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PRODUCTS OF BOUNDED DARBOUX AND ALMOST CONTINUOUS FUNCTIONS

Abstract

We characterize both the family of the sums of nonnegative Darboux/almost continuous functions (in the sense of Stallings) and the family of the products of bounded Darboux/almost continuous functions.

1 Preliminaries

The letters $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the real line, the set of integers, and the set of positive integers, respectively. The word *interval* means a nondegenerate interval. The word *function* denotes a mapping from a subset of $\mathbb{R}$ into $\mathbb{R}$. For brevity, no distinction is made between a function and its graph. If $F \subset \mathbb{R}^2$, then $\operatorname{dom} F$ and $\operatorname{rng} F$ denote the x-projection and the y-projection of F, respectively.

For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, χ_A, and $|A|$ to denote the interior, the closure, the characteristic function, and the cardinality of A, respectively. We write $\mathfrak{c} = |\mathbb{R}|$, and consider $\mathfrak{c}$ as the ordinal not in one-to-one correspondence with the smaller ordinals.

Let $f \colon \mathbb{R} \to \mathbb{R}$. For every $y \in \mathbb{R}$, let $[f < y] = \big\{x \in \mathbb{R} : f(x) < y\big\}$; the sets $[f \leq y]$, $[f > y]$, etc., we define analogously. If $A \subset \mathbb{R}$ and $|A| = \mathfrak{c}$, then let

$$\mathfrak{c}\text{-inf}(f, A) = \inf\big\{y \in \mathbb{R} : |A \cap [f < y]| = \mathfrak{c}\big\},$$
$$\mathfrak{c}\text{-sup}(f, A) = -\,\mathfrak{c}\text{-inf}(-f, A).$$

For each $x \in \mathbb{R}$, let

$$\mathfrak{c}\text{-}\underline{\lim}(f, x^-) = \lim_{\delta \to 0^+} \mathfrak{c}\text{-inf}\big(f, (x - \delta, x)\big),$$

$$\mathfrak{c}\text{-}\overline{\lim}(f, x^-) = -\mathfrak{c}\text{-}\underline{\lim}(-f, x^-).$$

The symbols $\mathfrak{c}\text{-}\underline{\lim}(f, x^+)$ and $\mathfrak{c}\text{-}\overline{\lim}(f, x^+)$ we define analogously. For each $x \in \mathbb{R}$ and $\delta > 0$ we let

$$\mathrm{R}_{[}(f,])x\delta = [(x - \delta, x + \delta) \times (f(x) - \delta, f(x) + \delta)].$$

Finally let

$$\mathcal{B}_f = \big\{ x \in \mathbb{R} : \mathfrak{c}\text{-}\underline{\lim}(|f - f(x)|, x^-) > 0 \text{ or } \mathfrak{c}\text{-}\underline{\lim}(|f - f(x)|, x^+) > 0 \big\}.$$

Let I and J be intervals (possibly unbounded), and $g \colon I \to J$. We say that g is *Darboux*, if it has the intermediate value property. We say that g is *almost continuous* in the sense of Stallings [19], if for every set $V \supset g$ open in $I \times J$, there is a continuous function $h \colon I \to J$ with $h \subset V$. The properties of almost continuous functions were studied by many authors. I recommend the survey papers [18], [6], and [7].

Observe that if g is <u>not</u> almost continuous, then there exists a set F closed in $I \times J$ such that $F \cap g = \emptyset$ and $F \cap h \neq \emptyset$ for each continuous function $h \colon I \to J$. Every such set is called a *blocking set for g*. If no proper subset of F is a blocking set for g, then F is called a *minimal blocking set for g*.

If $F \subset \mathbb{R}^2$ is a (minimal) blocking set for some function $g \colon \mathbb{R} \to \mathbb{R}$, then F is called a *(minimal) blocking set in $\mathbb{R}^2$*. Evidently, a function $g \colon \mathbb{R} \to \mathbb{R}$ is almost continuous if and only if it intersects every blocking set in $\mathbb{R}^2$ [10]. Recall that the domain of every minimal blocking set is an interval [9].

It is well-known that each function (from $\mathbb{R}$ into $\mathbb{R}$) can be written as the sum of two Darboux functions [11], while not every function is the product of Darboux functions [16]. The problem of characterizing of the family of the products of Darboux functions was solved in 1982 by J. G. Ceder [2]. He showed that a function f is the product of Darboux functions if and only if

$$\text{for all } x < t, \text{ if } f(x)f(t) < 0, \text{ then } [f = 0] \cap (x, t) \neq \emptyset. \tag{1}$$

Since almost continuity implies the Darboux property [19] and each function is the sum of two almost continuous functions [9], we can ask whether condition (1) above yields that f is the product of almost continuous functions. In 1991 T. Natkaniec showed that if the additivity of the σ-ideal of meager

sets equals $\mathfrak{c}$, then the answer is affirmative [17, Theorem 2]. I showed in 1997 that the extra set-theoretical assumption is redundant [14].

In 1995 I proved that each bounded function can be written as the sum of two bounded Darboux functions [12]. This result was generalized in 1998 by K. Ciesielski and A. Maliszewski [5]. We showed that each bounded function can be written as the sum of two bounded almost continuous functions [12]. So, it is natural to ask whether each bounded function which fulfills condition (1) is the product of bounded Darboux (or almost continuous) functions. Alas, the answer is no. It is easy to show that the function $\chi_{\mathbb{R}\setminus\{0\}}$ is not the product of <u>bounded</u> Darboux functions, though it is nonnegative [3, p. 79]. In my dissertation [13] (sold out), I characterized the products of at most k bounded, Darboux functions for each $k > 1$. The main goal of this paper is to show that for each $k > 1$, the family of the products of at most k bounded, almost continuous functions and the family of the products of at most k bounded, Darboux functions coincide.

2   Criteria of Almost Continuity

For brevity, we introduce a new notation. Let $g\colon \mathbb{R} \to \mathbb{R}$, let $P \subset \mathbb{R}$ be perfect, and $T_1 \neq T_2$. We will write that g *is of type $\langle T_1, T_2 \rangle$ with respect to P*, if

> for each open interval I with $P \cap I \neq \emptyset$ and any $y_1, y_2 \in \mathbb{R}$ such that $T_1 < y_1 < y_2 < T_2$ or $T_1 > y_1 > y_2 > T_2$, there are $x_1, x_2 \in I$ with (2) $x_1 < x_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$, and $P \cap [x_1, x_2] = \emptyset$.

Proposition 2.1. *Let $P \subset \mathbb{R}$ be perfect, $T > 0$, and $g\colon \mathbb{R} \to [-T, T]$. Suppose moreover that $g{\restriction}\operatorname{cl} I$ is almost continuous for each component I of $\mathbb{R} \setminus P$, and that g is both of type $\langle -T, T \rangle$ and of type $\langle T, -T \rangle$ with respect to P. Then g is almost continuous.*

PROOF. Let $V \subset \mathbb{R}^2$ be an open set containing g. Fix an $a \in \mathbb{R}$. Denote by S the set of all $b \geq a$ with the property that there exists a continuous function $h\colon [a, b] \to [-T, T]$ such that $h \subset V$ and $h(x) = g(x)$ for $x \in \{a, b\}$. Observe that $a \in S$, and let S' be a connected component of S containing a. Set $s = \sup S'$.

The rest of the proof of the proposition consists of several auxiliary claims. The end of the proof of each claim will be marked with $\lhd$.

Claim 1. If $s < \infty$, then $s \in S$.

The assertion is obvious if $s = a$. So, let $s > a$. We consider two cases. If there is a $b \in (a, s)$ such that $P \cap (b, s) = \emptyset$, then let h_1 correspond to $b \in S$.

By assumption, $g{\upharpoonright}[b,s]$ is almost continuous. So by [18, Lemma 6.2], there is a continuous function $h_2\colon [b,s]\to[-T,T]$ such that $h_2\subset V$ and $h_2(x)=g(x)$ for $x\in\{b,s\}$. Now the function $h=h_1\cup h_2$ proves $s\in S$.

In the other case let $\varepsilon\in(0,s-a)$ be such that $\mathrm{R}_{[}(g,])s\varepsilon\subset V$. (Recall that V is open and $V\supset g$.) Pick a $y_1\in(g(s)-\varepsilon,g(s)+\varepsilon)\cap(-T,T)$. Since g is of type $\langle-T,T\rangle$ with respect to P, there is an $x_1\in(s-\varepsilon,s)$ such that $g(x_1)=y_1$. Let h_1 correspond to $x_1\in S$, and let h_2 be the line segment connecting points $\langle x_1,y_1\rangle$ and $\langle s,g(s)\rangle$. Then the function $h=h_1\cup h_2$ proves $s\in S$. $\lhd$

Claim 2. We have $s=\infty$.

By way of contradiction, suppose that this is not the case. Let h_1 correspond to $s\in S$. (Cf. Claim 1.) Let $\varepsilon>0$ be such that $\mathrm{R}_{[}(g,])s\varepsilon\subset V$. We will show that $s+\varepsilon/2\in S'$, which contradicts the definition of s.

Fix a $b\in(s,s+\varepsilon/2]$. Define

$$c=\min\bigl\{x\in[s,b]:\ P\cap(x,b)=\emptyset\bigr\}. \tag{3}$$

If $c=b$, then put $h_2=\emptyset$, otherwise use the fact that $g{\upharpoonright}[c,b]$ is almost continuous, and construct a continuous function $h_2\colon[c,b]\to[-T,T]$ such that $h_2\subset V$ and $h_2(x)=g(x)$ for $x\in\{c,b\}$.

If $c=s$, then the function $h=h_1\cup h_2$ proves $b\in S$. In the opposite case let $\delta\in(0,c-s)$ be such that $\mathrm{R}_{[}(g,])c\delta\subset V$. Let $y_1\in(g(s)-\varepsilon,g(s)+\varepsilon)\cap(-T,T)$ and $y_2\in(g(c)-\delta,g(c)+\delta)\cap(-T,T)\setminus\{y_1\}$. Since g is both of type $\langle-T,T\rangle$ and of type $\langle T,-T\rangle$ with respect to P, there are $x_1,x_2\in(c-\delta,c)$ with $x_1<x_2$ such that $g(x_1)=y_1$, $g(x_2)=y_2$, and $P\cap[x_1,x_2]=\emptyset$. By the virtue of almost continuity of $g{\upharpoonright}[x_1,x_2]$, there is a continuous function $h_3\colon[x_1,x_2]\to[-T,T]$ such that $h_3\subset V$ and $h_3(x)=g(x)$ for $x\in\{x_1,x_2\}$. Let h_4 (respectively h_5) be the line segment connecting points $\langle s,g(s)\rangle$ and $\langle x_1,y_1\rangle$ (points $\langle x_2,y_2\rangle$ and $\langle c,g(c)\rangle$, respectively). Now the function $h=h_1\cup\cdots\cup h_5$ proves $b\in S$. $\lhd$

Since $a\in\mathbb{R}$ was arbitrary, it is easy to construct a continuous function $h\colon\mathbb{R}\to\mathbb{R}$ with $h\subset V$. This completes the proof. $\square$

Proposition 2.2. *Let $P\subset\mathbb{R}$ be perfect, $T>0$, and $g\colon\mathbb{R}\to[-T,T]$. Suppose moreover that $g{\upharpoonright}\operatorname{cl}I$ is almost continuous for each component I of $\mathbb{R}\setminus P$, that g is simultaneously of type $\langle-T,0\rangle$, of type $\langle0,-T\rangle$, of type $\langle T,0\rangle$, and of type $\langle0,T\rangle$ with respect to P, and that the set $P\cap[g=0]$ is dense in P. Then g is almost continuous.*

PROOF. Let $V\subset\mathbb{R}^2$ be an open set containing g. Fix an $a\in\mathbb{R}$. Denote by S the set of all $b\geq a$ with the property that there exists a continuous

function $h\colon [a,b] \to [-T,T]$ such that $h \subset V$ and $h(x) = g(x)$ for $x \in \{a,b\}$. Observe that $a \in S$, and let S' be a connected component of S containing a. Set $s = \sup S'$. The proof of the first claim is a repetition of the argument used in Claim 1 in Proposition 2.1.

Claim 1. If $s < \infty$, then $s \in S$. $\triangleleft$

Claim 2. We have $s = \infty$.

By way of contradiction, suppose that this is not the case. Let h_1 correspond to $s \in S$. Let $\varepsilon > 0$ be such that $\mathrm{R}_{[}(g,])s\varepsilon \subset V$. We will show that $s + \varepsilon/2 \in S'$, which contradicts the definition of s.

Fix a $b \in (s, s + \varepsilon/2]$. Define c by (3). If $c = b$, then put $h_2 = \emptyset$, otherwise use the fact that $g{\restriction}[c,b]$ is almost continuous, and construct a continuous function $h_2\colon [c,b] \to [-T,T]$ such that $h_2 \subset V$ and $h_2(x) = g(x)$ for $x \in \{c,b\}$.

If $c = s$, then the function $h = h_1 \cup h_2$ proves $b \in S$. In the opposite case let $\delta \in (0, c - s)$ be such that $\mathrm{R}_{[}(g,])c\delta \subset V$. Choose a $d \in P \cap [g = 0] \cap (c - \delta, c)$. Let $\eta \in (0, \min\{d - c + \delta, c - d\})$ be such that $[(d - \eta, d + \eta) \times (-\eta, \eta)] \subset V$, and let $e \in P \cap (d - \eta, d + \eta)$ be a bilateral limit point of P. Take an arbitrary

$$y_1 \in (g(s) - \varepsilon, g(s) + \varepsilon) \cap (-T, T) \setminus \{0\}.$$

Pick a y_2 with $|y_2| \le \min\{|y_1|, \eta\}$ such that $y_1 y_2 > 0$. Since $P \cap (d - \eta, e) \ne \emptyset$ and g is both of type $\langle -T, 0 \rangle$ and of type $\langle T, 0 \rangle$ with respect to P, we can choose $x_1, x_2 \in (d - \eta, e)$ with $x_1 < x_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$, and $P \cap [x_1, x_2] = \emptyset$. By the virtue of almost continuity of $g{\restriction}[x_1, x_2]$, there is a continuous function $h_3\colon [x_1, x_2] \to [-T,T]$ such that $h_3 \subset V$ and $h_3(x) = g(x)$ for $x \in \{x_1, x_2\}$.

Similarly, take an arbitrary

$$y_4 \in (g(c) - \delta, g(c) + \delta) \cap (-T, T) \setminus \{0\}.$$

Pick a y_3 with $|y_3| \le \min\{|y_4|, \eta\}$ such that $y_3 y_4 > 0$. Choose $x_3, x_4 \in (e, d + \eta)$ with $x_3 < x_4$ such that $g(x_3) = y_3$, $g(x_4) = y_4$, and $P \cap [x_3, x_4] = \emptyset$. As above, there is a continuous function $h_4\colon [x_3, x_4] \to [-T,T]$ such that $h_4 \subset V$ and $h_4(x) = g(x)$ for $x \in \{x_3, x_4\}$.

Let h_5 (respectively h_6 and h_7) be the line segment connecting points $\langle s, g(s) \rangle$ and $\langle x_1, y_1 \rangle$ (points $\langle x_2, y_2 \rangle$ and $\langle x_3, y_3 \rangle$, points $\langle x_4, y_4 \rangle$ and $\langle c, g(c) \rangle$, respectively). Now the function $h = h_1 \cup \cdots \cup h_7$ proves $b \in S$. $\triangleleft$

Since $a \in \mathbb{R}$ was arbitrary, it is easy to construct a continuous function $h\colon \mathbb{R} \to \mathbb{R}$ with $h \subset V$. This completes the proof. $\square$

Proposition 2.3. *Let $P \subset \mathbb{R}$ be perfect, $T > 0$, and $g \colon \mathbb{R} \to [-T, T]$. Suppose moreover that $g{\restriction}\operatorname{cl} I$ is almost continuous for each component I of $\mathbb{R} \setminus P$, and that $g \cap F \neq \emptyset$ for each closed set $F \subset \big[\mathbb{R} \times [-T, T]\big]$ with $P \cap \operatorname{int}(\operatorname{dom} F) \neq \emptyset$. Then g is almost continuous.*

PROOF. Let $F \subset \big[\mathbb{R} \times [-T, T]\big]$ be a minimal blocking set. Then $I_0 = \operatorname{int}(\operatorname{dom} F)$ is an open interval. If $P \cap I_0 \neq \emptyset$, then by assumption, we have $g \cap F \neq \emptyset$. Otherwise $\operatorname{dom} F$ is contained in the closure of a single component of $\mathbb{R} \setminus P$, say I_1. But $g{\restriction}\operatorname{cl} I_1$ is almost continuous, so $g \cap F \neq \emptyset$. Thus g is almost continuous. $\qquad\square$

3   Main Results

It is easy to see that the problem of characterizing the products of bounded, positive Darboux (almost continuous) functions can be reduced to the problem of characterizing of the sums of nonnegative Darboux (almost continuous) functions. Therefore we first solve the latter problem.

Theorem 3.1. *Let $k > 1$. For each $f \colon \mathbb{R} \to \mathbb{R}$ the following are equivalent*:

   i) *f is the sum of k nonnegative Darboux functions*;

   ii) *f is nonnegative and it fulfills the following condition*:

$$\min\big\{\mathfrak{c}\text{-}\overline{\lim}(f, x^-), \mathfrak{c}\text{-}\overline{\lim}(f, x^+)\big\} \geq f(x)/k \quad \text{for each } x \in \mathbb{R}; \quad (4)$$

   iii) *f is the sum of k nonnegative almost continuous functions.*

PROOF. i) $\Rightarrow$ ii). Let $f = g_1 + \cdots + g_k$, where $g_1, \ldots, g_k$ are nonnegative Darboux functions. Fix an $x \in \mathbb{R}$. Then $g_i(x) \geq f(x)/k$ for some $i \leq k$. Since $f \geq g_i$ on $\mathbb{R}$ and g_i is Darboux, we obtain

$$\mathfrak{c}\text{-}\overline{\lim}(f, x^-) \geq \mathfrak{c}\text{-}\overline{\lim}(g_i, x^-) \geq g_i(x) \geq f(x)/k.$$

Similarly $\mathfrak{c}\text{-}\overline{\lim}(f, x^+) \geq f(x)/k$. The nonnegativity of f is obvious.

   ii) $\Rightarrow$ iii). Set

$$U = \bigcup_{x \in \mathbb{R}} \big[\{x\} \times [0, f(x)]\big] = \big\{\langle x, y \rangle \in \mathbb{R}^2 : 0 \leq y \leq f(x)\big\}.$$

Denote by $\mathcal{K}$ the family of all closed sets $K \subset \mathbb{R}^2$ such that $\big|\operatorname{dom}(K \cap U)\big| = \mathfrak{c}$. Arrange the elements of $\mathcal{K}$ in a transfinite sequence, $\{K_\xi : \xi < \mathfrak{c}\}$, so that $\big|\{\xi < \mathfrak{c} : K_\xi = K\}\big| = k$ for each $K \in \mathcal{K}$. Proceeding by transfinite induction

choose for each $\xi < \mathfrak{c}$ a point $\langle x_\xi, y_\xi \rangle \in K_\xi \cap U$ such that $x_\xi \notin \{x_\zeta : \zeta < \xi\} \cup \mathcal{B}_f$. (Recall that by [4, Lemma 4], we have $|\mathcal{B}_f| < \mathfrak{c}$.)

Let $x \in \mathbb{R}$. If $x = x_\xi$ for some $\xi < \mathfrak{c}$, then put $i = |\{\zeta < \xi : K_\zeta = K_\xi\}| + 1$, and define $g_i(x) = y_\xi$ and $g_j(x) = (f(x) - y_\xi)/(k-1)$ for $j \neq i$. Otherwise for each i define $g_i(x) = f(x)/k$,

We defined functions $g_1, \ldots, g_k \colon \mathbb{R} \to [0, \infty)$ such that $f = g_1 + \cdots + g_k$ on $\mathbb{R}$. Fix an $i \leq k$. We will prove that g_i is almost continuous. Let $V \subset \mathbb{R}^2$ be an open set containing g_i. Notice that $\mathbb{R}^2 \setminus V \notin \mathcal{K}$. So, if

$$E = \mathrm{dom}(U \setminus V) = \big\{ x \in \mathbb{R} : \; [\{x\} \times [0, f(x)]] \not\subset V \big\},$$

then $|E| < \mathfrak{c}$. Define

$$F = \mathrm{dom}\big([\mathbb{R} \times \{0\}] \setminus V\big) = \big\{ x \in \mathbb{R} : \; \langle x, 0 \rangle \notin V \big\}.$$

Clearly $F \subset E$, whence $|F| < \mathfrak{c}$. But F is closed in $\mathbb{R}$, so it is at most countable.

Let $\mathcal{J}$ be the family of all intervals $[a, b]$ for which there exists a continuous function $h \colon [a, b] \to [0, \infty)$ such that $h \subset V$ and $h(a) = h(b) = 0$. Moreover let G be the set of all $x \in \mathbb{R}$ for which there exists a $\delta_x > 0$ such that $[a, b] \in \mathcal{J}$ whenever $a, b \in (x - \delta_x, x + \delta_x) \setminus E$ and $a < b$. The first claim is obvious.

Claim 1. If $[a_0, a_1] \in \mathcal{J}$ and $[a_1, a_2] \in \mathcal{J}$, then $[a_0, a_2] \in \mathcal{J}$. $\quad\triangleleft$

Claim 2. If $a, b \notin E$, $a < b$, and $J = [a, b] \subset G$, then $J \in \mathcal{J}$.

Indeed, the compactness of J and the relation $J \subset \bigcup_{x \in J}(x - \delta_x, x + \delta_x)$ imply that there exist $x_1, \ldots, x_p \in J$ such that $J \subset \bigcup_{i=1}^{p}(x_i - \delta_{x_i}, x_i + \delta_{x_i})$. Consequently, we can find nonoverlapping compact intervals $J_1, \ldots, J_l \in \mathcal{J}$ with $J = \bigcup_{j=1}^{l} J_j$. By Claim 1, we obtain $J \in \mathcal{J}$. $\quad\triangleleft$

Claim 3. We have $G = \mathbb{R}$.

First notice that G is open and that $\mathbb{R} \setminus F \subset G$. By way of contradiction suppose that the set $P = \mathbb{R} \setminus G \subset F$ is nonempty. Then P is scattered, so it contains an isolated point, say s. Let $\delta > 0$ be such that $P \cap (s - \delta, s + \delta) = \{s\}$. We will show that $s \in G$, which is impossible.

Let $a, b \in (s - \delta, s + \delta) \setminus E$ and $a < b$. By Claim 2, we may assume that $a < s < b$. Choose an $\varepsilon \in (0, s - a)$ such that $\mathrm{R}_{[}(g_,)\,i]s\varepsilon \subset V$. Set $y = \max\big\{g_i(s) - \varepsilon/2, 0\big\}$. Observe that if $s \in \mathcal{B}_f$, then by (4), we obtain

$$\mathfrak{c}\text{-}\overline{\lim}(f, s^-) \geq f(s)/k = g_i(s),$$

while $s \notin \mathcal{B}_f$ yields

$$\mathfrak{c}\text{-}\overline{\lim}(f, s^-) \geq f(s) \geq g_i(s).$$

Hence we can choose a $t \in [f \geq y] \cap (s - \varepsilon, s) \setminus E$. Put $L = [\{t\} \times [0, f(t)]]$. Since $t \notin E$, we have $L \subset V$. But L is compact, so there is an $\eta \in (0, s - t)$ such that $[(t - \eta, t + \eta) \times [0, f(t)]] \subset V$. Let h_1 correspond to $[a, t] \in \mathcal{J}$. (We use Claim 2.) Let h_2 (respectively h_3) be the line segment connecting points $\langle t, 0 \rangle$ and $\langle t + \eta, y \rangle$ (points $\langle t + \eta, y \rangle$ and $\langle s, g_i(s) \rangle$, respectively). Define $\tilde{h} = h_1 \cup h_2 \cup h_3$.

Analogously, we can construct a continuous function $\bar{h} \colon [s, b] \to \mathbb{R}$ such that $\bar{h} \subset V$, $\bar{h}(s) = g_i(s)$, and $\bar{h}(b) = 0$. Define $h = \tilde{h} \cup \bar{h}$. Clearly this function proves $[a, b] \in \mathcal{J}$. Hence $s \in G$, an impossibility. $\triangleleft$

Using Claims 3 and 2 it is easy to show that there exists a continuous function $h \colon \mathbb{R} \to \mathbb{R}$ with $h \subset V$.

iii) $\Rightarrow$ i). This implication is evident. $\square$

Remark 3.1. Let g_i be the function constructed in the proof of Theorem 3.1 ($i \leq k$). Observe that if J is a compact interval and $y \in [0, \mathfrak{c}\text{-sup}(f, J)]$, then by definition, we have $[J \times \{y\}] \in \mathcal{K}$, whence $y \in g_i[J]$. Thus, in particular,

- for each $x \in \mathbb{R}$, the left cluster set of g_i at x equals $[0, \mathfrak{c}\text{-}\overline{\lim}(f, x^-)]$,

- for each $x \in \mathbb{R}$, the right cluster set of g_i at x equals $[0, \mathfrak{c}\text{-}\overline{\lim}(f, x^+)]$,

- for each interval I and each $y \in [0, \mathfrak{c}\text{-sup}(f, I)]$, there exist $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$ such that $g(x_1) = g(x_3) = 0$ and $g(x_2) = y$.

The next lemma will be useful in the proof of Theorem 3.3.

Lemma 3.2. *Let $P \subset \mathbb{R}$ be nowhere dense and perfect, let $\mathcal{I}$ be the family of all components of $\mathbb{R} \setminus P$, and let $f \colon \mathbb{R} \to [0, \infty)$ be such that $\mathfrak{c}\text{-inf}(f, I \setminus P) = 0$ for each open interval I with $P \cap I \neq \emptyset$. Then there are pairwise disjoint families $\mathcal{I}_1, \ldots, \mathcal{I}_4 \subset \mathcal{I}$ such that $\mathfrak{c}\text{-inf}\left(f, I \cap \bigcup \mathcal{I}_j\right) = 0$ for each open interval I with $P \cap I \neq \emptyset$ and each $j \leq 4$.*

PROOF. First observe that by our assumption,

> for each locally finite family $\mathcal{I}' \subset \mathcal{I}$ and each open interval J, if $P \cap J \neq \emptyset$, then $\mathfrak{c}\text{-inf}\left(f, J \setminus (P \cup \bigcup \mathcal{I}')\right) = 0$.   (5)

Next we proceed by induction. Define $\mathcal{I}_{1,0} = \cdots = \mathcal{I}_{4,0} = \emptyset$. Assume that for some $n \in \mathbb{N}$ we have already defined pairwise disjoint locally finite families $\mathcal{I}_{1,n-1}, \ldots, \mathcal{I}_{4,n-1} \subset \mathcal{I}$. Let

$$S_n = \big\{ s \in \mathbb{Z} \colon P \cap ((s-1)/n, (s+1)/n) \neq \emptyset \big\}.$$

For each $s \in S_n$, by (5), we can choose distinct intervals

$$I_{1,n,s}, \ldots, I_{4,n,s} \in \mathcal{I} \setminus \bigcup_{j=1}^4 \mathcal{I}_{j,n-1}$$

such that $I_{j,n,s} \subset ((s-1)/n, (s+1)/n)$ and $\mathfrak{c}\text{-inf}(f, I_{j,n,s}) < n^{-1}$ for each j. For each $j \leq 4$ define $\mathcal{I}_{j,n} = \mathcal{I}_{j,n-1} \cup \{I_{j,n,s} : s \in S_n\}$.

For $j \leq 4$ put $\mathcal{I}_j = \bigcup_{n \in \mathbb{N}} \mathcal{I}_{j,n}$. Clearly these families are pairwise disjoint. To complete the proof fix an open interval I with $P \cap I \neq \emptyset$ and a $j \leq 4$. Fix an $\varepsilon > 0$. Let $n > \varepsilon^{-1}$ and $s \in S_n$ fulfill $((s-1)/n, (s+1)/n) \subset I$. Then

$$\mathfrak{c}\text{-inf}\big(f, I \cap \bigcup \mathcal{I}_j\big) \leq \mathfrak{c}\text{-inf}(f, I_{j,n,s}) < n^{-1} < \varepsilon.$$

Consequently, $\mathfrak{c}\text{-inf}\big(f, I \cap \bigcup \mathcal{I}_j\big) = 0.$ □

Theorem 3.3. *Let $k > 1$. For each $f \colon \mathbb{R} \to \mathbb{R}$ the following are equivalent:*

   i) *f is the product of k bounded Darboux functions;*

   ii) *f is bounded, it fulfills condition (1), and there is a $T \geq \sqrt[k]{\sup|f|[\mathbb{R}]}$ such that for each $x \in \mathbb{R}$,*

$$\max\big\{\mathfrak{c}\text{-}\underline{\lim}(|f|, x^-), \mathfrak{c}\text{-}\underline{\lim}(|f|, x^+)\big\} \leq T^{k-1} \cdot \sqrt[k]{|f(x)|}; \qquad (6)$$

   iii) *f is the product of k bounded almost continuous functions.*

PROOF. i) $\Rightarrow$ ii). Let $f = g_1 \ldots g_k$, where $g_1, \ldots, g_k$ are bounded Darboux functions. Define $T = \max\big\{\sup|g_i|[\mathbb{R}] : i \leq k\big\}$ and fix an $x \in \mathbb{R}$. Then $|g_i(x)| \leq \sqrt[k]{|f(x)|}$ for some $i \leq k$. Since g_i is Darboux, we obtain

$$\mathfrak{c}\text{-}\underline{\lim}(|f|, x^-) \leq \mathfrak{c}\text{-}\underline{\lim}(T^{k-1} \cdot |g_i|, x^-) \leq T^{k-1} \cdot |g_i(x)| \leq T^{k-1} \cdot \sqrt[k]{|f(x)|}.$$

Similarly $\mathfrak{c}\text{-}\underline{\lim}(|f|, x^+) \leq T^{k-1} \cdot \sqrt[k]{|f(x)|}$. The boundedness of f is obvious, and condition (1) follows by [2].

ii) $\Rightarrow$ iii). We will introduce a few new notations. We will write that a function g *has property* $(\star)$ *on an open interval* I if

> for each $y \in \big(\mathfrak{c}\text{-inf}(|f|, I)/T^{k-1}, T\big]$, there exist $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$ such that $g(x_1) = g(x_3) = T$ and $g(x_2) = y$.

Let $\mathcal{J}$ be the family of all intervals $J = [a, b]$ for which there exist almost continuous functions $g_1, \ldots, g_k \colon J \to [-T, T]$ such that $f = g_1 \ldots g_k$ on J, and for each i: $g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1-\operatorname{sgn}(i-1)}$ for $x \in \{a, b\}$, and either g_i or $-g_i$ has property $(\star)$ on (a, b). Moreover let G be the set of all $x \in \mathbb{R}$ for which there is a $\delta_x > 0$ such that $[a, b] \in \mathcal{J}$ whenever $a, b \in (x - \delta_x, x + \delta_x)$ and $a < b$.

Claim 1. Suppose that $Q \subset \mathbb{R}$ is perfect, $\mathfrak{c}\text{-inf}(|f|, Q \cap I) = 0$ for each open interval I with $Q \cap I \neq \emptyset$, and the set $Q \cap [f = 0]$ is dense in Q. There exist functions $g_1, \ldots, g_k \colon Q \to [-T, T]$ such that $f = g_1 \ldots g_k$ on Q, and for each i: $g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1-\operatorname{sgn}(i-1)}$ for $x \in \mathcal{B}\chi_Q$, and $g_i \cap F \neq \emptyset$ for each closed set $F \subset \big[\mathbb{R} \times [-T, T]\big]$ with $Q \cap \operatorname{int}(\operatorname{dom} F) \neq \emptyset$.

Let $B \subset Q \cap [f = 0]$ be countable and dense in Q. Denote by $\mathcal{K}$ the family of all compact sets $K \subset \big[\mathbb{R} \times ([-T, T] \setminus \{0\})\big]$ such that $Q \cap \operatorname{int}(\operatorname{dom} K) \neq \emptyset$. Arrange the elements of $\mathcal{K}$ in a transfinite sequence, $\{K_\xi : \xi < \mathfrak{c}\}$, so that $\big|\{\xi < \mathfrak{c} : K_\xi = K\}\big| = k$ for each $K \in \mathcal{K}$. Next we proceed by transfinite induction. Fix a $\xi < \mathfrak{c}$. Since K_ξ is compact,

$$z = \min\big\{-\max \operatorname{rng}(K_\xi \cap [\mathbb{R} \times (-\infty, 0)]), \min \operatorname{rng}(K_\xi \cap [\mathbb{R} \times (0, \infty)])\big\} > 0.$$

By assumption, $\big|[|f| < T^{k-1}z] \cap Q \cap \operatorname{int}(\operatorname{dom} K_\xi)\big| = \mathfrak{c}$. Choose a point $\langle x_\xi, y_\xi \rangle \in K_\xi$ so that $x_\xi \in Q \setminus (\{x_\zeta : \zeta < \xi\} \cup B \cup \mathcal{B}\chi_Q)$ and $\sqrt[k-1]{|f(x_\xi)|/z} \leq T$.

Let $x \in Q$. If $x = x_\xi$ for some $\xi < \mathfrak{c}$, then put $i = \big|\{\zeta < \xi : K_\zeta = K_\xi\}\big| + 1$, and define $g_1(x), \ldots, g_k(x)$ so that $g_i(x) = y_\xi$, $|g_j(x)| = \sqrt[k-1]{|f(x)/y_\xi|}$ for $j \neq i$, and $f(x) = g_1(x) \ldots g_k(x)$. If $x \notin \{x_\xi : \xi < \mathfrak{c}\}$, then for each i define $g_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1-\operatorname{sgn}(i-1)}$.

We defined functions $g_1, \ldots, g_k \colon Q \to [-T, T]$ such that $f = g_1 \ldots g_k$ on Q. Fix an $i \leq k$ and a closed set $F \subset \big[\mathbb{R} \times [-T, T]\big]$ with $Q \cap \operatorname{int}(\operatorname{dom} F) \neq \emptyset$. If $F \cap [B \times \{0\}] \neq \emptyset$, then clearly $g_i \cap F \neq \emptyset$. (Observe that $[B \times \{0\}] \subset g_i$.) Otherwise recall that B is dense in Q, and choose an $x \in B \cap \operatorname{int}(\operatorname{dom} F)$. There is an $\varepsilon > 0$ such that $J = [x - \varepsilon, x + \varepsilon] \subset \operatorname{dom} F$ and $F \cap [J \times (-\varepsilon, \varepsilon)] = \emptyset$. Then $K = F \cap [J \times \mathbb{R}] \in \mathcal{K}$, whence $g_i \cap F \supset g_i \cap K \neq \emptyset$. $\lhd$

Claim 2. If $J \subset \operatorname{cl}[f = 0]$ is a compact interval, then $J \in \mathcal{J}$.

By (6), we can conclude that $\mathfrak{c}\text{-inf}(|f|, I) = 0$ for each interval $I \subset J$. Now our assertion follows by Claim 1 and Proposition 2.3. $\lhd$

Claim 3. If $a < b$ and $(a, b) \subset [f > 0]$ or $(a, b) \subset [f < 0]$, then $[a, b] \in \mathcal{J}$.

Without loss of generality we may assume that $(a, b) \subset [f > 0]$. Define $\tilde{f}(x) = k \ln T - \ln f(x)$ if $x \in (a, b)$, and $\tilde{f}(x) = 0$ if $x \in \mathbb{R} \setminus (a, b)$. By (6), for each $x \in (a, b)$ we have

$$\mathfrak{c}\text{-}\overline{\lim}\big(\tilde{f}, x^-\big) = k \ln T - \mathfrak{c}\text{-}\underline{\lim}(\ln \circ f, x^-) \geq \ln T - k^{-1} \ln f(x) = \tilde{f}(x)/k,$$

and similarly $\mathfrak{c}\text{-}\overline{\lim}\big(\tilde{f}, x^+\big) \geq \tilde{f}(x)/k$. Since $\tilde{f}$ is nonnegative and $\tilde{f}$ vanishes outside of (a, b), we can use Theorem 3.1. So, there are nonnegative almost

continuous functions $\tilde{g}_1, \ldots, \tilde{g}_k \colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{f} = \tilde{g}_1 + \cdots + \tilde{g}_k$ on $\mathbb{R}$. Moreover we can conclude that for each i the conditions listed in Remark 3.1 hold.

Let $i \le k$. Define $g_i(x) = T/\exp(\tilde{g}_i(x))$ if $x \in (a, b)$, and $g_i(x) = \sqrt[k]{f(x)}$ if $x \in \{a, b\}$. (Notice that by (1), we have $\{a, b\} \subset [f \ge 0]$.) Then the left cluster set of g_i at b equals $[\mathfrak{c}\text{-}\underline{\lim}(f, b^-)/T^{k-1}, T]$, the right cluster set of g_i at a equals $[\mathfrak{c}\text{-}\underline{\lim}(f, a^+)/T^{k-1}, T]$, and g_i has property $(\star)$ on (a, b). So by (6), g_i is almost continuous. (See [8] or [18, Theorem 2.4].) ◁

Claim 4. Put $P = \mathbb{R} \setminus G$ and let A be the boundary of $\mathrm{cl}[f = 0]$. Then $P \subset A$.

If $x \notin A$, then there is a $\delta > 0$ such that either $(x - \delta, x + \delta) \subset \mathrm{cl}[f = 0]$ or $(x - \delta, x + \delta) \subset [f \ne 0]$. Let $J \subset (x - \delta, x + \delta)$ be a compact interval. Then by Claim 2 or by (1) and Claim 3, we obtain $J \in \mathcal{J}$. Thus $x \in G = \mathbb{R} \setminus P$. ◁

Claims 5 and 6 are easy to prove. (Cf. also Claim 2 in Theorem 3.1.)

Claim 5. If $[a_0, a_1] \in \mathcal{J}$ and $[a_1, a_2] \in \mathcal{J}$, then $[a_0, a_2] \in \mathcal{J}$. ◁

Claim 6. If $a < b$ and $[a, b] \subset G$, then $[a, b] \in \mathcal{J}$. ◁

Claim 7. If $a < b$ and $(a, b) \subset G$, then $[a, b] \in \mathcal{J}$.

Let $c \in (a, b)$. By Claim 5, it suffices to show that $[a, c], [c, b] \in \mathcal{J}$. We will verify only that $[a, c] \in \mathcal{J}$, the proof of the other relation being analogous.

If $[f = 0] \cap (a, d] = \emptyset$ for some $d \in (a, c)$, then by (1), either $(a, d] \subset [f > 0]$ and $f(a) \ge 0$, or $(a, d] \subset [f < 0]$ and $f(a) \le 0$. By Claims 3, 6, and 5, we get $[a, d] \in \mathcal{J}$, $[d, c] \in \mathcal{J}$, and finally $[a, c] \in \mathcal{J}$.

So assume that there exists a sequence $(a_n) \subset [f = 0] \cap (a, c)$ such that $a_n \searrow a$. Put $a_0 = c$. Fix $m \in \mathbb{N}$ and $j \in \{1, \ldots, 4\}$. Let $n = 4m - j + 1$ and let $g_{1,n}, \ldots, g_{k,n}$ correspond to $[a_n, a_{n-1}] \in \mathcal{J}$. (We use Claim 6.) Choose $t_{1,n}, \ldots, t_{k,n} \in \{-1, 1\}$ such that $t_{1,n} \ldots t_{k,n} = 1$ and

- if $j = 1$, then $-t_{1,n} g_{1,n}$ has property $(\star)$ on $[a_n, a_{n-1}]$;

- if $j = 2$, then $t_{1,n} g_{1,n}$ has property $(\star)$ on $[a_n, a_{n-1}]$;

- if $j = 3$, then $-t_{i,n} g_{i,n}$ has property $(\star)$ on $[a_n, a_{n-1}]$ for each $i > 1$;

- if $j = 4$, then $t_{i,n} g_{i,n}$ has property $(\star)$ on $[a_n, a_{n-1}]$ for each $i > 1$.

Let $i \leq k$. Define $g_i(x) = t_{i,n} g_{i,n}(x)$ if $x \in [a_n, a_{n-1}]$ for some $n \in \mathbb{N}$, and let $g_i(a) = \sqrt[k]{|f(a)|} \cdot (\operatorname{sgn} f(a))^{1-\operatorname{sgn}(i-1)}$. Notice that for each n we have

$$\{-T, T\} \subset (t_{i,n} g_{i,n})\big[[a_n, a_{n-1}]\big] \cup \cdots \cup (t_{i,n+3} g_{i,n+3})\big[[a_{n+3}, a_{n+2}]\big].$$

So, the right cluster set of g_i at a equals $[-T, T]$. Hence g_i is almost continuous and $[a, c] \in \mathcal{J}$. $\triangleleft$

Claim 8. The set P is perfect.

Clearly G is open. If $P \cap (s - \delta, s + \delta) = \{s\}$ for some $s \in P$ and $\delta > 0$, then by Claims 7 and 5, we obtain $s \in G$, an impossibility. $\triangleleft$

Claim 9. The set $P \cap [f = 0]$ is dense in P.

By way of contradiction suppose that this is not the case. Let $s < t$ be such that $\emptyset \neq P \cap (s, t) \subset [f \neq 0]$. We will show that $(s, t) \subset G$, which is impossible.

Let $J \subset (s, t)$ be a compact interval. By Claims 8, 7, and 5, we may assume that $P \cap J$ is perfect, and $\min J, \max J \in P$. Let $\mathcal{J}$ be the family of all components of $J \setminus P$. Since $P \subset A \subset \operatorname{cl}[f = 0]$, there are pairwise disjoint families $\mathcal{J}_1, \ldots, \mathcal{J}_4 \subset \mathcal{J}$ such that $P \cap J \subset \operatorname{cl} \bigcup \mathcal{J}_j$ for $j \leq 4$ and for each $I \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_4$ there is an $x_I \in [f = 0] \cap I$. For $I \in \mathcal{J} \setminus (\mathcal{J}_1, \cup \cdots \cup \mathcal{J}_4)$ let $g_{1,I}, \ldots, g_{k,I}$ correspond to $\operatorname{cl} I \in \mathcal{J}$. (See Claim 7.)

Let $j \leq 4$ and $I = (c, d) \in \mathcal{J}_j$. Let $g_{1,I}^0, \ldots, g_{k,I}^0$ (respectively $g_{1,I}^1, \ldots, g_{k,I}^1$) correspond to $[c, x_I] \in \mathcal{J}$ (respectively $[x_I, d] \in \mathcal{J}$). Choose $t_{i,I}^s \in \{-1, 1\}$ ($i \leq k$, $s \in \{0, 1\}$) so that $t_{1,I}^0 \ldots t_{k,I}^0 = t_{1,I}^1 \ldots t_{k,I}^1 = 1$ and

- if $j = 1$, then $-T \in (t_{1,I}^0 g_{1,I}^0)\big[[c, x_I]\big]$ and $T \in (t_{1,I}^1 g_{1,I}^1)\big[[x_I, d]\big]$;

- if $j = 2$, then $T \in (t_{1,I}^0 g_{1,I}^0)\big[[c, x_I]\big]$ and $-T \in (t_{1,I}^1 g_{1,I}^1)\big[[x_I, d]\big]$;

- if $j = 3$, then $-T \in (t_{i,I}^0 g_{i,I}^0)\big[[c, x_I]\big]$ and $T \in (t_{i,I}^1 g_{i,I}^1)\big[[x_I, d]\big]$ for each $i > 1$;

- if $j = 4$, then $T \in (t_{i,I}^0 g_{i,I}^0)\big[[c, x_I]\big]$ and $-T \in (t_{i,I}^1 g_{i,I}^1)\big[[x_I, d]\big]$ for each $i > 1$.

Let $i \leq k$. Define $\tilde{g}_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1-\operatorname{sgn}(i-1)}$ if $x \in J \setminus \bigcup_{I \in \mathcal{J}} \operatorname{cl} I$, and

$$g_i = \tilde{g}_i \cup \bigcup_{I \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_4} (t_{i,I}^0 g_{i,I}^0 \cup t_{i,I}^1 g_{i,I}^1) \cup \bigcup_{I \in \mathcal{J} \setminus (\mathcal{J}_1 \cup \cdots \cup \mathcal{J}_4)} g_{i,I}.$$

Observe that g_i both of type $\langle -T, T \rangle$ and of type $\langle T, -T \rangle$ with respect to $P \cap J$. (See definition (2).) Hence by Proposition 2.1, g_i is almost continuous. Thus $J \in \mathcal{J}$ and $P \cap G \supset P \cap (s, t) \neq \emptyset$, an impossibility. $\triangleleft$

Claim 10. If $P \cap (s, t) \neq \emptyset$, then $\mathfrak{c}\text{-inf}\big(|f|, P \cap (s, t)\big) = 0$.

By way of contradiction suppose that this is not the case. Let $s < t$ be such that $P \cap (s, t) \neq \emptyset$ and $\mathfrak{c}\text{-inf}\big(|f|, P \cap (s, t)\big) > 0$. We will show that $(s, t) \subset G$, which is impossible.

Let $J \subset (s, t)$ be a compact interval. We may assume that $P \cap J$ is perfect, and $\min J, \max J \in P$. Let $\mathfrak{I}$ be the family of all components of $J \setminus P$. Then Claim 9, condition (6), and our supposition imply that $\mathfrak{c}\text{-inf}(|f|, I \setminus P) = 0$ for each open interval I with $P \cap J \cap I \neq \emptyset$. So by Lemma 3.2, there are pairwise disjoint families $\mathfrak{I}_1, \ldots, \mathfrak{I}_4 \subset \mathfrak{I}$ such that $\mathfrak{c}\text{-inf}\big(|f|, I \cap \bigcup \mathfrak{I}_j\big) = 0$ for each open interval I with $P \cap J \cap I \neq \emptyset$ and each $j \leq 4$. Clearly we may assume that $\mathfrak{I} = \mathfrak{I}_1 \cup \cdots \cup \mathfrak{I}_4$.

Fix a $j \leq 4$ and an $I \in \mathfrak{I}_j$. Let $g_{1,I}, \ldots, g_{k,I}$ correspond to $\operatorname{cl} I \in \mathfrak{J}$. Choose $t_{1,I}, \ldots, t_{k,I} \in \{-1, 1\}$ so that $t_{1,I} \ldots t_{k,I} = 1$ and

- if $j = 1$, then $-t_{1,I} g_{1,I}$ has property $(\star)$ on I;

- if $j = 2$, then $t_{1,I} g_{1,I}$ has property $(\star)$ on I;

- if $j = 3$, then $-t_{i,I} g_{i,I}$ has property $(\star)$ on I for each $i > 1$;

- if $j = 4$, then $t_{i,I} g_{i,I}$ has property $(\star)$ on I for each $i > 1$.

Let $i \leq k$. Set $\tilde{g}_i(x) = \sqrt[k]{|f(x)|} \cdot (\operatorname{sgn} f(x))^{1 - \operatorname{sgn}(i-1)}$ if $x \in J \setminus \bigcup_{I \in \mathfrak{I}} \operatorname{cl} I$, and let $g_i = \tilde{g}_i \cup \bigcup_{I \in \mathfrak{I}} (t_{i,I} g_{i,I})$. Observe that g_i is simultaneously of type $\langle -T, 0 \rangle$, of type $\langle 0, -T \rangle$, of type $\langle T, 0 \rangle$, and of type $\langle 0, T \rangle$ with respect to $P \cap J$. Hence by Claim 9 and Proposition 2.2, g_i is almost continuous. Thus $J \in \mathfrak{J}$ and $P \cap G \supset P \cap (s, t) \neq \emptyset$, an impossibility. ◁

Claim 11. We have $G = \mathbb{R}$.

By way of contradiction suppose that $P \neq \emptyset$. We will show that $P \subset G$, which is impossible.

Let J be a compact interval. We may assume that $P \cap J$ is perfect, and $\min J, \max J \in P$. Let $\mathfrak{I}$ be the family of all components of $J \setminus P$. Apply Claim 1 with $Q = P \cap J$ to construct functions $\tilde{g}_1, \ldots, \tilde{g}_k : P \cap J \to [-T, T]$ fulfilling properties listed there. (Cf. Claims 8–10.) For each $I \in \mathfrak{I}$ let $g_{1,I}, \ldots, g_{k,I}$ correspond to $\operatorname{cl} I \in \mathfrak{J}$.

Let $i \leq k$. Define $g_i = \tilde{g}_i \cup \bigcup_{I \in \mathfrak{I}} g_{i,I}$. Then by Proposition 2.3, g_i is almost continuous. Thus $J \in \mathfrak{J}$ and $P \cap G = P \neq \emptyset$, an impossibility. ◁

Using Claims 11 and 5 one can easily construct almost continuous functions $g_1, \ldots, g_k : \mathbb{R} \to [-T, T]$ such that $f = g_1 \ldots g_k$ on $\mathbb{R}$.

iii) $\Rightarrow$ i). This implication is evident. □

For each $k > 1$ and $\alpha \geq 1$, if a Baire α function $f\colon \mathbb{R} \to \mathbb{R}$ fulfills condition ii) of Theorem 3.3, then f is the product of k bounded Darboux Baire α functions. (See [13], [15].) So, I would like to ask the following question. (Recall that in Baire class one Darboux property and almost continuity are equivalent [1].)

Problem 3.1. Let $k > 1$ and $\alpha \geq 2$. If a Baire α function $f\colon \mathbb{R} \to \mathbb{R}$ fulfills condition ii) of Theorem 3.3, is f the product of k bounded bounded almost continuous Baire α functions?

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