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THE IDEAL OF SIERPINSKI-ZYGMUND SETS ON THE PLANE

Abstract

We say that a set $X \subseteq \mathbb{R}^2$ is *Sierpiński-Zygmund* (or *SZ-set* for short) if it does not contain a partial continuous function of cardinality continuum $\mathfrak{c}$. We observe that the family of all such sets is $\mathrm{cf}(\mathfrak{c})$-additive ideal. Some examples of such sets are given. We also consider *SZ-shiftable sets*; that is, sets $X \subseteq \mathbb{R}^2$ for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + X$ is a SZ-set. Some results are proved about SZ-shiftable sets. In particular, we show that the union of two SZ-shiftable sets does not have to be SZ-shiftable.

The terminology is standard and follows [2]. The symbol $\mathbb{R}$ stands for the set of all real numbers. The cardinality of a set X we denote by $|X|$. In particular, $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. Given a cardinal κ, we let $\mathrm{cf}(\kappa)$ denote the cofinality of κ. We say that a cardinal κ is regular provided that $\mathrm{cf}(\kappa) = \kappa$.

A set $M \subseteq \mathbb{R}^n$ is called *Marczewski measurable* if every perfect set P has a perfect subset Q such that $Q \subseteq M$ or $Q \cap M = \emptyset$. If every perfect set P has a perfect subset Q such that $Q \cap M = \emptyset$, then M is called *Marczewski null*.

We consider only real-valued functions unless stated otherwise. No distinction is made between a function and its graph. For any planar set Y, we denote its x-projection by $\mathrm{dom}(Y)$. For any two partial real functions f, g we write $f+g$, $f-g$ for the sum and difference functions defined on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. The class of all functions from a set X into a set Y is denoted by Y^X. We write $f|A$ for the restriction of $f \in Y^X$ to the set $A \subseteq X$. For any function $g \in \mathbb{R}^X$, any family of functions $F \subseteq \mathbb{R}^X$, and any set $A \subseteq X \times \mathbb{R}$ we define $g + F = \{g + f \colon f \in F\}$ and $g + A = \{\langle x, g(x) + y\rangle \colon \langle x, y\rangle \in A\}$. The image

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and preimage of a set B under the function h are denoted by $h[B]$ and $h^{-1}[B]$, respectively.

Let us recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *Sierpiński-Zygmund* ($f \in \mathrm{SZ}$) if for every set $X \subseteq \mathbb{R}$ of cardinality continuum $\mathfrak{c}$, $f|X$ is discontinuous. This definition is generalized onto subsets of $\mathbb{R}^2$. (See [8].)

Definition 1. A set $X \subseteq \mathbb{R}^2$ is called a *Sierpiński-Zygmund* set (or simply *SZ-set*), if for every partial real continuous function f we have $|f \cap X| < \mathfrak{c}$.

We denote the family of all SZ-sets by $\mathcal{J}_{SZ}$. Since every Sierpiński-Zygmund function is also a SZ-set we have that the family $\mathcal{J}_{SZ}$ is not empty.

The next fact follows directly from this definition.

Fact 2. $\mathcal{J}_{SZ}$ is a $\mathrm{cf}(\mathfrak{c})$-additive ideal.

Proof. It is obvious that $\mathcal{J}_{SZ}$ is closed under the operation of taking subsets. We will show that $\mathcal{J}_{SZ}$ is $\mathrm{cf}(\mathfrak{c})$-additive. Take a $\kappa < \mathrm{cf}(\mathfrak{c})$. Let $\{X_\xi\colon \xi < \kappa\} \subseteq \mathcal{J}_{SZ}$ and $f \subseteq \bigcup_{\xi<\kappa} X_\xi$ be a partial continuous function. Since X_ξ is a SZ-set, we have that $|f \cap X_\xi| < \mathfrak{c}$ for each $\xi < \kappa$. Consequently, $|f \cap \bigcup_{\xi<\kappa} X_\xi| = |\bigcup_{\xi<\kappa}(f \cap X_\xi)| < \mathfrak{c}$. □

The question that one could ask here is how "big" a SZ-set can be. An example of the SZ-set that can be considered "big" in some sense is given in [8].

Lemma 3. [8, Lemma 19] *There exists a SZ-set $X \subseteq \mathbb{R}^2$ such that for every $x \in \mathbb{R}$, $|\mathbb{R} \setminus X_x| < \mathfrak{c}$ where $X_x = \{y \in \mathbb{R}\colon \langle x, y\rangle \in X\}$.*

Observe that the complement of every vertical section of the set X has size less than $\mathfrak{c}$. In particular, if MA holds then every vertical section is residual in $\mathbb{R}$. Moreover, under CH, the complement of every vertical section of X is countable. It turns out that the existence of such SZ-set (i.e., with co-countable vertical sections) is equivalent to CH. As is stated in the following proposition.

Proposition 4. CH *is equivalent to the existence of a SZ-set $X \subseteq \mathbb{R}^2$ such that $|\mathbb{R} \setminus X_x| \le \omega$ for every $x \in \mathbb{R}$.*

Proof. The existence of the desired set under the assumption of CH follows from the previous discussion. So we need to prove the opposite implication. Assume, by the way of contradiction, that the desired set X exists and CH does not hold, e.g. $\mathfrak{c} > \omega_1$. Since X is an SZ-set we get

($*$) $X^y = \{x \in \mathbb{R}\colon \langle x, y\rangle \in X\}$ has cardinality less than $\mathfrak{c}$ for every $y \in \mathbb{R}$.

We claim that there exists an $A \in [\mathbb{R}]^{\omega_1}$ such that $|\bigcup_{y\in A} X^y| < \mathfrak{c}$. The following two cases are possible.

Case 1. There exists a $\kappa < \mathfrak{c}$ such that $Z_\kappa = \{y\colon |X^y| = \kappa\}$ is uncountable.

Then we choose $A \in [Z_\kappa]^{\omega_1}$. Obviously, $|\bigcup_{y\in A} X^y| = \kappa\omega_1 < \mathfrak{c}$.

Case 2. $|Z_\kappa| \le \omega$ for every cardinal $\kappa < \mathfrak{c}$.

Put $Z = \{|X^y|\colon y \in \mathbb{R}\}$ and observe that $\mathbb{R} = \bigcup_{\kappa\in Z} Z_\kappa$. It follows from $(*)$ that if $\kappa \in Z$, then $\kappa < \mathfrak{c}$. Consequently, since the union of less than continuum many countable sets has size less than continuum, we conclude that $|Z| = \mathfrak{c}$. Let λ be the ω_1-st element of Z. We define $A = \{y\colon |X^y| < \lambda\}$. Clearly, $|\bigcup_{y\in A} X^y| = |\bigcup_{\kappa<\lambda} Z_\kappa| \le \lambda\omega < \mathfrak{c}$.

Now choose an $x \in \mathbb{R} \setminus \bigcup_{y\in A} X^y$ and notice that $(\{x\} \times A) \cap X = \emptyset$. So $A \subseteq \mathbb{R} \setminus X_x$. This is in contradiction with the fact that every vertical section of X is co-countable. □

It is worth remarking here that SZ-sets with the Baire property or measurable are "small." It means that every measurable SZ-set has measure zero and every SZ-set with the Baire property is meager. This follows from Fubini Theorem and Kuratowski-Ulam Theorem, respectively. But do such "small" SZ-sets exist? The answer is yes. It is easy to construct a Sierpiński-Zygmund function (so also a SZ-set) contained in $\mathbb{R} \times \mathfrak{C}$, whose domain is the whole real line. $\mathfrak{C}$ is the standard linear Cantor set. Observe also that there are "big" SZ-sets in terms of outer measure. The set X from Lemma 3 is of full outer measure. To see this, choose a closed set $F \subseteq \mathbb{R}^2 \setminus X$. Based on the properties of X we conclude that every vertical section of F is countable. Hence F is of measure zero. This proves that X is of full outer measure.

The above discussion states that "good" SZ-sets (in terms of measure or Baire property) are "small". However, we have the following assertion.

Remark 5. There exists a SZ-set which is Marczewski measurable but not Marczewski null.

Proof. We claim that the set X from Lemma 3 is the desired set. Let us see why X is Marczewski measurable but not Marczewski null. Fix a perfect set $P \subseteq \mathbb{R}^2$. There are two possible cases. Either some vertical section P_a of P is perfect, or all vertical sections are countable. In the first case, there is a $Q \subseteq \{a\} \times P_a$ completely contained in X, because the complement of every vertical section of X has cardinality less than $\mathfrak{c}$. In the second case, we can find a partial continuous function $f \subseteq P$ defined on a perfect set. To see this consider a function $g\colon \mathrm{dom}(P) \to \mathbb{R}$ defined by $g(x) = \sup(P_x \cap (-\infty, 0])$. The function g is upper semi-continuous so also of Baire class one. Thus, g contains a continuous function defined on a perfect set. (See [6].)

Since $|f \cap X| < \mathfrak{c}$, the restriction of f to some perfect subset R of $\mathrm{dom}(f)$

is disjoint with X. Note that $f|R$ is a perfect set. Thus P contains a perfect subset disjoint with X.

It is obvious that X contains a perfect set (every vertical section contains a perfect set). So X is not Marczewski null. This completes the proof of our remark. $\square$

Another interesting observation is that the property of being a SZ-set is not preserved under the homeomorphic images. It is easy to see that any vertical line is a SZ-set, but after a rotation, for example about $\frac{\pi}{4}$, it is not a SZ-set any more. However, if $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism preserving vertical lines then $h[X]$ is a SZ-set for every $X \in \mathcal{J}_{SZ}$.

Fact 6. Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an homeomorphism such that $h[L]$ is a vertical line for every vertical line L. Then $h\{\mathcal{J}_{SZ}\} = \{h[X]\colon X \in \mathcal{J}_{SZ}\} = \mathcal{J}_{SZ}$.

PROOF. First we show the inclusion $h\{\mathcal{J}_{SZ}\} \subseteq \mathcal{J}_{SZ}$. It is easy to see that if $f\colon A \to \mathbb{R}$ is a partial continuous function then $h^{-1}[f]\colon A \to \mathbb{R}$ is also continuous. This implies that for every $X \in \mathcal{J}_{SZ}$, $h[X]$ is also in $\mathcal{J}_{SZ}$ since $h[X] \cap f = h[X \cap h^{-1}[f]]$.

Now to show the other inclusion, let us fix a $Y \in \mathcal{J}_{SZ}$. Note that h^{-1} also preserves all vertical lines. Thus, from the first part of the proof, $X = h^{-1}[Y] \in \mathcal{J}_{SZ}$. Hence $Y = h[X] \in h\{\mathcal{J}_{SZ}\}$. $\square$

As we mentioned at the beginning of this paper, the concept of Sierpiński-Zygmund sets is a generalization of the concept of Sierpiński-Zygmund functions. One of the questions related to the family SZ of Sierpiński-Zygmund functions is for how "big" families $F \subseteq \mathbb{R}^{\mathbb{R}}$ we can find a function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + F \subseteq$ SZ. (See e.g. [3].) Similar question can be asked in the case of Sierpiński-Zygmund sets. This leads to the following definition.

Definition 7. A set $X \subseteq \mathbb{R}^2$ is called SZ-*shiftable*, if there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + X$ is SZ-set.

We denote the family of all SZ-shiftable sets by $\mathcal{SZ}_{shift}$. Obviously $\mathcal{J}_{SZ} \subseteq \mathcal{SZ}_{shift}$, so $\mathcal{SZ}_{shift}$ is not empty.

Lemma 8. *Let $X \subseteq \mathbb{R}^2$. If for all $x \in \mathbb{R}$ and $A \in [\mathbb{R}]^{<\mathfrak{c}}$ there exists an $a \in \mathbb{R}$ such that $(a + A) \cap X_x = \emptyset$, then A is SZ-shiftable.*

PROOF. Let $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ be the sequences of all real numbers and all continuous functions defined on a G_δ subset of $\mathbb{R}$, respectively. We will define a function $f\colon \mathbb{R} \to \mathbb{R}$ which shifts X into $\mathcal{J}_{SZ}$, using transfinite induction. For every $\alpha < \mathfrak{c}$ we choose $f(x_\alpha) \in \mathbb{R}$ with the property that $(f(x_\alpha) + X_{x_\alpha}) \cap \{f_\xi(x_\alpha)\colon \xi < \alpha\} = \emptyset$. Such a choice is possible because of the assumptions on X. It is easy to see that $\operatorname{dom}((f+X)\cap f_\beta) \subseteq \{x_\xi : \xi < \beta\}$ for each $\beta < \mathfrak{c}$. Thus $f + X \in \mathcal{J}_{SZ}$. $\square$

Recall that under Martin's Axiom (MA) the union of less than $\mathfrak{c}$ meager sets is meager. Suppose that $A \in [\mathbb{R}]^{<\mathfrak{c}}$ and $B \subseteq \mathbb{R}$ is meager. Then the set $B - A = \bigcup_{x\in A}(B - x)$ is meager as a union of less than $\mathfrak{c}$ meager sets. Now, if we choose an $a \notin B - A$ then $(a + A) \cap B = \emptyset$. Notice that the same argument can be repeated for the sets of measure zero.

The above discussion and Lemma 8 immediately imply the following.

Corollary 9. (MA) *If each vertical section of a set $X \subseteq \mathbb{R}^2$ is meager or of measure zero, then $X \in \mathcal{SZ}_{shift}$.*

It may also be of interest to determine whether $\mathcal{SZ}_{shift}$ is closed under the union operation. Fact 2 states, in particular, that the union of two SZ-sets is also a SZ-set. Thus, the natural question that appears here is whether the same is true for SZ-shiftable sets. It turns out not to be the case.

Example 10. *There exist $A_1, A_2 \in \mathcal{SZ}_{shift}$ such that $A_1 \cup A_2 = \mathbb{R}^2 \notin \mathcal{SZ}_{shift}$.*

Proof. Put A_1 to be the set X from Lemma 3 and A_2 to be its complement. Based on Lemma 8 A_2 is SZ-shiftable. Next, notice that $A_1 \in \mathcal{J}_{SZ} \subseteq \mathcal{SZ}_{shift}$. Finally, $A_1 \cup A_2 = \mathbb{R}^2$ and obviously $\mathbb{R}^2$ is not in $\mathcal{SZ}_{shift}$. $\square$

Before we finish let us make a comment about [8, Theorem 2 (1)] which says: MA implies that for every finite family F of real functions there exists an almost continuous function g (each open subset of $\mathbb{R}^2$ containing the graph of g contains also the graph of a continuous function) such that $g + f$ is Sierpiński-Zygmund for every $f \in F$. Note that this result can be expressed using the notion of SZ-sets. Under MA the following holds:

> *If, for some fixed $n \in \omega$, every vertical section of the set $X \subseteq \mathbb{R}^2$ has at most n elements then there exists an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f + X \in \mathcal{J}_{SZ}$.*

We generalize the above result as follows.

Theorem 11. (MA) *If every vertical section of the set $X \subseteq \mathbb{R}^2$ is finite then there exists an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f + X \in \mathcal{J}_{SZ}$.*

Before we prove the theorem we need to cite some lemmas and recall some properties. First let us observe that a function $f : \mathbb{R} \to \mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*, i.e., a closed set $K \subseteq \mathbb{R}^2$ which meets every continuous function from $\mathbb{R}$ to $\mathbb{R}$ and is disjoint with at least one function from $\mathbb{R}^{\mathbb{R}}$. Next we give some definitions needed to state the lemmas. (See [8].) For $X \subseteq \mathbb{R}$ by $\mathrm{C}_{<\mathfrak{c}}(X)$ we denote the family of all functions $f\colon X \to \mathbb{R}$ which can be represented as a union of less than $\mathfrak{c}$-many partial continuous functions. The symbol $\mathrm{SZ}(X)$ denotes the family of all partial Sierpiński-Zygmund functions defined on X.

Let $A \subseteq \mathbb{R}$ be everywhere of second category, that is $A \cap I$ is of second category for every nontrivial interval I. We define $\mathcal{F}_A$ as a family of all $F \subseteq \mathbb{R}^{\mathbb{R}}$ whose union $\bigcup F$ (as a subset of $\mathbb{R}^2$) contains no function from $\mathrm{C}_{<\mathfrak{c}}(A \cap B)$ for any Borel set B of second category.

Lemma 12. [8, Lemma 12] (MA) *Let $F \in \mathcal{F}_A$ be a family such that $|F| \leq \mathfrak{c}$. There exists a $g \in \mathrm{SZ}(A)$ such that every extension $\bar{g}\colon \mathbb{R} \to \mathbb{R}$ of g is almost continuous and $g + F \subseteq \mathrm{SZ}(A)$.*

A slight modification of the proof of the above lemma gives a little stronger result. (See [7, Lemma 2.2.1].)

Lemma 13. (MA) *Let $F \in \mathcal{F}_A$ be a family such that $|F| \leq \mathfrak{c}$. There exists a $g \in \mathrm{SZ}(A)$ such that $g + F \subseteq \mathrm{SZ}(A)$ and for every blocking set $B \subseteq \mathbb{R}^2$ there is a non-empty open interval $I_B \subseteq \mathrm{dom}(B)$ with the property that $\mathrm{dom}(B \cap g)$ is dense in I_B.*

Lemma 14. [8, Lemma 13] (MA) *Let $\{f_i\}_1^n \subseteq \mathbb{R}^{\mathbb{R}}$, $n = 1, 2, \ldots$. There exists $\{f_i'\}_1^n \in \mathcal{F}_A$ such that $f_i|A_i \in \mathrm{C}_{<\mathfrak{c}}(A_i)$, where $A_i = \{x\colon f_i(x) \neq f_i'(x)\}$.*

Note that Lemmas 13 and 14 imply the following.

($\star$) (MA) Assume that $F \subseteq \mathbb{R}^{\mathbb{R}}$ is finite and $A \subseteq \mathbb{R}$ is everywhere of second category. Then there exists a function $g\colon A \to \mathbb{R}$ such that $g + F \subseteq \mathrm{SZ}(A)$ and for every blocking set B, $\mathrm{dom}(g \cap B)$ is dense in some non-empty open interval I_B.

PROOF. Let us consider the partition $\{H_n\colon n \in \omega\}$ of $\mathbb{R}$, where H_n is defined by $H_n = \{x \in \mathbb{R}\colon |X_x| = n\}$. Let $G_n \subseteq \mathbb{R}$ be a maximal open set such that H_n is everywhere of second category in G_n. Such a set can be easily constructed. Simply define G_n as the interior of the set $\mathbb{R} \setminus \bigcup_{I \in \mathcal{I}_n} I$, where $\mathcal{I}_n$ is the set of all open intervals in which H_n is meager.

We claim that for every $n < \omega$, there exists a function $g_n\colon (G_n \cap H_n) \to \mathbb{R}$ such that $g_n + X = \{\langle x, g_n(x) + y\rangle\colon x \in (G_n \cap H_n),\ \langle x, y\rangle \in X\} \in \mathcal{J}_{SZ}$ and $\bigcup_{n<\omega} g_n$ intersects every blocking set B.

First observe that this claim implies the conclusion of the theorem. Put $g\colon \mathbb{R} \to \mathbb{R}$ to be an extension of $\bigcup_{n<\omega} g_n$ such that $\big(g|(\mathbb{R} \setminus \bigcup_{n<\omega} G_n \cap H_n)\big) + X$ is an SZ-set. This extension exists based on Corollary 9. Thus, $g + X$ is the union of countable many SZ-sets. Consequently, $g + X \in \mathcal{J}_{SZ}$. Clearly, g intersects every blocking set, so g is almost continuous.

To complete the proof we need to show the above claim. Fix an $n<\omega$ and put $A_n=(G_n\cap H_n)\cup\bigcup_{I\in\mathcal{I}_n} I$. The set A_n is everywhere of second category. Notice also that the part of X contained in $(G_n\cap H_n)\times\mathbb{R}$ can be covered by n functions $f_1,\ldots,f_n$ from $\mathbb{R}$ to $\mathbb{R}$. So, by $(\star)$, there exists a function $g_n'\colon A_n\to\mathbb{R}$ such that $g_n'+\{f_1,\ldots,f_n\}\subseteq \mathrm{SZ}(A_n)$ and for every blocking set B, $\mathrm{dom}(g_n'\cap B)$ is dense in some non-empty open interval I_B. Thus, if we define $g_n=g_n'|(G_n\cap H_n)$ then $g_n+X\in\mathcal{J}_{SZ}$.

What remains to prove is that $\bigcup_{n<\omega} g_n$ intersects every blocking set B. Notice that $I_B\cap G_n\neq\emptyset$ for some n. Thus, $g_n\cap B\neq\emptyset$. Consequently, $\emptyset\neq B\cap\bigcup_{n<\omega} g_n\subseteq B\cap g$. This finishes the proof. □

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