \to \infty$ as $n\to\infty$.

Let P_n' and P_n'' be sequences of points on the graphs of functions f_1 and f_2 respectively, constructed analogously as in the proof of Theorem 2.2 while constructing the zigzagging function f there. Using these points we define an absolutely continuous function $f_a(t)$ as follows. Its graph starts from the origin O along the graph of f_1 to the point P_1' chosen below the graph of f_2, then proceed with the line with the slope N to the point P_1'' on the graph of f_2, then continues along the graph of f_2 until P_2'' (note that we skipped P_2'), next along the horizontal line to the point P_3' on the graph of f_1, then along the line with the slope N to the point P_3'' on the graph of f_2, now along the graph of f_2 to P_4'' (we skipped P_4'), and then along horizontal line to P_5', and so on.

We construct the function $f_b(t)$ which is in a sense complementary to f_a. We define its graph to be the same as f_2 from the origin to P_1'' (we skipped P_1'), and then go horizontally to the point P_2' on the graph of f_1, then along line with slope N to the point P_2'' on the graph of f_2, and now along the graph of f_2 to the point P_3'' (we skipped P_3'), then horizontally to P_4', then along the line with slope N until P_4'', and so on.

Let A and B be two sets generated by swarming functions f_a and f_b, (see Lemma 2.1). It is clear from the construction that $\max\{f_a(n), f_b(n)\} =$

$Nn - \sqrt{n}$. Using Lemma 2.3 we conclude that

$$\dim_B(A \cup B) = \lim_{n\to\infty} \frac{\max\{f_a(n), f_b(n)\}}{n} = N.$$

On the other hand, $f_a(t'_n) = \sqrt{t'_n}$ for infinitely many n's and the same for f_b, and using (11) we obtain that $\underline{\dim}_B A = \underline{\dim}_B B = 0$.

(b) We construct piecewise linear, continuous functions $f_a(t)$ and $f_b(t)$, $t \geq 1$, as follows. Let us start with auxiliary functions $f_1(t) = \sqrt{t}$ and $f_2(t) = Nt - \sqrt{t}$. The graphs of functions f_a and f_b will be defined so as to zigzag between f_1 and f_2, and so that $f_a(t) + f_b(t) = Nt$. First define $f_a(t)$. Let f_a be equal to f_2 from the origin to a point P_1 on the graph of f_2 chosen above the graph of f_1, next continue horizontally to the point P_2 on the graph of f_1, then with the slope N to the point P_3 on the graph of f_2, then horizontally to the point on the graph of f_1, and so on.

Let $f_b(t) := Nt - f_a(t)$. It is easy to see that f_a and f_b satisfy conditions of Lemma 2.1, so that they generate subsets A and B of $[0,1]^N$. Since the graphs of f_a and f_b meet the graph of $f_1(t) = \sqrt{t}$ along an unbounded sequences of values $t_n(a)$ and $t_n(b)$ respectively, the lower box dimensions of A and B are both equal to zero. On the other hand, using Lemma 2.3 we have

$$\underline{\dim}_B(A \times B) = \liminf_{n\to\infty} \frac{f_A(n) + f_B(n)}{n} = N.$$

The claim related to the upper box dimension is obtained by modifying the above proof. $\square$

Remark 2.6. Using a slight modification in the proof we can find two subsets A and B of $[0,1]^N$ such that $\underline{\dim}_B A = \underline{\dim}_B B = 0$, $\underline{\dim}_B(A \times B) = N$ and $\overline{\dim}_B(A \times B) = 2N$.

Remark 2.7. Motivated by Theorems 2.2 and 2.4 it seems to make sense to speak about "zigzagging fractals". By this we mean that the corresponding swarming function $f(t)$, $t \geq 1$ (obtained as a continuous and piecewise linear extension of $f(n) = \log_2 \omega_n(A)$) zigzags between two prescribed functions $f_1(t)$ and $f_2(t)$, such that $f_1(t) \leq f(t) \leq f_2(t)$ for $t \geq t_0$, and such that f meets each of the graphs of f_1 and f_2 at divergent sequence of values t'_n and t''_n respectively. If we assume that there exists $\lim_{t\to\infty} \frac{f_i(t)}{t}$, $i = 1, 2$, then

$$\underline{\dim}_B A = \lim_{t\to\infty} \frac{f_1(t)}{t} \text{ and } \overline{\dim}_B A = \lim_{t\to\infty} \frac{f_2(t)}{t}. \tag{12}$$

With a suitable choice of bounding functions f_1 and f_2 it is possible to have control not only over box dimensions, but also over Minkowski contents of A, (see Corollaries 3.3 and 3.4 below).

3 Bounds and Separation of Minkowski Contents.

In this section we obtain upper and lower bounds of Minkowski contents of a given bounded subset A of $\mathbb{R}^N$ in terms of the swarming sequence $s_n := \log_2 \omega_n(A)$, (see Theorem 3.1). Recall that s-dimensional lower and upper Minkowski contents of A, $s \geq 0$, are defined by

$$\mathcal{M}_*^s(A) = \liminf_{r\to 0} \frac{|A_r|}{r^{N-s}} \text{ and } \mathcal{M}^{*s}(A) = \limsup_{r\to 0} \frac{|A_r|}{r^{N-s}},$$

where A_r is the Minkowski sausage of A; that is, A_r is Euclidean r-neighborhood of A; $A_r = \{y \in \mathbb{R}^N : d(y,A) < r\}$, (see e.g. Mattila [19]). By $|A_r|$ we denote N-dimensional Lebesgue measure of A_r. We say that A is Minkowski measurable if there exists $d = \dim_B A$ and $0 < \mathcal{M}_*^d(A) = \mathcal{M}^{*d}(A) < \infty$. We shall occasionally deal with sets indexed with integers, say A_1 and A_2, which should not be confused with Minkowski sausages.

The following theorem enables us to obtain necessary and sufficient conditions on A to have nondegenerate Minkowski contents for some $d \in [0, N]$; that is,

$$0 < \mathcal{M}_*^d(A) \leq \mathcal{M}^{*d}(A) < \infty, \tag{13}$$

see Theorem 3.2(c). Note that this condition implies $d = \dim_B A$. The crucial role in Theorem 3.1 is played by two natural parameters $\underline{D}$ and $\overline{D}$ associated with the fractal set A that we call lower and upper dispersions of A. The reason for this name is that information about these parameters enables partial control of the Minkowski contents, (see (15) and (16)). In particular, if the difference $\overline{D} - \underline{D}$ of dispersion parameters is large enough, then A is not Minkowski measurable, (see (19)).

Theorem 3.1. (Bounds of Minkowski contents) *Let A be a bounded subset of $\mathbb{R}^N$, $s_n := \log_2 \omega_n(A)$, and let $\underline{d} := \underline{\dim}_B A = \liminf_n s_n/n$, $\overline{d} := \overline{\dim}_B A = \limsup_n s_n/n$, see (2). Let*

$$\underline{D} := \liminf_{n\to\infty} (s_n - \underline{d}\, n) \text{ and } \overline{D} := \limsup_{n\to\infty} (s_n - \overline{d}\, n), \tag{14}$$

that we call lower and upper dispersions of A respectively. Then we have:

$$2^{-N} C_N \cdot 2^{\overline{D}} \leq \mathcal{M}^{*\overline{d}}(A) \leq 2^{N-\overline{d}} D_N \cdot 2^{\overline{D}}, \tag{15}$$

$$2^{\underline{d}-2N} C_N \cdot 2^{\underline{D}} \leq \mathcal{M}_*^{\underline{d}}(A) \leq D_N \cdot 2^{\underline{D}}, \tag{16}$$

where C_N and D_N are the N-dimensional Lebesgue measure of the unit ball and of the 1-neighborhood of the unit cube respectively.

PROOF. (a) To prove the left-hand side inequality of (15), take any of $\omega_n(A)$ closed cubes of the 2^{-n}-grid meeting A. The 2^{-n}-neighborhood of a point $x \in A$ in any such cube occupies the smallest part of the cube if and only if x is in a vertex of the cube. In this case the volume of this part is equal to $2^{-N}C_N(2^{-N})^n$. Hence, $|A_{2^{-n}}| \geq \omega_n(A) \cdot 2^{-N}C_N(2^{-N})^n$, and from this we have

$$\frac{|A_{2^{-n}}|}{(2^{-n})^{N-\overline{d}}} \geq 2^{-N}C_N \cdot 2^{s_n - n\overline{d}}.$$

Therefore,

$$\mathcal{M}^{*\overline{d}}(A) = \limsup_{r\to 0} \frac{|A_r|}{r^{N-\overline{d}}} \geq \limsup_{n\to\infty} \frac{|A_{2^{-n}}|}{(2^{-n})^{N-\overline{d}}} \geq 2^{-N}C_N \cdot 2^{\overline{D}}.$$

(b) To prove the right-hand side inequality of (16), note that the r neighborhood of the set of points of A contained in any of $\omega_n(A)$ cubes of the 2^{-n} grid is a subset of the 2^{-n}-neighborhood of the cube. The volume of the 2^{-n}-neighborhood of the cube is $D_N(2^{-n})^N$; hence

$$\frac{|A_{2^{-n}}|}{(2^{-n})^{N-\underline{d}}} \leq \frac{\omega_n(A) \cdot D_N 2^{-nN}}{(2^{-n})^{N-\underline{d}}} = D_N 2^{s_n - n\underline{d}}.$$

From this we conclude that

$$\mathcal{M}_*^{\underline{d}}(A) = \liminf_{r\to 0} \frac{|A_r|}{r^{N-\overline{d}}} \leq \liminf_{n\to\infty} \frac{|A_{2^{-n}}|}{(2^{-n})^{N-\underline{d}}} \leq D_N \cdot 2^{\underline{D}}.$$

(c) Let us fix any $r \in (0,1)$. There exists a unique integer $n \geq 1$ such that $2^{-n} \leq r < 2^{-(n-1)}$. As in (b) we obtain that

$$\frac{|A_r|}{r^{N-\overline{d}}} \leq \frac{|A_{2^{-(n-1)}}|}{(2^{-n})^{N-\overline{d}}} \leq \frac{\omega_{n-1}(A) \cdot D_N 2^{-(n-1)N}}{2^{-n(N-\overline{d})}} = 2^{N-\overline{d}} D_N \cdot 2^{s_{n-1}-(n-1)\overline{d}}.$$

The right-hand side inequality of (15) follows in much the same way as in (b).

(d) Fixing $r \in (0,1)$ and finding n as in (c), we obtain as in (a) that

$$\frac{|A_r|}{r^{N-\underline{d}}} \geq \frac{|A_{2^{-n}}|}{(2^{-(n-1)})^{N-\underline{d}}} \geq \frac{\omega_n(A) \cdot 2^{-N}C_N 2^{-nN}}{2^{-(n-1)(N-\underline{d})}} = 2^{\underline{d}-2N} C_N \cdot 2^{s_n - n\underline{d}}.$$

The left-hand side inequality of (16) follows as in (a). $\square$

Remark 3.1. It is well known that $C_N = 2\pi^{N-2}N^{-1}\Gamma(N/2)^{-1}$, for example, $C_1 = 2$, $C_2 = \pi$, $C_3 = 4\pi/3$, and $C_N \to 0$ as $N \to \infty$. For the values of D_N see Gardner [8, pp. 348–349 and Steiner's formula A.28]. It is easy to see that $D_1 = 3$, $D_2 = 5 + \pi$, $D_3 = 7 + 13\pi/3$, and clearly, $D_N \leq 3^N$.

Remark 3.2. Note that $\overline{D} = \limsup_n n(\frac{s_n}{n} - \overline{d})$, so that $\overline{D}$ is a measure of the speed of accumulation of the sequence s_n/n at $\overline{d}$: the larger $\overline{D}$, the slower the accumulation rate. Similarly for $\underline{D}$.

Remark 3.3. It is natural to view the value of

$$2^{\overline{D}} = 2^{\limsup_n (s_n - \overline{d}\, n)} = \limsup_{n\to\infty} \omega_n(A) \cdot (2^{-n})^{\overline{d}}$$

appearing in (15) as the d-dimensional upper Minkowski *grid* content $\mathcal{M}^{*\overline{d}}(A, 2)$ of A, with base $b = 2$. Inequality (15) shows that the upper Minkowski grid content is equivalent to the usual upper Minkowski content of A. Similarly for the lower Minkowski grid content. Compare with Falconer [4, discussion about the quantity $\nu(F)$ on p. 43], and also with the notion of Jordan content of A, (see Folland [7, pp. 71–73]), which is a special case of the above Minkowski grid content of A. Estimates (15) and (16) are also closely related to Martio and Vuorinen [18, (3.1) Lemma and (3.11) Theorem].

It is possible to describe a large class of bounded sets A in $\mathbb{R}^N$ that are not Minkowski measurable (see Theorem 3.2(b)). Also, using dispersion parameters $\underline{D}$ and $\overline{D}$ of A it is possible to characterize sets with nondegenerate Minkowski contents as those with nondegenerate dispersions (see Theorem 3.2(c)).

Theorem 3.2. (Separation of Minkowski contents) *Let the conditions of Theorem 3.1 be satisfied. Then:*
(a)

$$\mathcal{M}^{*\overline{d}}(A) = 0 \text{ or } \infty \Longleftrightarrow \overline{D} = -\infty \text{ or } \infty \text{ respectively,} \tag{17}$$

$$\mathcal{M}_*^{\underline{d}}(A) = 0 \text{ or } \infty \Longleftrightarrow \underline{D} = -\infty \text{ or } \infty \text{ respectively.} \tag{18}$$

(b) If $d := \underline{d} = \overline{d}$, then the Minkowski contents can be separated, provided $\overline{D} - \underline{D}$ is large enough:

$$\overline{D} - \underline{D} > N + \log_2(D_N/C_N) \Longrightarrow \mathcal{M}_*^d(A) < \mathcal{M}^{*d}(A). \tag{19}$$

More precisely, assuming the lower bound on $\overline{D} - \underline{D}$ in (19) is fulfilled, we have

$$\mathcal{M}_*^d(A) \le D_N \cdot 2^{\underline{D}} < 2^{-N} C_N \cdot 2^{\overline{D}} \le \mathcal{M}^{*d}(A). \tag{20}$$

(c) If $d := \underline{d} = \overline{d}$, then we have the following characterization of nondegeneracy of Minkowski contents of A.

$$0 < \mathcal{M}_*^d(A) \le \mathcal{M}^{*d}(A) < \infty \Longleftrightarrow \underline{D}, \overline{D} \in (-\infty, \infty). \tag{21}$$

PROOF. Assuming that $d = \underline{d} = \overline{d}$, the separation of the Minkowski contents is obviously secured by the condition $D_N \cdot 2^{\underline{D}} < 2^{-N} C_N \cdot 2^{\overline{D}}$ (see Theorem 3.1), which is equivalent to $\overline{D} - \underline{D} > N + \log_2(D_N/C_N)$. Properties (17), (18), and (21) follow immediately from (15) and (16). □

Remark 3.4. Due to property (19) it is natural to define $\operatorname{ampl} A := \overline{D} - \underline{D}$, the amplitude of the set A. The larger the amplitude of A, the larger the quotient of the upper and lower Minkowski contents. More precisely, if $\operatorname{ampl} A > N + \log_2(D_N/C_N)$, then, (see (20)),

$$\mathcal{M}^{*d}(A)/\mathcal{M}_*^d(A) \geq \frac{C_N}{2^N D_N} \cdot 2^{\operatorname{ampl} A}. \tag{22}$$

It is worth noting that Cantor's classical middle third set is not Minkowski measurable, although its amplitude is zero, (see (29) below).

The following two easy consequences of Theorem 3.1 provide numerous examples of fractal sets in $[0,1]^N$ with partial control over their Minkowski contents.

Corollary 3.3. *Let $\underline{d} \leq \overline{d}$ be given real numbers in $[0, N]$, and let $\underline{\delta}$ and $\overline{\delta}$ be arbitrary real numbers (if $\underline{d} = \overline{d}$, we assume that $\underline{\delta} \leq \overline{\delta}$). Let $f_1(t) = \underline{d} \cdot t + \underline{\delta}$ and $f_2(t) = \overline{d} \cdot t + \overline{\delta}$, $t \geq 1$. Assume that $f(t)$ is an absolutely continuous function satisfying $0 \leq f'(t) \leq N$ a.e., $f(1) \in [0, N+1)$, and there exists $s_0 \geq 1$ such that $f_1(t) \leq f(t) \leq f_2(t)$ for $t \geq s_0$. Assume also that there exist two unbounded sequences of points with integer abscissae n' and n'', lying on the graph of f_1 and f_2 respectively, such that they also belong to the graph of f. Let A be a subset of $[0,1]^N$ generated by the swarming function f, (see Lemma 2.1). Then $\lfloor f(n) \rfloor \leq \log_2 \omega_n(A) \leq \lfloor f(n) \rfloor + N \log_2 3$, $\underline{\dim}_B A = \underline{d}$, $\overline{\dim}_B A = \overline{d}$, and for the Minkowski contents we have the estimates*

$$2^{-N} C_N \cdot 2^{\overline{\delta}-1} \leq \mathcal{M}^{*\overline{d}}(A) \leq 3^N 2^{N-\overline{d}} D_N \cdot 2^{\overline{\delta}}, \tag{23}$$

$$2^{\underline{d}-2N} \cdot 2^{\underline{\delta}-1} \leq \mathcal{M}_*^{\underline{d}}(A) \leq 3^N D_N \cdot 2^{\underline{\delta}}. \tag{24}$$

If $d := \underline{d} = \overline{d}$ and $\overline{\delta} - \underline{\delta} > N + 1 + \log_2(3^N D_N/C_N)$, then $\mathcal{M}^{*d}(A) > \mathcal{M}_*^d(A)$.

PROOF. Bounds for $s_n := \log_2 \omega_n(A)$ follow from Lemma 2.1, and the assumption about points with integer abscissae implies that $\underline{\dim}_B = \underline{d}$ and $\overline{\dim}_B A = \overline{d}$. We use Theorem 3.1 by noting that $s_n - \overline{d} \cdot n \leq \lfloor \overline{d} \cdot n + \overline{\delta} \rfloor - \overline{d} \cdot n \leq \overline{\delta} + N \log_2 3$ for all $n \geq s_0$, and $s_{n''} - \overline{d} \cdot n_{n''} \geq \lfloor \overline{d} \cdot n'' + \overline{\delta} \rfloor - \overline{d} \cdot n_{n''}$; hence $\overline{\delta} - 1 \leq \overline{D} \leq \overline{\delta} + N \log_2 3$, and similarly $\underline{\delta} - 1 \leq \underline{D} \leq \underline{\delta} + N \log_2 3$. □

Remark 3.5. The existence of a function f satisfying the conditions of Corollary 3.3 (given the functions f_1 and f_2) is easy to verify. It suffices to modify a swarming function such that $f'(t) = 0$ or $f'(t) = N$ a.e., described in the proof of Theorem 2.2, in such a way that it is equal to f_1 in a 1-neighborhood of each P_i', and equal to f_2 in a 1-neighborhood of each P_i''.

Example 3.1. We can describe a class of fractals with degenerate Minkowski contents. It suffices for example that $f_1(t) = \underline{d} \cdot t + \varphi_1(t)$ where $\varphi_1(t)$ is an absolutely continuous function such that $\varphi_1(t) = o(t)$, $\varphi_1'(t) \geq -\underline{d}$ for a.e. t, and $\varphi_1(t) \to \infty$ (or $-\infty$) as $t \to \infty$, and similarly for f_2. Namely, in this case we have that $\underline{D} = \liminf_n (s_n - \underline{d}\, n) = \liminf_n \varphi_1(n) = \infty$ (or $-\infty$ respectively), and the claim follows from Theorem 3.2. We confine ourselves to formulate a special case, rather than the most general possible result.

Corollary 3.4. (Degenerate Minkowski contents) *Let us change the swarming functions f_1 and f_2 appearing in Corollary 3.3 to*

$$f_1^{\pm}(t) = \underline{d} \cdot t \pm \sqrt{t} \text{ and } f_2^{\pm}(t) = \overline{d} \cdot t \pm \sqrt{t}.$$

In the case when $\underline{d} = \overline{d}$ we do not allow the combination of f_1^+ and f_2^-. Let a swarming function f be constructed in the same way as in Corollary 3.3. If $A = A(f_1^{\pm}, f_2^{\pm})$ is a subset of $[0,1]^N$ generated by f (see Lemma 2.1), then $\underline{\dim}_B A = \underline{d}$, $\overline{\dim}_B A = \overline{d}$, and the corresponding Minkowski contents are degenerate:

$$\begin{aligned}
&\text{(a) } \mathcal{M}_*^{\underline{d}}(A(f_1^-, f_2^-)) = 0, && \mathcal{M}^{*\overline{d}}(A(f_1^-, f_2^-)) = 0;\\
&\text{(b) } \mathcal{M}_*^{\underline{d}}(A(f_1^-, f_2^+)) = 0, && \mathcal{M}^{*\overline{d}}(A(f_1^-, f_2^+)) = \infty;\\
&\text{(c) } \mathcal{M}_*^{\underline{d}}(A(f_1^+, f_2^-)) = \infty, && \mathcal{M}^{*\overline{d}}(A(f_1^+, f_2^-)) = 0;\\
&\text{(d) } \mathcal{M}_*^{\underline{d}}(A(f_1^+, f_2^+)) = \infty, && \mathcal{M}^{*\overline{d}}(A(f_1^+, f_2^+)) = \infty.
\end{aligned}$$

Remark 3.6. It is possible to combine for example a linear function $f_1(t) = \underline{d} \cdot t + \underline{\delta}$ with a nonlinear function $f_2(t) = \overline{d} \cdot t + \sqrt{t}$, in order to obtain a swarming function of a fractal set with nondegenerate Minkowski contents at dimension $\underline{\dim}_B A = \underline{d}$ and degenerate Minkowski contents at dimension $\overline{\dim}_B A = \overline{d}$, etc.

Remark 3.7. It is clear that Theorem 3.1 can be extended to arbitrary base $b \geq 2$. Indeed, defining $\omega_n(A, b)$ as the number of closed cubes of b^{-n}-grid meeting a bounded set $A \subseteq \mathbb{R}^N$, and $s_n := \log_b \omega_n(A, b)$, then again $\overline{d} := \overline{\dim}_B A = \limsup_n s_n/n$, $\underline{d} := \underline{\dim}_B A = \liminf_n s_n/n$, and defining as before $\overline{D} := \limsup_n (s_n - \overline{d}\, n)$, : $\underline{D} = \liminf_n (s_n - \underline{d}\, n)$, we have

$$2^{-N} C_N \cdot b^{\overline{D}} \leq \mathcal{M}^{*\overline{d}}(A) \leq b^{N-\overline{d}} D_N \cdot b^{\overline{D}}, \tag{25}$$

$$2^{-N} b^{\underline{d}-N} C_N \cdot b^{\underline{D}} \leq \mathcal{M}_*^{\underline{d}}(A) \leq D_N \cdot b^{\underline{D}}. \tag{26}$$

Now we estimate the Minkowski contents of graphs of some continuous, rapidly oscillating functions $f : \Omega \to \mathbb{R}$ defined on an open set Ω in $\mathbb{R}^N$. We shall need the notion of oscillation of f on a subset B of Ω, defined by $\operatorname{osc}_B f := \sup_B f - \inf_B f$. Recall that for a given $\alpha \in (0, 1]$ we have that f is α-Hölderian if and only if there exists $\overline{c} > 0$ such that

$$\operatorname*{osc}_{Q(\delta)\cap\Omega} f \le \overline{c} \cdot \delta^\alpha \text{ for all } r > 0 \text{ and all } x \in \Omega,$$

where $Q(\delta)$ is any closed cube with side δ. We say that a function $f : \Omega \to \mathbb{R}$ is α-anti-Hölderian, $\alpha \in (0, 1]$, if by definition

$$\operatorname*{osc}_{B_r(x)\cap\Omega} f \ge \underline{c} \cdot r^\alpha \text{ for all } r > 0 \text{ and all } x \in \Omega,$$

for some $\underline{c} > 0$, (see Tricot [23, Section 12.5] and also Falconer [4, Corollary 11.2(b)]). The main example of a function which is both α-Hölderian and α-anti-Hölderian is the Weierstrass function.

Theorem 3.5. *(a) Let Ω be a bounded open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ a function which is α-Hölderian and α-anti-Hölderian, $\alpha \in (0, 1)$. If A is the graph of f in $\mathbb{R}^{N+1}$, then $d := \dim_B A = N + 1 - \alpha$ and*

$$2^{-N-1} C_{N+1}\, \underline{c}\, |\Omega| \le \mathcal{M}^{*d}(A) \le D_{N+1} 2^\alpha\, \overline{c}\, |\overline{\Omega}|,$$
$$2^{-\alpha-N-1} C_{N+1} \underline{c}\, |\Omega| \le \mathcal{M}_*^d(A) \le D_{N+1}\, \overline{c}\, |\overline{\Omega}|.$$

For $\alpha = 1$ these estimates hold with $\overline{c}$ changed to $2 + \overline{c}$.

(b) Let A be the graph of the Weierstrass function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sum_{k=1}^\infty \lambda^{-k\alpha} \cos(\lambda^k x)$, where $\lambda > 1$ is sufficiently large and $\alpha \in (0, 1)$. Then $d := \dim_B A = 2 - \alpha$ and

$$\frac{\pi}{4}\, \underline{c} \le \mathcal{M}^{*d}(A) \le (5 + \pi) 2^\alpha\, \overline{c},$$
$$2^{-\alpha-2} \pi\, \underline{c} \le \mathcal{M}_*^d(A) \le (5 + \pi)\, \overline{c},$$

where $\underline{c} = \frac{1}{20} \lambda^{-\alpha}$, $\overline{c} = (1 - \lambda^{-\alpha-1})^{-1} + 2(1 - \lambda^{-\alpha})^{-1}$. For the Knopp (or Takagi) function $f(x) = \sum_k 2^{-k\alpha} g(2^k x)$, $x \in [0, 1]$, where $g : \mathbb{R} \to \mathbb{R}$ is a periodic piecewise linear, continuous function passing through the origin, with values in $[0, 1]$, $g'(x) = \pm 2$ for a.e. x, the above estimates hold with $\underline{c} = 1$ and $\overline{c} = 2(2^{1-\alpha} - 1)^{-1} + (2^\alpha - 1)^{-1}$.

(c) Let A be the graph of the self-affine function $f : [0, 1] \to \mathbb{R}$ generated by affine transformations $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, \dots, m$,

$$S_i \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \frac{1}{m} & 0 \\ a_i & c_i \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} + \begin{bmatrix} \frac{i-1}{m} \\ b_i \end{bmatrix}$$

where $1/m < c_i \le 1$. *Let* $c := \min_i c_i$, $a := \max_i |a_i|$, h *be the height of* A*; that is, the length of its projection onto the vertical axis,* v *be the vertical distance from* q_2 *to the segment* $[q_1, q_3]$ *(see Falconer [4, Example 11.4]). Then* $d := \dim_B F = 1 + \log_m(c_1 + \cdots + c_m)$*, and*

$$\frac{\pi v}{4} \le \mathcal{M}^{*d}(A) \le \frac{(5+\pi)m}{c_1 + \cdots + c_m}\left(\frac{ma}{1-(mc)^{-1}} + h\right),$$
$$\frac{\pi v}{4m}(c_1 + \cdots + c_m) \le \mathcal{M}_*^d(A) \le (5+\pi)\left(\frac{ma}{1-(mc)^{-1}} + h\right).$$

(d) Let A *be the McMullen set in* $[0,1]^2$ *defined by parameters* p, q, *and* r, $p > q$, $r < pq$. *Let the set* A_1 *be obtained by choosing of* r *out of* pq *rectangles in* $[0,1]^2$*, taken from the natural* $p^{-1} \times q^{-1}$*-rectangular grid. We assume that in any of* q *rows there is at least one chosen rectangle. The set* A *is obtained by intersection of iterated sets* A_n. *Then* $d := \dim_B A = 1 + \log_p \frac{r}{q}$ *(see Falconer [4, Example 9.11] with* $p_1 := p$ *and* $\sum_j N_j =: r$*, with columns instead of rows, and* $p < q$*), and*

$$\frac{\pi}{4} \le \mathcal{M}^{*d}(A) \le (5+\pi)\frac{pq}{r},$$
$$\frac{\pi r}{4pq}4 \le \mathcal{M}_*^d(A) \le 5+\pi.$$

(e) Let A *be the set of all real numbers* $x = 0.x_1x_2x_3\cdots \in [0,1]$ *in binary representation (hence* $x_i \in \{0,1\}$*), such that there are no two consecutive* 1*'s in the sequence of digits* (x_i)*, and there is no infinite sequence of consecutive zeros. Then* $d := \dim_B A = \log_2 \frac{1+\sqrt{5}}{2}$ *and*

$$\frac{\sqrt{5}+2}{\sqrt{4}+1} \le \mathcal{M}^{*d}(A) \le 12 \cdot \frac{\sqrt{5}+2}{(\sqrt{5}+1)^2}$$
$$\text{and } \frac{\sqrt{5}+2}{(\sqrt{5}+1)^2} \le \mathcal{M}_*^d(A) \le 3\frac{\sqrt{5}+2}{\sqrt{5}+1}. \tag{27}$$

PROOF.

(a) We shall need the following lemma which is an easy extension of Falconer [4, Proposition 11.1].

Lemma 3.6. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *and* $f : \Omega \to \mathbb{R}$ *a continuous function. Denote its graph in* $\mathbb{R}^{N+1}$ *by* A. *Let* $\delta \in (0,1)$ *be a given number and* N_δ *the number of closed cubes of a* δ*-grid in* $\mathbb{R}^{N+1}$ *that meet* A. *Let* $C(\Omega, \delta)$ *be the collection of closed cubes* Q *of the corresponding* δ*-grid in* $\mathbb{R}^N$*, which meet*

Ω. Let $C_0(\Omega,\delta)$ be the collection of all cubes from $C(\Omega,\delta)$ that are contained in Ω. Then

$$\delta^{-1}\sum_{Q\in C_0(\Omega,\delta)}\operatorname*{osc}_Q f \le N_\delta \le 2\cdot \#C(\Omega,\delta)+\delta^{-1}\sum_{Q\in C(\Omega,\delta)}\operatorname*{osc}_{Q\cap\Omega} f.$$

PROOF. First, we have that

$$\lim_{\delta\to 0}\#C_0(\Omega,\delta)\cdot\delta^N=|\Omega| \text{ and } \lim_{\delta\to 0}\#C(\Omega,\delta)\cdot\delta^N=|\overline{\Omega}| \tag{28}$$

(see Folland [7, p. 73]). These limits are called lower and upper Jordan contents of Ω respectively (see Folland [7, pp. 72–73]). For all $\delta>0$ we have $\#C_0(\Omega,\delta)\cdot\delta^N\le|\Omega|\le|\overline{\Omega}|\le\#C(\Omega,\delta)\cdot\delta^N$, with the left-hand side and the right-hand side being nondecreasing and nonincreasing functions of δ respectively. Let $c_1>|\overline{\Omega}|$ be given. Since $\#C(\Omega,\delta)\le c_1\delta^{-N}$ for δ sufficiently small (see (28)) using Lemma 3.6 and $\alpha\in(0,1)$ we have

$$N_\delta\le 2c_1\delta^{-N}+\delta^{-1}\cdot c_1\delta^{-N}\cdot\overline{c}\delta^\alpha\le c_2\cdot\delta^{-(N+1-\alpha)},$$

where c_2 is larger than and arbitrarily close to $|\overline{\Omega}|\cdot\overline{c}$, provided δ is sufficiently small. Let us consider the standard 2^{-n}-grid; that is, $\delta=2^{-n}$. Then from

$$s_n:=\log_2\omega_n(A)=\log_2 N_{2^{-n}}\le\log_2 c_2+n(N+1-\alpha)$$

we have $\overline{D}:=\limsup_n(s_n-nd)\le\log_2 c_2$. If we let $c_2\to|\overline{\Omega}|\cdot\overline{c}$, we obtain $\overline{D}\le\log_2(|\overline{\Omega}|\cdot\overline{c})$. Analogously, $\underline{D}\ge\log_2(|\Omega|\underline{c})$, and the claim follows from Theorem 3.1. The case $\alpha=1$ is treated analogously.

(b) The claim follows from (a) (see also Falconer [4, Example 11.3] or Tricot [23, Section 12.7]).

(c) We sketch the proof. Here we consider the base $b:=m$ and the corresponding m^{-n} grid. Using the analysis from Falconer [4, Example 11.4] we obtain that $\log_m v+nd\le s_n\le\log_m r_1^+ +nd$, where r_1^+ is larger than and arbitrarily close to $r_1:=h+ma/(1-(mc)^{-1})$, provided n is sufficiently large. From this we conclude that

$$\log_m v\le\underline{D}\le\overline{D}\le\log_m r_1,$$

and the claim follows from estimates (25) and (26).

(d) Take any closed interval of length p^{-n} belonging to the natural p^{-n}-grid on the base interval of the square $[0,1]^2$. The number of closed squares in $[0,1]^2$ with base p^{-n} lying in the column above the chosen interval, that meet A, is between $(p/q)^n$ and $(p/q)^n+1$. Hence, $(rp/q)^n\le\omega_n(A,p)\le r^n((p/q)^n+1)$, and from this we get

$$nd\le s_n:=\log_p\omega_n(A,p)\le\log_p[r^n((p/q)^n+1)].$$

Since $0 \le s_n - nd \le \log_p[1 + (q/p)^n] \to 0$ as $n \to \infty$, we obtain $D = 0$. The claim follows from (25) and (26) with $b := p$.

(e) We have that $\omega_n(A) = \|B^{n-1}\|$, where B is the matrix

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

and for any 2×2 matrix B' we take the norm $\|B'\| = \sum_{i,j} |b'_{ij}|$. Using (2) and the fact that the sequence $\|B^n\|^{1/n}$ converges to the spectral radius of B (which is equal to $(1 + \sqrt{5})/2$), we conclude that $\dim_B A = \log_2 \frac{1+\sqrt{5}}{2}$, see Bishop [1, Chapter 1, Example 2.3] for details. Here we have that $s_n := \log_2 \omega_n(A) = \log_2 \|B^{n-1}\|$. Since

$$B^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$$

where F_n are Fibonacci numbers, $F_n = \frac{1}{2}(\alpha^n - \beta^n)$, $n = 0, 1, 2, \ldots$, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $\beta = \frac{1}{2}(1 - \sqrt{5})$, and $|\beta| < 1$, we obtain

$$\begin{aligned} \underline{D} &= \liminf_{n\to\infty}(s_n - n\,d) = \lim_{n\to\infty}(\log_2 \|B^{n-1}\| - n\,d) \\ &= \log_2[\lim_{n\to\infty} \|B^{n-1}\|\, \alpha^{-n}] \\ &= \log_2[\lim_{n\to\infty} (F_n + 2F_{n-1} + F_{n-2})\alpha^{-n}] \\ &= \log_2[\lim_{n\to\infty} \frac{1}{2}(\alpha^n + 2\alpha^{n-1} + \alpha^{n-2} + O(\beta^{n-1}))\alpha^{-n}] \\ &= \log_2(1 + 2\alpha^{-1} + \alpha^{-2}) = \log_2 \frac{\sqrt{5}+2}{\sqrt{5}+1}. \end{aligned}$$

Using Theorem 3.1 we obtain (27). □

Remark 3.8. Regarding Theorem 3.5(a), it is possible to construct an open and bounded set Ω such that $|\Omega| < |\overline{\Omega}|$, and even with arbitrarily small $|\Omega|$ and arbitrarily large $|\overline{\Omega}|$. For example, it suffices to take a ball $B_R(0)$ with large R, a countable dense set S in the ball, and cover S with a sequence of balls contained in $B_R(0)$ such that their union Ω has arbitrarily small Lebesgue measure. Clearly, $\overline{\Omega} = \overline{B}_R(0)$. Furthermore, it is possible to construct an open set Ω such that $|\Omega|$ is arbitrarily small and $\overline{\Omega} = \mathbb{R}^N$. It suffices to take S dense in $\mathbb{R}^N$ and proceed as before.

Remark 3.9. We do not know if any of the sets appearing in Theorem 3.5 is Minkowski measurable. It is known that the box dimension of the graph

A of the function $y := x^\alpha \cos x^{-\beta}$, $x \in (0,1)$, where α and β are positive constants such that $\alpha < \beta$, is equal to $d := 2-(\alpha+1)/(\beta+1)$ (see Tricot [23, pp. 111, 112]). It can be shown that the d-dimensional Minkowski contents of A are nondegenerate (personal information by Dr. Mervan Pašić). Also, estimates of Minkowski contents of graphs of rapidly oscillating solutions of nonlinear ordinary differential equations are studied for the first time in Pašić and Županović [21], using methods developed in Pašić [20]. In Martio and Vuorinen [18, (4.20) Remark] it is shown that any isotropic self-similar fractal set A which satisfies the open set condition is Minkowski nondegenerate; that is, has nondegenerate d-dimensional Minkowski contents, where $d = \dim_B A$.

Example 3.2. We show that the converse of separation property (19) in Theorem 3.2 does not hold. Let us consider standard Cantor's middle third set $A \subseteq [0,1]$. Here we take the natural base $b = 3$, for which it is easy to see that $\omega_n(A,3) = 5 \cdot 2^{n-1}$. (Recall that we count *closed* intervals of 3^{-n}-grid of the real line, meeting A; the number of open such intervals is 2^n.) Hence, $s_n := \log_3 \omega_n(A,3) = \log_3 \frac{5}{2} + nd$, where $d = \log_3 2$, and therefore

$$D := \overline{D} = \underline{D} = \log_3 \frac{5}{2}, \ \ \operatorname{ampl} A := \overline{D} - \underline{D} = 0. \tag{29}$$

Using bounds (25) and (26) we obtain

$$\frac{5}{2} \le \mathcal{M}^{*d}(A) \le \frac{45}{4} \text{ and } \frac{5}{3} \le \mathcal{M}_*^d(A) \le \frac{15}{2}.$$

This is in accordance with precise values of Minkowski contents of Cantor's set found in Lapidus, Pomerance [15, Theorem 2.4]:

$$\mathcal{M}^{*d}(A) = (\log_{3/2} 9)\,(\log_4 3/2)^{\log_3 2} \approx 2.58304,$$
$$\mathcal{M}_*^d(A) = 2^{2-\log_3 2} \approx 2.49497.$$

Minkowski nonmeasurability of self-similar fractal strings of lattice type, originally proved by Lapidus [12], is studied in Lapidus and van Frankenhuysen [17, Section 6.3.1], in the context of associated zeta functions and complex fractal dimensions.

4 Minkowski Contents and the Lebesgue Integral.

In this section we study the Lebesgue integrability of functions of the form $d(x,A)^{-\gamma}$ in a neighborhood of A, where A is a given fractal set and $\gamma > 0$. The first result known to us in which the Minkowski content condition was employed to study this kind of problems, has been published by Harvey and Polking [9, p. 42] in 1970, see also a brief historical overview in [27].

Let Ω be a given open subset of $\mathbb{R}^N$ and $A \subseteq \overline{\Omega}$ a nonempty subset, not necessarily bounded. Assume that $h_i : (0, r_0) \to \mathbb{R}$, $i = 1, 2$, are two positive, continuous functions such that $h_i(r) \to 0$ as $r \to 0$. We define generalized lower and upper Minkowski contents of A relative to Ω, and with respect to gauge functions h_i:

$$\mathcal{M}_*(h_1, A, \Omega) := \liminf_{r \to 0} \frac{|A_r \cap \Omega|}{r^N} h_1(r), \tag{30}$$

$$\mathcal{M}^*(h_2, A, \Omega) := \limsup_{r \to 0} \frac{|A_r \cap \Omega|}{r^N} h_2(r). \tag{31}$$

Relative Minkowski contents were introduced in this generality in Žubrinić [26], motivated by He and Lapidus [10]. If $\Omega = \mathbb{R}^N$ (or if $A_\varepsilon \subseteq \Omega$ for some $\varepsilon > 0$), then instead of (30) and (31) we write $\mathcal{M}_*(h_1, A)$ and $\mathcal{M}^*(h_2, A)$. Note that for $h_i(r) = r^s$, $i = 1, 2$, with fixed $s \geq 0$, we obtain the s-dimensional lower and upper Minkowski contents $\mathcal{M}_*^s(A, \Omega)$ and $\mathcal{M}^{*s}(A, \Omega)$ of A relative to Ω, introduced also in [26]:

$$\mathcal{M}_*^s(A, \Omega) := \liminf_{r \to 0} \frac{|A_r \cap \Omega|}{r^{N-s}} \text{ and } \mathcal{M}^{*s}(A, \Omega) := \limsup_{r \to 0} \frac{|A_r \cap \Omega|}{r^{N-s}}$$

It is easy to see that there exists a unique value $s = \underline{d}$ at which the function $s \mapsto \mathcal{M}_*^s(A, \Omega)$ has infinite jump from infinity down to zero. This value is called the lower box dimension of A relative to Ω (or relative box dimension), and is denoted by $\underline{\dim}_B(A, \Omega)$. Similarly for the upper box dimension of A relative to Ω. If $\underline{\dim}_B(A, \Omega) = \overline{\dim}_B(A, \Omega)$, the common value is denoted by $\dim_B(A, \Omega)$, and we call it the relative box dimension of A with respect to Ω. For $\Omega = \mathbb{R}^N$ we obtain classical Minkowski contents and classical box dimensions. Note also that $A_r = (\overline{A})_r$, so that $\mathcal{M}^*(h_1, A, \Omega) = \mathcal{M}^*(h_1, \overline{A}, \Omega)$ and $\overline{\dim}_B(A, \Omega) = \overline{\dim}_B(\overline{A}, \Omega)$. As an example, an open set $\Omega \subset \mathbb{R}^2$ containing a (one-dimensional!) ray A can be constructed such that $\dim_B(A, \Omega) > 1$, see [26, Example 2.1].

In this section the identity obtained in Žubrinić [27, Lemma 3] will play a crucial role. Its first version appeared in [26, Theorem 2.9(a)].

Lemma 4.1. *Let $u : (0, r) \to \mathbb{R}$ be a decreasing, nonnegative, C^1 diffeomorphism from $(0, r)$ onto its range. Then*

$$\int_{A_r \cap \Omega} u(d(x, A))\, dx = u(r)\, |A_r \cap \Omega| + \int_0^r |A_t \cap \Omega|\, |u'(t)|\, dt. \tag{32}$$

In particular, for all $\gamma > 0$ we have

$$\int_{A_r\cap\Omega} d(x,A)^{-\gamma}dx = r^{-\gamma}|A_r\cap\Omega| + \gamma\int_0^r |A_t\cap\Omega|\, t^{-\gamma-1}dt. \tag{33}$$

We state a result providing necessary and sufficient conditions for integrability of a class of singular functions, which extends [26, Theorem 2.9 and Corollary 2.5]. Due to Lemma 4.1 its proof is very simple.

Theorem 4.2. *(a) Assume that $\mathcal{M}_*(h_1, A, \Omega) > 0$. Then for all $r > 0$,*

$$I_r := \int_{A_r\cap\Omega} d(x,A)^{-\gamma}dx < \infty \Longrightarrow \int_0^r \frac{t^{N-\gamma-1}}{h_1(t)}\,dt < \infty.$$

Moreover, for any fixed $R > 0$ and for all $r \in (0,R)$ we have

$$I_r \geq \underline{c}\left(\frac{r^{N-\gamma}}{h_1(r)} + \gamma\int_0^r \frac{t^{N-\gamma-1}}{h_1(t)}\,dt\right) \text{ and } \underline{c} := \inf_{t\in(0,R)} \frac{|A_t\cap\Omega|}{t^N}h_1(t). \tag{34}$$

(b) Assume that $\mathcal{M}^*(h_2, A, \Omega) < \infty$. Then for all $r > 0$,

$$\int_0^r \frac{t^{N-\gamma-1}}{h_2(t)}\,dt < \infty \Longrightarrow I_r := \int_{A_r\cap\Omega} d(x,A)^{-\gamma}dx < \infty$$

Moreover, for any fixed $R > 0$ and for all $r \in (0,R)$ we have that

$$I_r \leq \underline{c}\left(\frac{r^{N-\gamma}}{h_2(r)} + \gamma\int_0^r \frac{t^{N-\gamma-1}}{h_2(t)}\,dt\right) \text{ and } \underline{c} := \sup_{t\in(0,R)} \frac{|A_t\cap\Omega|}{t^N}h_2(t).$$

PROOF. (a) It is clear that $|A_t\cap\Omega| \geq \underline{c}t^N h(t)^{-1}$ for all $t \in (0,R)$. Using identity (33) we obtain the inequality in (34). Since $I_r < \infty$, the desired inequality follows. Claim (b) can be proved analogously. $\square$

Remark 4.1. If $0 < \mathcal{M}_*(h_1, A, \Omega)$ and $\mathcal{M}^*(h_2, A, \Omega) < \infty$, then we obtain the following asymptotics of I_r as $r \to 0$. For every $\varepsilon > 0$ there exists $R > 0$ such that for $r \in (0,R)$ we have

$$(\mathcal{M}_*(h_1, A, \Omega) - \varepsilon)\cdot F_1(r) \leq I_r \leq (\mathcal{M}^*(h_2, A, \Omega) + \varepsilon)\cdot F_2(r),$$

where we define $F_1(r) := r^{N-\gamma}h_1(r)^{-1} + \gamma\int_0^r t^{N-\gamma-1}h_1(t)^{-1}\,dt$, and analogously $F_2(r)$. In particular, if $0 < \mathcal{M}_*^d(A,\Omega) \leq \mathcal{M}^{*d}(A,\Omega) < \infty$, then

$$(\mathcal{M}_*^d(A,\Omega) - \varepsilon)\cdot F(r) \leq I_r \leq (\mathcal{M}^{*d}(A,\Omega) + \varepsilon)\cdot F(r),$$

where $F(r) := \frac{N-d}{N-d-\gamma} r^{N-d-\gamma}$, and hence $I_r \asymp r^{N-d-\gamma}$ as $r \to 0$; that is, the quotient $I_r / r^{N-d-\gamma}$ is between two positive constants (both arbitrarily close to the corresponding Minkowski contents) for all $r > 0$ sufficiently small. If A is Minkowski measurable relatively to Ω; that is, if there exists $\mathcal{M}^d(A, \Omega) \in (0, \infty)$, then

$$I_r \sim \frac{N-d}{N-d-\gamma} \mathcal{M}^d(A, \Omega) \cdot r^{N-d-\gamma} \quad \text{as } r \to 0. \tag{35}$$

These estimates extend Žubrinić [26, Theorems 2.9(c) and 3.1(b)].

Theorem 4.3. *(a) Assume that $\mathcal{M}_*(h_1, A, \Omega) > 0$ with $h_1(t) = t^d g_1(t)$, where $d \in [0, N]$, and $g_1(t)$ is a bounded function near $t = 0$; that is, $\limsup_{t \to 0} g_1(t) < \infty$. Then for all $r > 0$,*

$$\int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx < \infty \Longrightarrow \gamma < N - d.$$

(b) Assume that $\mathcal{M}^*(h_2, A, \Omega) < \infty$ with $h_2(t) = t^d g_2(t)$, where $d \in [0, N]$, and let $\limsup_{t \to 0} t^\varepsilon g_2(t)^{-1} < \infty$ for all $\varepsilon > 0$. Then for all $r > 0$,

$$\gamma < N - d \Longrightarrow \int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx < \infty.$$

(c) Assume that $\mathcal{M}_*(h_1, A, \Omega) > 0$ and $\mathcal{M}^*(h_2, A, \Omega) < \infty$ with $h_i(t) = t^d g_i(t)$, $i = 1, 2$, where $d \in [0, N]$, and let the conditions $\limsup_{t \to 0} g_1(t) < \infty$ and $\limsup_{t \to 0} t^\varepsilon g_2(t)^{-1} < \infty$ be satisfied for all $\varepsilon > 0$. Then for all $r > 0$,

$$\int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx < \infty \Longleftrightarrow \gamma < N - d. \tag{36}$$

PROOF.

(a) The function $g(t)$ is bounded by a positive constant C near $t = 0$. Using Theorem 4.2(a) we obtain that for $r > 0$,

$$\infty > \int_0^r t^{N-\gamma-1} h_1(t)^{-1} dt \geq \frac{1}{C} \int_0^r t^{N-d-\gamma-1} dt;$$

hence $\gamma < N - d$.

(b) Let us take $\varepsilon > 0$ sufficiently small, so that $N - \gamma - d - \varepsilon > 0$. There exists a constant $M = M(\varepsilon) > 0$ such that $t^\varepsilon g_2(t)^{-1} \leq M$. We have that

$$\begin{aligned} \int_0^r \frac{t^{N-\gamma-1}}{h_2(t)} dt &= \int_0^r t^{N-d-\gamma-1} g_2(t)^{-1} dt \\ &= \int_0^r t^{N-d-\gamma-\varepsilon-1} t^\varepsilon g_2(t)^{-1} dt \leq M \frac{r^{N-d-\gamma-\varepsilon}}{N-d-\gamma-\varepsilon} < \infty. \end{aligned}$$

The conclusion follows from Theorem 4.2(b).

Claim (c) follows from (a) and (b). □

Note that in Theorem 4.3(c) it may happen that $\mathcal{M}_*(h_1, A, \Omega)$ is larger than $\mathcal{M}^*(h_2, A, \Omega)$, but by multiplying h_1 or h_2 by a positive constant we can easily achieve that $0 < \mathcal{M}_*(h_1, A, \Omega) \le \mathcal{M}^*(h_2, A, \Omega) < \infty$. An immediate consequence of Theorem 4.3(c) is the following result, which extends [26, Theorem 3.1(b)]. It was the principal motivation for our study of Minkowski contents of fractal sets in this paper.

Theorem 4.4. *Assume that Ω is an open subset of $\mathbb{R}^N$ and $A \subseteq \overline{\Omega}$, where A is not necessarily bounded and assume the relative box dimension $d := \dim_B(A, \Omega)$ exists. If the relative Minkowski contents of A are nondegenerate; that is,*

$$0 < \mathcal{M}_*^d(A, \Omega) \le \mathcal{M}^{*d}(A, \Omega) < \infty,$$

then for all $r > 0$,

$$\int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx < \infty \Longleftrightarrow \gamma < N - \dim_B(A, \Omega).$$

In particular, if $d := \dim_B A$ exists, and if the corresponding classical Minkowski contents are nondegenerate; that is,

$$0 < \mathcal{M}_*^d(A) \le \mathcal{M}^{*d}(A) < \infty, \tag{37}$$

then for all $r > 0$,

$$\int_{A_r} d(x, A)^{-\gamma} dx < \infty \quad \Longleftrightarrow \quad \gamma < N - \dim_B A. \tag{38}$$

Remark 4.2. The second part of Theorem 4.4 applies to all fractal sets appearing in Theorem 3.5. In Theorem 5.2 we will show that the nondegeneracy condition of Minkowski contents in (37) is indeed essential for the characterization of integrability (38) to hold in this form.

We can also improve [26, Theorem 2.3], by showing that the He - Lapidus condition (see [26, condition (2.5)]) is not necessary (see Theorem 4.5(a) below). The proof follows from Theorem 4.2 similarly as the proof of Theorem 4.3.

Theorem 4.5. *Let Ω be an open set in $\mathbb{R}^N$ and A a subset of $\overline{\Omega}$.*

(a) Assume that $\mathcal{M}_*(h_1, A, \Omega) > 0$, where $h_1(r) \le C \cdot r^\alpha$ for all $r > 0$ sufficiently small, and $\alpha > 0$ is a constant. Then for all $r > 0$,

$$\int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx < \infty \Longrightarrow \gamma < N - \alpha.$$

(b) Assume that $\mathcal{M}^*(h_2, A, \Omega) < \infty$, where $h_2(r) \geq C \cdot r^\beta$ for all $r > 0$ sufficiently small, and $\beta > 0$ is a constant. Then for all $r > 0$,

$$\gamma < N - \beta \Longrightarrow \int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx < \infty.$$

Remark 4.3. Using Lemma 4.1 we can easily generate other Lebesgue integrable functions, for example those having logarithmic singularity on a prescribed set $A \subset \mathbb{R}^N$. It suffices to consider the function $u(t) := \log t^{-1}$ for $t \in (0,1)$; that is, $x \mapsto \log d(x, A)^{-1}$, or analogous functions with iterated logarithms.

Theorem 4.6. *Let A be a subset of $\mathbb{R}^N$ and Ω an open set such that $A \subset \overline{\Omega} \subseteq \mathbb{R}^N$. Assume that $0 < \mathcal{M}_*^d(A, \Omega) \leq \mathcal{M}^{*d}(A, \Omega) < \infty$, $d < N$. Then the function $x \mapsto \log d(x, A)^{-1}$ is Lebesgue integrable on $A_r \cap \Omega$. Moreover, for any fixed $R < 1$ and $r \in (0, R)$ we have*

$$F_1(r) \leq \int_{A_r \cap \Omega} \log d(x, A)^{-1} dx \leq F_2(r), \tag{39}$$

where $F_1(r) = (\log r^{-1} + (N-d)^{-1}) \cdot \inf_{t \in (0,R)} \frac{|A_t \cap \Omega|}{t^{N-d}} \cdot r^{N-d}$, and $F_2(r)$ is defined analogously with sup *instead of* inf. *The corresponding function $u : \Omega \to \mathbb{R}$ defined by*

$$u(x) := \begin{cases} \log d(x, A)^{-1} & \text{for } x \in \Omega \cap A_r, \\ 0 & \text{for } x \in \Omega \setminus A_r, \end{cases}$$

is contained in the space $\cap_{1 \leq p < \infty} L^p(\Omega)$.

PROOF. Estimate (39) follows from Lemma 4.1. It is easy to see that for any $p \geq 1$ there exist $\gamma > 0$ and $C > 0$ such that $p\gamma < N - d$ and $u(x) \leq C \cdot d(x, A)^{-\gamma}$, for $x \in A_r$. The claim follows from Theorem 4.4. $\square$

5 Gauge Functions for Minkowski Contents.

Let us describe a class of d-dimensional fractal sets with degenerate d-dimensional Minkowski contents, but which possess gauge functions such that the corresponding Minkowski contents are nondegenerate.

Theorem 5.1. *Let a set in $A \subset [0,1]^N$ be defined by a swarming function $f(t) := d \cdot t + \varphi(t)$, $t \geq 1$ (see Lemma 2.1), where $d \in [0, N)$ is fixed, and $\varphi(t)$ is an absolutely continuous function such that $\varphi(1) \in [-d, N - d + 1)$, $0 \leq \varphi' \leq N - d$ for a.e. $t \geq 1$, and $\varphi(t) = o(t)$, $\varphi(t) \to \infty$ as $t \to \infty$, (for* *example, $\varphi(t) := \sqrt{t}$). Then $\dim_B A = d$ and A has degenerate Minkowski content with $\mathcal{M}^d(A) = \infty$, while for the gauge function $h(r) := r^d 2^{-\varphi(\log_2 r^{-1})}$ the corresponding h-Minkowski contents are nondegenerate:*

$$2^{d-2N-1} C_N \le \mathcal{M}_*(h, A) \le 3^N D_N$$
$$\text{and } 2^{-N-1} C_N \le \mathcal{M}^*(h, A) \le 3^N 2^{N-d} D_N.$$

Also, for all $r > 0$,

$$I_r := \int_{A_r} d(x, A)^{-\gamma} dx < \infty \Longleftrightarrow \gamma < N - d, \tag{40}$$

and for any $R > 0$ there exist positive constants $\underline{c}$ and $\overline{c}$ such that for all $r \in (0, R)$:

$$\underline{c}\, F(r) \le I_r \le \overline{c}\, F(r), \tag{41}$$

where $F(r) := r^{N-d-\gamma} 2^{\varphi(\log_2 r^{-1})} + \int_0^r t^{N-d-\gamma-1} 2^{\varphi(\log_2 t^{-1})}\, dt$.

PROOF. By (2) we have that $\dim_B A = \lim_n f(n)/n = d$. Also, since $s_n := \log_2 \omega_n(A) \ge \lfloor f(n) \rfloor$ (see Lemma 2.1) we conclude that

$$\underline{D} = \liminf_{n\to\infty} (s_n - nd) \ge \liminf_{n\to\infty} (\lfloor dn + \varphi(n) \rfloor - nd) = \infty;$$

hence by Theorem 3.2(a) the Minkowski content is degenerated: $\mathcal{M}^d(A) = \infty$. To find the appropriate gauge function of A which would generate nondegenerate Minkowski contents of A, let $r \in (0, 1)$ be fixed, and choose a natural number n such that $2^{-n} \le r < 2^{-(n-1)}$. Using the estimate from step (d) of the proof of Theorem 3.1 we have

$$\frac{|A_r|}{r^{N-d}} \ge 2^{d-2N} C_N 2^{s_n - nd} \ge 2^{d-2N} C_N 2^{\varphi(n)-1} = 2^{d-2N-1} C_N 2^{\varphi(\log_2 r^{-1})}.$$

Hence $\frac{|A_r|}{r^N} h(r) \ge 2^{d-2N-1} C_N$, and therefore $\mathcal{M}_*(h, A) \ge 2^{d-2N-1} C_N$. Using the estimate from step (c) of the proof of Theorem 3.1 we have

$$\begin{aligned}\frac{|A_r|}{r^{N-d}} &\le 2^{N-d} D_N 2^{s_{n-1} - (n-1)d} \le 2^{N-d} D_N 2^{\varphi(n-1) + N \log_2 3} \\ &= 3^N 2^{N-d} D_N 2^{\varphi(\log_2 r^{-1})},\end{aligned}$$

since φ is nondecreasing. Hence $\frac{|A_r|}{r^N} h(r) \le 3^N 2^{N-d} D_N$, and thus $\mathcal{M}^*(h, A) \le 3^N 2^{N-d} D_N$. Using steps (a) and (b) from the proof of Theorem 3.1 we have

$$\begin{aligned}\frac{|A_{2^{-n}}|}{(2^{-n})^{N-d}} &\ge 2^{-N} C_N 2^{s_n - nd} \ge 2^{-N-1} C_N 2^{\varphi(n)}, \\ \frac{|A_{2^{-n}}|}{(2^{-n})^{N-d}} &\le D_N 2^{s_n - nd} \le D_N 2^{\varphi(n) + N \log_2 3} = 3^N D_N 2^{\varphi(n)};\end{aligned}$$

hence,

$$\mathcal{M}^*(h,A) \geq \limsup_{n\to\infty} \frac{|A_{2^{-n}}|}{(2^{-n})^N} h(2^{-n}) \geq 2^{-N-1} C_N,$$
$$\mathcal{M}_*(h,A) \leq \liminf_{n\to\infty} \frac{|A_{2^{-n}}|}{(2^{-n})^N} h(2^{-n}) \leq 3^N D_N.$$

To prove (40) it suffices to use Theorem 4.3(c). We have that $g(t) := 2^{-\varphi(\log_2 t^{-1})} \to 0$ as $t \to \infty$, and for any $\varepsilon > 0$,

$$\lim_{t\to 0} t^\varepsilon g(t)^{-1} = \lim_{t\to 0} t^\varepsilon 2^{\varphi(\log_2 t^{-1})} = \lim_{p\to\infty} 2^{-p\varepsilon+\varphi(p)} = \lim_{p\to\infty} 2^{-p(\varepsilon-\varphi(p)/p)} = 0.$$

Estimate (41) for I_r follows from Theorem 4.2. □

Example 5.1. The conditions of Theorem 5.1 are satisfied if $\varphi_1(t) = t^\alpha$ with $\alpha \in (0,1)$ or $\varphi_2(t) = \beta \log_2 t$ (with $\beta > 0$ sufficiently small). The corresponding gauge functions are

$$h_1(r) = r^d 2^{-(\log_2 r^{-1})^\alpha} \text{ and } h_2(r) = r^d (\log_2 r^{-1})^{-\beta}. \tag{42}$$

Now we construct a class of fractal sets showing that nondegeneracy condition (37) is indeed important for characterization of integrability (38) to hold in Theorem 4.4. Note that in (45) we allow $\gamma = N - \gamma$, in contrast to (38).

Theorem 5.2. *Let A be a set in $[0,1]^N$ generated by a swarming function $f(t) := d \cdot t - \varphi(t)$, $t \geq 1$, where $d \in (0,N)$ is fixed, $\varphi(1) \in (-N+d-1, d]$, $0 \leq \varphi'(t) \leq N-d$ for a.e. $t \geq 1$, $\varphi(t) = o(t)$ and $\varphi(t) \to \infty$ as $t \to \infty$, and*

$$\int_1^\infty 2^{-\varphi(t)} dt < \infty. \tag{43}$$

Then $\dim_B A = d$, and A has degenerate Minkowski content,$\mathcal{M}^d(A) = 0$. For the gauge function

$$h(r) := r^d 2^{\varphi(\log_2 r^{-1})} \tag{44}$$

the lower and upper h-Minkowski contents of A are nondegenerate:

$$2^{d-2N-1} C_N \leq \mathcal{M}_*(h,A) \leq 3^N D_N \text{ and } 2^{-N-1} D_N \leq \mathcal{M}^*(h,A) \leq 3^N 4^{N-d} D_N.$$

Furthermore, we have that for all $r > 0$,

$$\int_{A_r} d(x,A)^{-\gamma} dx < \infty \Longleftrightarrow \gamma \leq N - d. \tag{45}$$

PROOF. Given $r \in (0,1)$ there exists a unique natural number n such that $2^{-n} \le r < 2^{-(n-1)}$. Using the estimate from step (c) in the proof of Theorem 3.1, and the fact that φ is nondecreasing, we obtain

$$\begin{aligned}\frac{|A_r|}{r^{N-d}} &\le 2^{N-d} D_N 2^{s_{n-1}-(n-1)d} \le 2^{N-d} D_N 2^{-\varphi(n-1)+N\log_2 3}\\ &= 2^{N-d} D_N 2^{-\varphi(\log_2 r^{-1}-1)}.\end{aligned}$$

Hence, since $\varphi(t)-\varphi(t-1) \le \int_{t-1}^{t} \varphi'(s)\,ds \le N-d$, we have $\varphi(t-1) \ge \varphi(t) - N + d$ for $t \ge 2$, and substituting $t = \log_2 r^{-1}$ we conclude that $\mathcal{M}^*(h,A) \le 3^N 4^{N-d} D_N$. The remaining estimates of h-Minkowski contents are obtained in the same way as in the proof of Theorem 5.1.

Defining $g(t) := 2^{\varphi(\log_2 t^{-1})}$ we have that for all $\varepsilon > 0$, $\lim_{t\to 0} t^{\varepsilon} g(t)^{-1} = 0$, hence by Theorem 4.3(b) we have that sufficiency part in (45) holds with $\gamma < N-d$.

Let us consider the case of $\gamma = N-d$. We use Theorem 4.2(b). It suffices to check that $H_r := \int_0^r t^{N-\gamma-1} h(t)^{-1} dt < \infty$. Indeed, we have

$$H_r = \int_0^r t^{-1} 2^{-\varphi(\log_2 t^{-1})} dt = \log 2 \int_{\log_2 r^{-1}}^{\infty} 2^{-\varphi(s)} ds < \infty.$$

To prove the necessity part in (45), we use Theorem 4.5(a) with $\Omega = \mathbb{R}^N$. Let us fix any $\alpha < d$. We first check that $h(r) \le r^{\alpha}$. Indeed, for t sufficiently large we have that $\frac{\varphi(t)}{t} \le d-\alpha$, and the desired inequality follows by substituting $t = \log_2 \frac{1}{r}$, with r sufficiently small. Using Theorem 4.5 we obtain $\gamma < N-\alpha$. Now letting $\alpha \to d$ we conclude that $\gamma \le N-d$. □

Remark 5.1. Note that for the gauge function in (44) there is no positive constant C such that $h(r) \le C \cdot r^d$. Hence, Theorem 5.2 shows that the growth condition on h_1 in Theorem 4.5(a) is in some sense optimal for the conclusion to hold there. Analogously for Theorem 4.3(a).

Example 5.2. It is easy to see that the integral condition (43) is fulfilled if the function $\varphi : (1,\infty) \to \mathbb{R}^N$ is increasing C^1 diffeomorphism onto its range, and the inverse function $\psi := \varphi^{-1}$ satisfies the condition

$$\int_{\varphi(1)}^{\infty} 2^{-s} \psi'(s)\,ds < \infty,$$

where we have used a change of variables. This is the case if for example the function ψ' has at most polynomial growth, or even more generally, if $\psi'(s) \le$

2^{cs} for some constant $c \in (0,1)$. Among examples that satisfy conditions of Theorem 5.2 are $\varphi_1(t) = t^\alpha$ with $\alpha \in (0,1)$, or $\varphi_2(t) = \beta \log_2 t$ with $\beta > 1$. The corresponding gauge functions are

$$h_1(r) = r^d 2^{(\log_2 r^{-1})^\alpha} \text{ and } h_2(r) = r^d (\log_2 r^{-1})^\beta. \tag{46}$$

It is also possible to construct fractals using iterated logarithms in φ_2, so that they will appear in h_2 as well. The same is possible in Example 5.1.

Example 5.3. Interesting geometric constructions of fractals yielding gauge functions of the form h_2 in (46) can be seen in Caetano [2], Evans and Harris [3, Section 6.2.2], or in He, Lapidus [10, Examples 7.5 and 7.7, see also the Appendix]. Here we can easily construct various open sets Ω in $\mathbb{R}^N$ with the boundary $\partial\Omega$ having prescribed inner box dimension in $(N-1, N)$, and with partial control over Minkowski contents of $\partial\Omega$ relative to Ω, and also with degenerate contents possessing gauge functions as above. If A is a compact fractal set in $\mathbb{R}^N$ as in Theorem 5.1, it suffices to define $\Omega := B_R(0) \setminus A$ with $\dim_B A =: d \in (N-1, N)$, and with R large enough so that $A \subset B_R(0)$. It is clear that $\partial\Omega = \partial B_R(0) \cup A$, and for the relative box dimension of the boundary with respect to Ω (also called inner box dimension in the literature) we have $\dim_B(\partial\Omega, \Omega) = \dim_B A = d$, classical Minkowski content is degenerate, $\mathcal{M}^d(\partial\Omega) = 0$, while generalized Minkowski contents $\mathcal{M}_*(h, \partial\Omega)$ and $\mathcal{M}^*(h, \partial\Omega)$ are nondegenerate. Analogous sets Ω can be obtained using Theorem 5.2. Using fractal set A described in Corollary 3.3 or 3.4 we can construct a domain Ω as above such that relative box dimensions $\underline{d}$ and $\overline{d}$ of the boundary with respect to Ω have prescribed values in $(N-1, N)$, and with partial control over the corresponding relative Minkowski contents $\mathcal{M}_*^{\underline{d}}(\partial\Omega, \Omega)$ and $\mathcal{M}^{*\overline{d}}(\partial\Omega, \Omega)$.

We also formulate a result about general properties of Minkowski contents, which extends Krantz, Parks [11, Theorem 3.3.6]. It enables to generate new fractals with nondegenerate Minkowski contents, and the corresponding new Lebesgue integrable functions. See also the excision property of the upper Minkowski content described in Lemma 6.6(a) below.

Proposition 5.3. *Let $h_i : (0, r_0) \to \mathbb{R}$, $i = 1, 2$, be gauge functions for pairs (A_i, Ω_i), where $A_i \subseteq \overline{\Omega}_i \subseteq \mathbb{R}^{N_i}$. We have*

$$\mathcal{M}^*(h_1 h_2, A_1 \times A_2, \Omega_1 \times \Omega_2) \le \mathcal{M}^*(h_1, A_1, \Omega_1) \cdot \mathcal{M}^*(h_2, A_2, \Omega_2).$$

Assuming that gauge functions h_i are nondecreasing, then

$$\mathcal{M}_*(h_1, A_1, \Omega_1) \cdot \mathcal{M}_*(h_2, A_2, \Omega_2) \le \sqrt{2}^{N_1+N_2} \mathcal{M}_*(h_1 h_2, A_1 \times A_2, \Omega_1 \times \Omega_2).$$

If $h := h_1 = h_2$ and $\Omega := \Omega_1 = \Omega_2$, then

$$\mathcal{M}^*(h, A_1 \cup A_2, \Omega) \le \mathcal{M}^*(h, A_1, \Omega) + \mathcal{M}^*(h, A_2, \Omega).$$

Analogous inequality holds for $\mathcal{M}_*$ provided $d(A_1, A_2) > 0$:

$$\mathcal{M}_*(h, A_1 \cup A_2, \Omega) \ge \mathcal{M}_*(h, A_1, \Omega) + \mathcal{M}_*(h, A_2, \Omega).$$

PROOF. Let us prove the second inequality only. It is clear that $(A_1)_r \times (A_2)_r \subseteq (A_1 \times A_2)_{\sqrt{2}\,r}$. Hence,

$$((A_1)_r \cap \Omega_1) \times ((A_2)_r \cap \Omega_2) \subset (A_1 \times A_2)_{\sqrt{2}\,r} \cap (\Omega_1 \times \Omega_2),$$

and from this it follows that

$$\frac{|(A_1)_r \cap \Omega_1|}{r^{N_1}} \cdot \frac{|(A_2)_r \cap \Omega_2|}{r^{N_2}} \le \sqrt{2}^{N_1+N_2} \frac{|(A_1 \times A_2)_{\sqrt{2}\,r} \cap (\Omega_1 \times \Omega_2)|}{(\sqrt{2}\,r)^{N_1+N_2}}.$$

The claim follows easily by multiplying this inequality with $h_1(r)h_2(r) \le h_1(\sqrt{2}\,r)h_2(\sqrt{2}\,r)$, and taking the liminf as $r \to 0$. $\square$

It is easy to see that classical Minkowski contents are positively homogeneous; that is, $\mathcal{M}^d_*(\lambda A) = \lambda^d \mathcal{M}^d_*(A)$ and $\mathcal{M}^{*d}(\lambda A) = \lambda^d \mathcal{M}^{*d}(A)$ for any $\lambda > 0$. This property can be extended to generalized Minkowski contents. We say that a gauge function $h : (0, r_0) \to \mathbb{R}$ is almost homogeneous with degree $d \ge 0$ if for all $\lambda > 0$,

$$h(\lambda r) \sim \lambda^d h(r) \quad \text{as } r \to 0. \tag{47}$$

Example 5.4. It is easy to see that functions having the form $h_1(r) := r^d 2^{\pm(\log_2 r^{-1})^\alpha}$, with $\alpha \in (0,1)$, or $h_2(r) := r^d (\log r^{-1})^\beta$, with $\alpha \in \mathbb{R}$ and $d \ge 0$, appearing in (42) and (46), are both almost homogeneous with degree d. Indeed, for h_1 the condition (47) reduces to verifying that $\log_2^\alpha (\lambda r)^{-1} - \log_2^\alpha r^{-1} \to 0$ as $r \to 0$, and this follows from $|u^\alpha - v^\alpha| \le \alpha \cdot u^{\alpha-1}|u - v|$ for any $0 < u \le v$, where $\alpha \in (0,1)$. Almost homogeneity of a gauge function implies homogeneity of the corresponding Minkowski contents. The proof of this fact is simple, and we omit it.

Proposition 5.4. *Assume that a gauge function h is almost homogeneous with degree $d \ge 0$. Then for any $\lambda > 0$*

$$\mathcal{M}^*(h, \lambda A) = \lambda^d \cdot \mathcal{M}^*(h, A) \text{ and } \mathcal{M}_*(h, \lambda A) = \lambda^d \cdot \mathcal{M}_*(h, A).$$

Remark 5.2. Probably the easiest geometrical way to construct a fractal S in $[0,1]$ with degenerate 1-dimensional Minkowski content (and equal to zero) is to start with a sequence of compact sets A_n in $\mathbb{R}$ such that $d_n := \dim_B A_n \in (0,1)$, $d_n \to 1$ as $n \to \infty$, and possessing nondegenerate d_n-dimensional Minkowski contents. Let us define a generalized fractal spray (generalizing standard fractals sprays in which all A_n coincide, see Lapidus and van Frankenhuysen [17]; fractal sprays were introduced by Lapidus and Pomerance [15]), $S := \bigcup_{n=1}^{\infty}(\lambda_n A_n + \mu_n)$. Here λ_n and μ_n are real numbers chosen so that each set $\lambda_n A_n + \mu_n$ is contained in the prescribed closed interval I_n in $[0,1]$, and the family of intervals I_n is disjoint. Note that S is negligible (i.e., its Lebesgue measure is zero) as a countable union of negligible sets. It is easy to see that $\dim_B A = 1$, since $\mathcal{M}^s(S) = \infty$ for $s < 1$ (for any such s there exists $d_n \in (s,1)$; hence $\mathcal{M}^s(\lambda_n A_n + \mu_n) = \lambda_n^s \mathcal{M}^s(A_n) = \infty$). Since the set S is trivially 1-rectifiable, using Federer [6, 3.2.39. Theorem] we have that $\mathcal{M}^1(A) = H^1(S) = 0$, where H^1 is the Hausdorff measure (equal to Lebesgue's measure in this case). This construction can be applied to the sequence of uniform Cantor sets $A_n := C^{(a_n)}$, where $d_n = \log_{1/a_n} 2$ and $a_n \to 1/2$, or to a-sets (in the terminology of Lapidus, van Frankenhuysen [17]) defined by $A_n := \{k^{-\alpha_n} : k \in \mathbb{N}\}$, where $d_n = 1/(1+\alpha_n)$ and $\alpha_n \to 0$. We do not know if these sets possess gauge functions which would yield nondegenerate generalized Minkowski contents of S. It is easy to extend the definition of generalized fractal spray to $\mathbb{R}^N$, defining $S := \cup_n(\lambda_n O_n A_n + \mu_n)$, where A_n are compact subsets of $\mathbb{R}^N$, O_n is a sequence of orthogonal matrices of order N, and $\lambda_n \in \mathbb{R}$, $\mu_n \in \mathbb{R}^N$. A generalized fractal spray generated by a sequence of uniform Cantor sets has been used in Žubrinić [27, Proposition 10] in order to construct maximally singular Lebesgue integrable function $u : (0,1) \to \mathbb{R}$, in the sense that Hausdorff's dimension of the set of its singular points is equal to 1.

6 Minkowski Measurable Spirals in the Plane.

In this section we consider two classes of nonrectifiable spirals in the plane defined in polar coordinates (r,φ). Let us first consider a class of spirals of the focus type.

Theorem 6.1. *Let A be a spiral of the focus type, defined by $r = \varphi^{-\alpha}$, where $\varphi \geq \varphi_0$, with $\varphi_0 > 0$ a fixed angle.*

(a) If $\alpha \in (0,1)$, then $d := \dim_B A = 2/(1+\alpha)$ (see Tricot [23, p. 121]) and A is Minkowski measurable,

$$\mathcal{M}^d(A) = \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} \frac{1+\alpha}{1-\alpha}. \tag{48}$$

(b) If $\alpha = 1$, then $d = 1$ and $\mathcal{M}^1(A) = \infty$, while with the gauge function $h(r) := r \cdot (\log r^{-1})^{-1}$ we have that $0 < \mathcal{M}_*(h, A) \le \mathcal{M}^*(h, A) < \infty$.

Remark 6.1. It is easy to see that the spiral $r = \varphi^{-\alpha}$, $\varphi \ge \varphi_0$ is rectifiable if and only if $\alpha > 1$. In this case $\dim_B A = 1$ and its 1-dimensional Minkowski content is equal to its length multiplied by 2.

Theorem 6.2. *Let A be a spiral of the limit cycle type, defined by $r = 1 - \varphi^{-\alpha}$, where $\alpha > 0$ is a fixed constant, and $\varphi \ge \varphi_0$, with $\varphi_0 \ge 1$ a fixed angle. Then A is Minkowski measurable: $d := \dim_B A = \frac{2+\alpha}{1+\alpha}$ and $\mathcal{M}^d(A) = 2\pi(1+\alpha)(\pi\alpha)^{-\alpha/(1+\alpha)}$.*

Remark 6.2. It is interesting to note that in the nonrectifiable case the corresponding d-dimensional Minkowski content of the spiral A does not depend on the initial angle φ_0.

Remark 6.3. If λA is the graph of the spiral $r = \lambda\varphi^{-\alpha}$, $\varphi \ge \varphi_0 > 0$, with fixed $\alpha \in (0, 1)$, scaled with respect to the spiral A in Theorem 6.1 with factor $\lambda > 0$, we obtain the corresponding Minkowski content using homogeneity, $\mathcal{M}^d(\lambda A) = \lambda^d \mathcal{M}^d(A)$. On the other hand, in the case of the spiral $A(\lambda)$ of the cycle type $r = 1 - \lambda\varphi^{-\alpha}$, $\varphi \ge \varphi_0$, we have

$$\mathcal{M}^{*d}(A(\lambda)) = 2\pi(1+\alpha)\lambda^{1/(1+\alpha)}(\pi\alpha)^{-\alpha/(1+\alpha)}.$$

Note that here λ does not have the role of scaling factor of $A(\lambda)$ with respect to the spiral A in Theorem 6.2. This value of Minkowski content is obtained by direct computation, repeating the proof of Theorem 6.2. It is interesting that for this class of spirals we have that $\mathcal{M}^d(A(\lambda)) = \lambda^{d-1}\mathcal{M}^d(A)$. Intuitively, this property is due to the fact that all radial sections of the spiral $A(\lambda)$ are sets of the form $\{(1-\lambda(\varphi_1+2k\pi)^{-\alpha}, \varphi_1) \in \mathbb{R}^2 : k \in \mathbb{N}\}$, and the box dimension of any such set (with fixed φ_1 and $\alpha > 0$) is equal to $\dim_B\{k^{-\alpha} : k \in \mathbb{N}\} = 1/(1+\alpha) = d - 1$ (see e.g. Tricot [23, p. 25]).

As a consequence of Theorem 4.3(c) we obtain new examples of Lebesgue integrable functions, with singular sets being equal to spirals.

Corollary 6.3. *Let $B_1(0)$ be the unit disk in the plane. Assume that either the assumptions of Theorem 6.1 or of Theorem 6.2 are satisfied, and let A be the corresponding spiral. Then for $\gamma > 0$,*

$$\int_{B_1(0)} d(x, A)^{-\gamma} dx < \infty \Longleftrightarrow \gamma < 2 - d.$$

Assuming that conditions of Theorem 6.1(a) or 6.2 are satisfied, we have

$$\lim_{r\to 0}\frac{1}{r^{2-d-\gamma}}\int_{A_r} d(x,A)^{-\gamma}dx = \frac{2-d}{2-d-\gamma}\cdot\begin{cases}\pi(\pi\alpha)^{-2\alpha/(1+\alpha)}\frac{1+\alpha}{1-\alpha},\\ 2\pi(1+\alpha)(\pi\alpha)^{-\alpha/(1+\alpha)},\end{cases}$$

respectively, see (35).

Theorems 6.1 and 6.2 can be reformulated in terms of simple dynamical systems as follows. By the trajectory of the dynamical system here we mean the set of the form $\{(r(t),\varphi(t))\in\mathbb{R}^2 : t\in(t_0,\infty)\}$, with t_0 corresponding to initial angle φ_0, where the functions $r(t)$ and $\varphi(t)$ satisfy the corresponding system of differential equations.

Theorem 6.4. *(a) Assume that A is any trajectory of dynamical system defined by $\dot r = -a\cdot r^b$, $\dot\varphi = 1$, where $a>0$, $b>2$, $0<r(t)<1$ and $\varphi(t)\ge\varphi_0$ with fixed $\varphi_0>0$. Then*

$$d := \dim_B A = 2-\frac{2}{b},\ \mathcal{M}^d(A) = \frac{\pi b}{b-2}(a\pi)^{-2/b}.$$

(b) Assume that A is any trajectory of dynamical system defined by $\dot r = a(1-r)^b$, $\dot\varphi = 1$, where $a>0$, $b>1$, $0<r(t)<1$ and $\varphi(t)\ge\varphi_0$ with fixed and sufficiently large $\varphi_0>0$. Then

$$d := \dim_B A = 2-\frac{1}{b},\ \mathcal{M}^d(A) = \frac{2\pi b}{b-1}(a\pi)^{-1/b}.$$

Remark 6.4. The larger the box dimension of a trajectory in Theorem 6.4 (i.e. the larger the value of b), the greater the "density" of the trajectory near its ω-limit set. This can nicely be seen by plotting the phase portrait. Hausdorff and box dimensions of nonrectifiable trajectories $A = A(\varphi_0,\infty)$ of general dynamical systems in $\mathbb{R}^N$ are studied in Pesin [22].

Let a spiral $r = r(\varphi)$, $\varphi>\varphi_0$, be given, and let A be the corresponding curve in the plane. For any two given angles $\varphi_1<\varphi_2$ we let $A(\varphi_1,\varphi_2)$ be the subset of A corresponding to angles $\varphi\in(\varphi_1,\varphi_2)$. We can write $A = A(\varphi_0,\infty)$.

The following lemma states that for large values of φ the set $A(\varphi,\varphi+2\pi)$ corresponding to the spiral $r=\varphi^{-\alpha}$ (or $r = 1-\varphi^{-\alpha}$) is almost indistinguishable from the circle.

Lemma 6.5. *Let $r=\varphi^{-\alpha}$, where $\alpha>0$ is a constant, $\varphi\ge\varphi_0$, and $\vec r(\varphi) := r(\varphi)(\cos\varphi\,\vec i+\sin\varphi\,\vec j)$. Then*

$$\angle(\vec r(\varphi),\vec r\,'(\varphi))\to\pi/2,\quad \angle(\vec r\,'(\varphi),\vec r\,'(\varphi+2\pi))\to 0,$$
$$\frac{r(\varphi)-r(\varphi+2\pi)}{r(\varphi)}\to 0,\quad R(\varphi)\sim r(\varphi),$$

as $\varphi \to \infty$, where $R(\varphi)$ is the radius of curvature of A at the point corresponding to φ. Analogous claim holds for the spiral $r = 1 - \varphi^{-\alpha}$ of the cycle type.

PROOF. We consider the spiral of the focus type only. Letting $A(\varphi) := \angle(\vec{r}(\varphi), \vec{r}\,'(\varphi))$ we have

$$\cos A(\varphi) = \frac{\vec{r}(\varphi) \cdot \vec{r}\,'(\varphi)}{|\vec{r}(\varphi)| \cdot |\vec{r}\,'(\varphi)|} = \frac{-\alpha}{\sqrt{\alpha^2 + \varphi^2}} \to 0 \quad \text{as } \varphi \to \infty;$$

hence, $A(\varphi) \to \pi/2$. Denoting $B(\varphi) := \angle(\vec{r}(\varphi), \vec{r}(\varphi + 2\pi))$ we have

$$\cos B(\varphi) = \frac{\vec{r}\,'(\varphi) \cdot \vec{r}\,'(\varphi + 2\pi)}{|\vec{r}\,'(\varphi)| \cdot |\vec{r}\,'(\varphi + 2\pi)|} = \frac{\alpha^2 \varphi^{-1}(\varphi + 2\pi)^{-1} + 1}{\sqrt{(\alpha^2\varphi^{-2} + 1)(\alpha^2(\varphi + 2\pi)^{-2} + 1)}} \to 1;$$

hence, $B(\varphi) \to 0$ as $\varphi \to \infty$. The fact that $(r(\varphi) - r(\varphi + 2\pi))/r(\varphi) \to 0$ follows from the Lagrange mean value theorem, $\varphi^{-\alpha} - (\varphi + 2\pi)^{-\alpha} \le 2\pi\alpha\varphi^{-\alpha-1}$.

For any curve $r = r(\varphi)$ of class C^2 in the plane we have

$$R(\varphi) = \frac{(r^2 + r'^2)^{3/2}}{|r^2 + 2r'^2 - rr''|},$$

see e.g. Gardner [8, p. 25]. By direct computation we have that

$$R(\varphi) = \varphi^{-\alpha} \frac{(1 + \alpha^2\varphi^{-2})^{3/2}}{1 - \alpha(1 - \alpha)\varphi^{-2}} \sim \varphi^{-\alpha} \text{ as } \varphi \to \infty. \qquad \square$$

Now we show that if the curve $A = A(\varphi_0, \infty)$ is smooth, then the upper d-dimensional Minkowski content of $A(\varphi_1, \infty)$ does not depend on φ_1, $\varphi_1 \ge \varphi_0$, provided $d > 1$.

Lemma 6.6. (Excision property of the upper Minkowski content)

(a) Let E and F be two bounded sets in $\mathbb{R}^N$, $E \subset F$. If $\mathcal{M}^{*d}(E) = 0$, then $\mathcal{M}^{*d}(F) = \mathcal{M}^{*d}(F \setminus E)$.

(b) Let $A = A(\varphi_0, \infty)$ be a spiral defined by a C^1 function $r = r(\varphi)$, $\varphi \ge \varphi_0$. Assume that $d := \dim_B A(\varphi_0, \infty) > 1$. Then for any $\varphi_1 > \varphi_0$ we have $\mathcal{M}^{*d}(A(\varphi_1, \infty)) = \mathcal{M}^{*d}(A(\varphi_0, \infty))$.

PROOF. (a) It is clear that $\mathcal{M}^{*d}(F \setminus E) \le \mathcal{M}^{*d}(F)$. To show the reverse inequality, note that from $E \subset F$ it follows that $F = E \cup (F \setminus E)$; hence $|F_\varepsilon| \le |E_\varepsilon| + |(F \setminus E)_\varepsilon|$ for all $\varepsilon > 0$. Dividing by ε^{N-d} and taking lim sup as $\varepsilon \to 0$ we obtain that

$$\mathcal{M}^{*d}(F) \le \mathcal{M}^{*d}(E) + \mathcal{M}^{*d}(F \setminus E) = \mathcal{M}^{*d}(F \setminus E).$$

(b) Let $E := A(\varphi_0, \varphi_1)$, where $\varphi_1 \in (\varphi_0, \infty)$, and $F := A(\varphi_0, \infty)$. Since E is a curve of finite length, we have $\dim_B E = \dim_B A(\varphi_0, \varphi_1) = 1$ (see Tricot [23, Theorem on p. 106]). Hence $d = \dim_B(F) > \dim_B E$, and in particular $\mathcal{M}^d(E) = 0$. The claim follows from (a). $\square$

Remark 6.5. We believe the claim in Lemma 6.6(a) is not true for lower Minkowski contents. It would be interesting to find an example.

PROOF OF THEOREM 6.1. (a1) Let $\varepsilon > 0$ and $\varphi_1 \in (\varphi_0, \infty)$ be fixed. We first consider the upper bound of $|A(\varphi_1, \infty)_\varepsilon|$. Let us find a value $\overline{\varphi}_2 = \overline{\varphi}_2(\varepsilon)$ sufficiently large, such that $\varphi^{-\alpha} - (\varphi + 2\pi)^{-\alpha} \leq 2\varepsilon$ for all $\varphi \in (\overline{\varphi}_2, \infty)$. Using the Lagrange mean value theorem we get that this inequality is satisfied with $\overline{\varphi}_2 := C\varepsilon^{-1/(1+\alpha)}$, where $C = (\pi\alpha)^{1/(1+\alpha)}$. We split $|A(\varphi_1, \infty)_\varepsilon|$ into two parts: the central disk of area $\overline{P}_1(\varepsilon)$ (covering the "nucleus" of the set $A(\overline{\varphi}_2, \infty)_\varepsilon$, corresponding to $\varphi \in (\overline{\varphi}_2, \infty)$, $r \leq \overline{\varphi}_2^{-\alpha}$) and the area $\overline{P}_2(\varepsilon)$ covering the "tail" corresponding to $\varphi \in (\varphi_1, \overline{\varphi}_2 + 2\pi)$ (we assume that ε is sufficiently small, so that $\overline{\varphi}_2 > \varphi_1$). We have that

$$\overline{P}_1(\varepsilon) := \pi \cdot r(\overline{\varphi}_2)^2 = \pi\overline{\varphi}_2^{-2\alpha} \sim \pi(\pi\alpha)^{-2\alpha/(1+\alpha)}\varepsilon^{2\alpha/(1+\alpha)} \quad \text{as } \varepsilon \to 0.$$

Now we approximate $A(\varphi_1, \overline{\varphi}_2)_\varepsilon$ by a "radial sausage" $A(\varphi_1, \overline{\varphi}_2)_{\overline{\varepsilon},rad}$ defined by

$$A(\varphi_1, \overline{\varphi}_2)_{\overline{\varepsilon},rad} := \{(\rho, \varphi) \in \mathbb{R}^2 : \varphi \in (\varphi_1, \overline{\varphi}_2),\ r(\varphi) - \overline{\varepsilon} < \rho < r(\varphi) + \overline{\varepsilon}\},$$

with suitably chosen constant $\overline{\varepsilon} > 0$. The idea is to expand the radial sausage corresponding to ε using auxiliary parameter $\overline{D}(\varphi_1)$ in order to cover the Minkowski sausage $A(\varphi_1, \overline{\varphi}_2)_\varepsilon$. From Lemma 6.5 it follows that there exists a constant $\overline{D}(\varphi_1) > 1$ independent of $\varepsilon > 0$, such that $\overline{D}(\varphi_1) \to 1$ as $\varphi_1 \to \infty$, and such that with $\overline{\varepsilon} := \overline{D}(\varphi_1) \cdot \varepsilon$ we have

$$A(\varphi_1, \overline{\varphi}_2)_\varepsilon \subseteq A(\varphi_1, \overline{\varphi}_2)_{\overline{\varepsilon},rad} \cup S_1(\overline{\varepsilon}) \cup S_2(\overline{\varepsilon}).$$

Here $S_1(\overline{\varepsilon})$ and $S_2(\overline{\varepsilon})$ are the corresponding semidiscs of radii $\overline{\varepsilon}$ attached at two points of A for $\varphi = \varphi_1$ and $\varphi = \overline{\varphi}_2$, covering the set $A(\varphi_1, \overline{\varphi}_2)_\varepsilon \setminus A(\varphi_1, \overline{\varphi}_2)_{\overline{\varepsilon},rad}$. Therefore (taking into account the uncovered part of $A(\varphi_1, \infty)_\varepsilon$ in $P_1(\varepsilon)$ corresponding to $\varphi \in (\overline{\varphi}_2, \overline{\varphi}_2 + 2\pi)$ and $r \geq \varphi^{-\alpha}$) we obtain

$$\begin{aligned}
\overline{P}_2(\varepsilon) &\leq |A(\varphi_1, \overline{\varphi}_2 + 2\pi)_{\overline{\varepsilon},rad}| + |S_1(\overline{\varepsilon})| + |S_2(\overline{\varepsilon})| \\
&= \frac{1}{2} \int_{\varphi_1}^{\overline{\varphi}_2 + 2\pi} [(r(\varphi) + \overline{\varepsilon})^2 - (r(\varphi) - \overline{\varepsilon})^2]\, d\varphi + \pi\overline{\varepsilon}^2 \\
&= \overline{D}(\varphi_1) K(\varepsilon) \varepsilon^{2\alpha/(1+\alpha)} + o(\varepsilon^{2-d}),
\end{aligned}$$

where we have used that $\alpha \in (0,1)$. Since $K(\varepsilon) \to K := \frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}$ as $\varepsilon \to 0$, we conclude that $\limsup_{\varepsilon\to 0} \frac{\overline{P}_2(\varepsilon)}{\varepsilon^{2-d}} \le \overline{D}(\varphi_1)\cdot K$, Hence, using excision Lemma 6.6(b) (note that $A = A(\varphi_0,\infty)$ and $d := 2/(1+\alpha) > 1$, see Tricot [23, p. 121], and $A(\varphi_1,\infty) \subset A(\varphi_0,\infty)$) we get

$$\mathcal{M}^{*d}(A) = \mathcal{M}^{*d}(A(\varphi_1,\infty)) \le \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} + \overline{D}(\varphi_1)\cdot\frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}.$$

Letting $\varphi_1 \to \infty$ we obtain

$$\begin{aligned}\mathcal{M}^{*d}(A) &\le \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} + \frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}\\ &= \pi(\pi\alpha)^{-2\alpha/(1+\alpha)}\frac{1+\alpha}{1-\alpha}.\end{aligned} \tag{49}$$

(a2) Now we proceed to obtain the lower bound. Let us again fix $\varepsilon > 0$ and $\varphi_1 \in (\varphi_0,\infty)$. Using the Lagrange mean value theorem we see that inequality $|r(\varphi) - r(\varphi+2\pi)| \ge 2\varepsilon$ is satisfied with $\varphi \le \underline{\varphi}_2 := C\varepsilon^{-1/(1+\alpha)} - 2\pi$, where $C := (\pi\alpha)^{1/(1+\alpha)}$. From Lemma 6.5 it follows that there exists a constant $\underline{D}(\varphi_1) < 1$ independent of $\varepsilon > 0$, such that $\underline{D}(\varphi_1) \to 1$ as $\varphi_1 \to \infty$, and such that with $\underline{\varepsilon} := \underline{D}(\varphi_1)\cdot\varepsilon$ we have $A(\varphi_1,\underline{\varphi}_2)_{\underline{\varepsilon},rad} \subseteq A(\varphi_1,\underline{\varphi}_2)_\varepsilon$, and the radial sausage $A(\varphi_1,\underline{\varphi}_2)_{\underline{\varepsilon},rad}$ is non-self-intersecting; that is, for any fixed $\varphi \in (\varphi_1,\underline{\varphi}_2)$ the family of intervals $(r(\varphi-2k\pi)-\underline{\varepsilon}, r(\varphi-2k\pi)+\underline{\varepsilon})$, indexed with $k \in \mathbb{N}\cup\{0\}$, is disjoint. Let us first estimate the area of the "nucleus" $A(\underline{\varphi}_2,\infty)_\varepsilon$ from below:

$$\underline{P}_1(\varepsilon) : \pi\cdot r(\underline{\varphi}_2+4\pi)^2 = \pi(\underline{\varphi}_2+4\pi)^{-2\alpha} \sim \pi C^{-2\alpha}\varepsilon^{2\alpha/(1+\alpha)} \quad \text{as } \varepsilon \to 0.$$

Also, assuming that ε is small enough so that $\varphi_1 < \underline{\varphi}_2$, we have

$$\begin{aligned}\underline{P}_2(\varepsilon) &:= |A(\varphi_1,\underline{\varphi}_2)_\varepsilon| \ge |A(\varphi_1,\underline{\varphi}_2)_{\underline{\varepsilon},rad}|\\ &= \frac{1}{2}\int_{\varphi_1}^{\underline{\varphi}_2}[(r(\varphi)+\underline{\varepsilon})^2 - (r(\varphi)-\underline{\varepsilon})^2]\,d\varphi\\ &= \underline{D}(\varphi_1)K(\varepsilon)\varepsilon^{2\alpha/(1+\alpha)} + o(\varepsilon^{2-d}).\end{aligned}$$

Since $K(\varepsilon) \to K := \frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}$ as $\varepsilon \to 0$, we conclude that

$$\liminf_{\varepsilon\to 0}\frac{\underline{P}_2(\varepsilon)}{\varepsilon^{2-d}} \ge \underline{D}(\varphi_1)\cdot K.$$

Hence

$$\mathcal{M}^d_*(A(\varphi_1,\infty)) = \liminf_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{2-d}} \geq \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} + \underline{D}(\varphi_1)\cdot\frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}.$$

From $A(\varphi_1,\infty)\subset A = A(\varphi_0,\infty)$ it follows that

$$\mathcal{M}^d_*(A) \geq \mathcal{M}^d_*(A(\varphi_1,\infty)) \geq \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} + \underline{D}(\varphi_1)\cdot\frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}.$$

Letting $\varphi_1\to\infty$ we obtain

$$\begin{aligned}\mathcal{M}^d_*(A) &\geq \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} + \frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}\\ &= \pi(\pi\alpha)^{-2\alpha/(1+\alpha)}\frac{1+\alpha}{1-\alpha}.\end{aligned}$$

This together with (49) finishes the proof of (48).

(b) For $\alpha = 1$ the corresponding gauge function is obtained by repeating the same computations as in (a). □

The proof of Theorem 6.2 can be obtained analogously. The notions of nucleus and tail for spirals that we used in the proof of Theorem 6.1 have been introduced in Tricot [23, pp. 121, 122].

Remark 6.6. As seen from the proof of Theorem 6.1, the d-dimensional Minkowski content of a spiral A of the focus type, defined by $r=\varphi^{-\alpha}$, is the sum of two quantities: nucleus Minkowski content of A which is equal to

$$\mathcal{M}^d(A,n) := \pi(\pi\alpha)^{-2\alpha/(1+\alpha)},$$

and tail Minkowski content of A equal to

$$\mathcal{M}^d(A,t) := \frac{2}{1-\alpha}(\pi\alpha)^{(1-\alpha)/(1+\alpha)}.$$

Analogously for the spiral $r = 1-\varphi^{-\alpha}$ of the cycle type, its nucleus content is $2\pi(\pi\alpha)^{-\alpha/(1+\alpha)}$, while its tail content is $2(\pi\alpha)^{1/(1+\alpha)}$. It is easy to define d-dimensional (upper and lower) nucleus and tail Minkowski contents for graphs A of general curves of spiral type defined in polar coordinates by $r=f(\varphi)$, $\varphi\geq\varphi_0$, provided $d:=\dim_B A$ exists, the function f is strictly monotone and $\lim_{\varphi\to\infty} f(\varphi)$ is finite. Namely, in this case for any $\varepsilon>0$ the maximal value of $\varphi_2=\varphi_2(\varepsilon)$ exists and is such that $A(\varphi_0,\varphi_2)_{\varepsilon,rad}$ is non-self-intersecting (tail of $A_{\varepsilon,rad}$). The set $A(\varphi_2,\infty)_{\varepsilon,rad}$ contains the ball (nucleus of $A_{\varepsilon,rad}$) $B_{f(\varphi_2+2\pi)}(0)$. See also Tricot [23, Figure 10.4 on p. 122].

Remark 6.7. Following the idea of directional Minkowski sausage due to Tricot [23, Chapter 16], it is natural to define radial distance $d_{rad}(x, A)$ from a point x to the spiral A by

$$d_{rad}(x, A) := d(x, \mathbb{R}_+\{x\} \cap \overline{A}),$$

where we define $d(x, \emptyset) := \infty$, and $\mathbb{R}_+\{x\} := \{tx : t \geq 0\}$. Using radial Minkowski sausage $A_{\varepsilon,rad} := \{y \in \mathbb{R}^N : d_{rad}(x, A) < \varepsilon\}$ one can define radial Minkowski contents by changing $|A_\varepsilon|$ to $|A_{\varepsilon,rad}|$ in the usual definition of Minkowski contents. In this way one obtains *radial* upper and lower box dimensions of spirals. For spirals in Theorems 6.1 and 6.2 these radial dimensions coincide with the usual ones (see Tricot [23, p. 249]). Radial Minkowski contents of these spirals also coincide with the usual Minkowski contents, which can be seen from the proof of Theorem 6.1. Also, we have the same integrability criteria of the function $d_{rad}(x, A)^{-\gamma}$ involving radial distance function from the set A as in Corollary 6.3. All integrability results in this paper involving Euclidean distance functions can be formulated also in terms of suitable directional distances and the corresponding directional Minkowski contents.

Acknowledgements

I express my gratitude to the referee for many useful comments and suggestions.

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