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Aleksander Maliszewski,* Department of Mathematics, Bydgoszcz Academy, pl. Weyssenhoffa 11, 85–072 Bydgoszcz, Poland. e-mail: `AMal@ab-byd.edu.pl`

COMPARABLE ALMOST CONTINUOUS FUNCTIONS

Abstract

The main goal of this paper is to characterize the family of functions which are the averages of two comparable almost continuous functions.

1 Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals. For each $A \subset \mathbb{R}$ let $|A|$ denote the cardinality of A. We write $\mathfrak{c} = |\mathbb{R}|$ and use the symbol $\operatorname{cf}(\mathfrak{c})$ to denote the cofinality of $\mathfrak{c}$. The projection of a set $A \subset \mathbb{R}^2$ onto the x-axis will be denoted by $\operatorname{dom}(A)$.

The word *function* denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. We say that functions φ and ψ are *comparable* if either $\varphi < \psi$ on $\mathbb{R}$ or $\varphi > \psi$ on $\mathbb{R}$. Functions will be identified with their graphs. We say that a function f is *Darboux* and write $f \in \mathbf{D}$ if the set $f[J]$ is connected for every interval J. We will say that a function f is *almost continuous* in the sense of Stallings [6] and write $f \in \mathbf{A}$ if for every open set $U \subset \mathbb{R}^2$ containing f there is a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ with $h \subset U$.

Let f be a function and $x \in \mathbb{R}$. We define

$$\mathfrak{c}\text{-}\underline{\lim}(f, x^-) = \lim_{\delta \to 0^+} \inf\bigl\{y \in \mathbb{R}\colon \bigl|\{t \in (x-\delta, x)\colon f(t) < y\}\bigr| = \mathfrak{c}\bigr\}$$

and $\mathfrak{c}\text{-}\overline{\lim}(f, x^-) = -\,\mathfrak{c}\text{-}\underline{\lim}(-f, x^-)$. The symbols $\mathfrak{c}\text{-}\underline{\lim}(f, x^+)$ and $\mathfrak{c}\text{-}\overline{\lim}(f, x^+)$ we define analogously.

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The symbol $\square$ denotes the end of the proof of a theorem or of a corollary. We use the symbol $\lhd$ to denote the end of the proof of an auxiliary claim within such a proof.

2 Introduction

In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson proved that a function f is the average of two comparable Darboux functions φ and ψ if and only if for each $x \in \mathbb{R}$ we have both $\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} < \infty$ and $\min\{\mathfrak{c}\text{-}\overline{\lim}(f, x^-), \mathfrak{c}\text{-}\overline{\lim}(f, x^+)\} > -\infty$ [1, Theorem 2]. A similar problem is to determine, for a given function f on $\mathbb{R}$, a necessary and sufficient condition for there to exist a Darboux function ψ such that $\psi > f$ on $\mathbb{R}$. (The answer to this question can be easily obtained using the proof of [1, Theorem 2].) In both cases we ask whether there is a positive function g such that both $f + g$ and $(-f) + g$ are Darboux (the first problem) or such that $f + g$ is Darboux (the second problem). It suggests a similar problem for classes of functions consisting of more than two functions. Several results related to this problem can be found in [4].

It is well-known that $\mathbf{A} \subset \mathbf{D}$ [6], and that the algebraic properties of the classes $\mathbf{A}$ and $\mathbf{D}$ are very similar. (See, e.g., [5] or [3].) In this paper I will show that most results proved in [4] hold true if we replace the family $\mathbf{D}$ with $\mathbf{A}$. In particular, Corollary 3.4 is a significant improvement both of the main part of [1, Theorem 2] and of [5, Theorem 7.2].

Recall that by [2, Example 4.1], there exists a family of functions, $\mathfrak{A}$, such that $|\mathfrak{A}| = \operatorname{cf}(\mathfrak{c})$, $|f| \le 1$ on $\mathbb{R}$ for each $f \in \mathfrak{A}$ (so, in particular, condition ii) of Theorem 3.2 holds), but for each nonnegative function g there is an $f \in \mathfrak{A}$ with $f + g \notin \mathbf{D}$. Thus we cannot weaken the assumption "$|\mathfrak{A}| < \operatorname{cf}(\mathfrak{c})$" in Theorem 3.2. However, I do not know whether the assumption "$|\mathfrak{A}| \le \mathfrak{c}$" is necessary in Theorem 3.1.

3 Main results

Theorem 3.1. *Let $\mathfrak{A}$ be a family of functions with $|\mathfrak{A}| \le \mathfrak{c}$. The following are equivalent:*

i) *there is a nonnegative function g such that $f + g \in \mathbf{D}$ for each $f \in \mathfrak{A}$;*

ii) *there is a nonnegative function $\bar{g}$ such that for each $f \in \mathfrak{A}$ and each $x \in \mathbb{R}$ we have*

$$\max\{\mathfrak{c}\text{-}\underline{\lim}(f + \bar{g}, x^-), \mathfrak{c}\text{-}\underline{\lim}(f + \bar{g}, x^+)\} \le (f + \bar{g})(x); \qquad (1)$$

iii) *there is a positive function g such that $f+g\in\mathbf{A}$ for each $f\in\mathfrak{A}$.*

Proof. The implications iii) ⇒ i) and i) ⇒ ii) are evident. To prove the implication ii) ⇒ iii) define for each $f\in\mathfrak{A}$

$$U_f=\bigcup_{x\in\mathbb{R}}\left[\{x\}\times((f+\bar{g})(x),\infty)\right]=\left\{\langle x,y\rangle\in\mathbb{R}^2\colon y>(f+\bar{g})(x)\right\}$$

and

$$\mathcal{K}_f=\left\{K\subset\mathbb{R}^2\colon K\text{ is closed and }\left|\operatorname{dom}(K\cap U_f)\right|=\mathfrak{c}\right\}.$$

Let $\left\{\langle f_\xi,K_\xi\rangle\colon \xi<\mathfrak{c}\right\}$ be an enumeration of all pairs $\langle f,K\rangle$ such that $f\in\mathfrak{A}$ and $K\in\mathcal{K}_f$. Proceeding by transfinite induction choose for each $\xi<\mathfrak{c}$ a point $\langle x_\xi,y_\xi\rangle\in K_\xi\cap U_{f_\xi}$ such that $x_\xi\neq x_\zeta$ for $\zeta<\xi$. Define the function $g\colon\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}y_\xi-f_\xi(x) & \text{if } x=x_\xi \text{ for some } \xi<\mathfrak{c},\\ \bar{g}(x)+1 & \text{otherwise.}\end{cases}$$

Fix an $f\in\mathfrak{A}$. We will prove that $f+g\in\mathbf{A}$. Let $V\subset\mathbb{R}^2$ be an open set containing $f+g$. Notice that $K\cap(f+g)\neq\emptyset$ for each $K\in\mathcal{K}_f$. Hence $\mathbb{R}^2\setminus V\notin\mathcal{K}_f$. So if

$$E=\operatorname{dom}(U_f\setminus V)=\left\{x\in\mathbb{R}\colon\left[\{x\}\times((f+\bar{g})(x),\infty)\right]\not\subset V\right\},$$

then $|E|<\mathfrak{c}$. Let $\mathcal{J}$ be the family of all intervals $J=[a,b]$ for which there is a continuous function $h\colon J\to\mathbb{R}$ such that $h\subset V$ and $h=f+g$ on $\{a,b\}$. The first claim is obvious.

Claim 1. If $[a,b]\in\mathcal{J}$ and $[b,c]\in\mathcal{J}$, then $[a,c]\in\mathcal{J}$. ◁

Claim 2. Let $a<b$. There are $a',b'\in(a,b)\setminus E$ such that $[a',b]\in\mathcal{J}$ and $[a,b']\in\mathcal{J}$.

Choose an $\varepsilon\in(0,b-a)$ with $\left[(b-\varepsilon,b+\varepsilon)\times((f+g)(b)-\varepsilon,(f+g)(b)+\varepsilon)\right]\subset V$. (Recall that V is open.) By (1), there is an $a'\in(b-\varepsilon,b)\setminus E$ such that

$$(f+\bar{g})(a')<(f+\bar{g})(b)+\varepsilon/2<(f+g)(b)+\varepsilon/2.$$

Let L be the interval with end points $(f+g)(a')$ and $(f+g)(b)+\varepsilon/2$. Since $a'\notin E$, we have $\left[\{a'\}\times L\right]\subset V$. Thus there is a $\delta\in(0,b-a')$ such that

$$\left[(a'-\delta,a'+\delta)\times L\right]\subset V.$$

Define $h=f+g$ on $\{a',b\}$, $h(a'+\delta)=(f+g)(b)+\varepsilon/2$, and let h be linear in intervals $[a',a'+\delta]$ and $[a'+\delta,b]$. Clearly h proves $[a',b]\in\mathcal{J}$.

Similarly we can show that there is a $b'\in(a,b)\setminus E$ such that $[a,b']\in\mathcal{J}$. ◁

Claim 3. Let $(b_n) \subset \mathbb{R} \setminus E$ and $b_n \nearrow b$. If $[a, b_n] \in \mathfrak{J}$ for each $n \in \mathbb{N}$, then $[a, b] \in \mathfrak{J}$.

By Claim 2, we can find an $a' \in (a, b) \setminus E$ with $[a', b] \in \mathfrak{J}$. Let $n \in \mathbb{N}$ be such that $b_n > a'$, and let $h_1 \colon [a, b_n] \to \mathbb{R}$ and $h_2 \colon [a', b] \to \mathbb{R}$ correspond to $[a, b_n] \in \mathfrak{J}$ and $[a', b] \in \mathfrak{J}$, respectively. We consider three cases.

Case 1. If $h_1(a') \geq h_2(a')$, then let $\varepsilon > 0$ be such that

$$S = \big[(a' - \varepsilon, a' + \varepsilon) \times (h_1(a') - \varepsilon, h_1(a') + \varepsilon)\big] \subset V.$$

(Recall that $h_1 \subset V$.) Choose an $\eta \in (0, a' - a)$ with $h_1 \restriction [a' - \eta, a'] \subset S$. Put $L = [h_2(a'), h_1(a')]$. Since $a' \notin E$, we have $\big[\{a'\} \times L\big] \subset V$. Thus there is a $\delta \in (0, \eta)$ such that $\big[(a' - \delta, a' + \delta) \times L\big] \subset V$. Define $h = h_1$ on $[a, a' - \delta]$, $h = h_2$ on $[a', b]$, and let h be linear in $[a' - \delta, a']$. Clearly h proves $[a, b] \in \mathfrak{J}$.

Case 2. Proceeding similarly we can show that $h_2(b_n) \geq h_1(b_n)$ implies $[a, b] \in \mathfrak{J}$.

Case 3. Finally suppose that $h_1(a') < h_2(a')$ and $h_1(b_n) > h_2(b_n)$. Since h_1 and h_2 are continuous on $[a', b_n]$, there is a $c \in (a', b_n)$ with $h_1(c) = h_2(c)$. Define $h = h_1$ on $[a, c]$ and $h = h_2$ on $[c, b]$. Clearly h proves $[a, b] \in \mathfrak{J}$. $\lhd$

Claim 4. For each $a < b$ we have $[a, b] \in \mathfrak{J}$.

Put $B = \big\{b' \in (a, b) \setminus E \colon [a, b'] \in \mathfrak{J}\big\}$. By Claim 2, B is nonempty. Set $s = \sup B$. By Claim 3, we obtain $[a, s] \in \mathfrak{J}$. If $s < b$, then by Claim 2, there is a $b' \in (s, b) \setminus E$ with $[s, b'] \in \mathfrak{J}$. But then Claim 1 implies $b' \in B$, an impossibility. $\lhd$

Using Claim 4 one can easily construct a continuous function $h \colon \mathbb{R} \to \mathbb{R}$ contained in V. This completes the proof. $\square$

The proofs of the results below mimic the arguments used in [4].

Theorem 3.2. *Let $\mathfrak{A}$ be a family of functions with $|\mathfrak{A}| < \operatorname{cf}(\mathfrak{c})$. The following are equivalent*:

i) *there is a positive function g such that $f + g \in \mathbf{D}$ for each $f \in \mathfrak{A}$*;

ii) *the inequality* $\sup\big\{\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} - f(x) \colon f \in \mathfrak{A}\big\} < \infty$ *holds for each $x \in \mathbb{R}$*;

iii) *there is a positive function g such that $f + g \in \mathbf{A}$ for each $f \in \mathfrak{A}$.* $\square$

Corollary 3.3. *For each function f the following are equivalent*:

i) *there is a function $\psi \in \mathbf{D}$ such that $\psi > f$ on $\mathbb{R}$*;

ii) *for each* $x \in \mathbb{R}$ *we have* $\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} < \infty$;

iii) *there is a function* $\psi \in \mathbf{A}$ *such that* $\psi > f$ *on* $\mathbb{R}$ *and* $\psi - f \in \mathbf{A}$. □

Corollary 3.4. *For each function* f *the following are equivalent*:

i) *there are functions* $\varphi, \psi \in \mathbf{D}$ *such that* $\varphi < f < \psi$ *on* $\mathbb{R}$;

ii) *for every* $x \in \mathbb{R}$ *we have both* $\max\{\mathfrak{c}\text{-}\underline{\lim}(f, x^-), \mathfrak{c}\text{-}\underline{\lim}(f, x^+)\} < \infty$ *and* $\min\{\mathfrak{c}\text{-}\overline{\lim}(f, x^-), \mathfrak{c}\text{-}\overline{\lim}(f, x^+)\} > -\infty$;

iii) *there are functions* $\varphi, \psi \in \mathbf{A}$ *such that* $\varphi < f < \psi$ *and* $f = (\varphi + \psi)/2$ *on* $\mathbb{R}$, *and* $f - \varphi = \psi - f \in \mathbf{A}$. □

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