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COMMON FIXED POINTS FOR COMMUTING COURNOT MAPS

Abstract

We study some conditions to guarantee the existence of common fixed points of two commuting Cournot maps $F(x, y) = (f_2(y), f_1(x))$, $G(x, y) = (g_2(y), g_1(x))$, defined from $I^2 = [0, 1]^2$ into itself. In particular, we prove that Jungck's Theorem and Jachymski's equivalent conditions can be only partially proved in this setting.

1 Introduction

In the fifties the problem of proving whether two commuting continuous interval maps share fixed points was posed independently by E. Dyer, A. Shields and L. Dubins. This problem has a positive solution in the case of polynomials, as J. F. Ritt pointed out in the 1920's (see [22]). Moreover, this problem has a positive answer in particular cases, under restrictive conditions (for instance, see [12], [13], [26], [9], [8], [23]). Finally, it is known that Boyce ([4]) and Huneke ([16]) found simultaneously counterexamples which show that in general the answer is negative.

Since then the results in this subject were focused on the following directions. First, instead of two commuting functions, a family of commuting functions was considered ([5], [20], [6]). Second, the problem was extended to other compact metric spaces and to particular classes of continuous maps ([18], [19], [17], [14], [15]). Third, the problem has been also posed in terms of sharing periodic points which are not necessarily fixed points (see [1], [2], [27]).

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Now we investigate whether the results on common fixed points of commuting functions work for a special class of two-dimensional continuous maps, Cournot maps, whose form is $F(x, y) = (g(y), f(x))$. This class of maps models the Cournot duopoly ([10]), an economical process in which two competitive firms produce an identical commodity, and their profits are given in terms of the levels of production of the rival firm in the last step. (See [11] and [21] for a detailed explanation of the model.)

The paper is organized as follows. In Section 2 we present known results on shared fixed points for commuting interval maps and triangular maps, which we will try to extend to the Cournot case. In Section 3 we introduce definitions and notation used throughout the paper. Moreover, we give basic properties on fixed points for (commuting) Cournot maps, and connect these maps with the compositions of their coordinate maps. In the next sections we state our main results on common fixed points for Cournot maps defined on the unit square.

2 Preliminaries. Results on Common Fixed Points

The space of continuous maps from a compact metric space X into itself is denoted by $C(X, X)$. Let $f \in C(X, X)$. We define the n-th *iterate* of f by $f^n = f \circ f^{n-1}$, $n \geq 1$, $f^0 =$ Identity. The *orbit* of $x \in X$ is the set $\{f^n(x)\}_{n=0}^{\infty}$. We say that $x \in X$ is a *periodic point* of f whenever $f^n(x) = x$ for some nonnegative integer n. The smallest of these values n is called the *order* or period of the periodic point. If $f(x) = x$, then x is a *fixed point*. $\mathrm{Per}(f)$, $\mathrm{P}(f)$ and $\mathrm{Fix}(f)$ denote the sets of periods, periodic points and fixed points of f, respectively.

If $f \in C(I, I)$, with $I = [0, 1]$, we say that f is an interval map. A map $G \in C(I^n, I^n)$ is called a *triangular map* if it has the form

$$G(x_1, x_2, ..., x_n) = (g_1(x_1), g_2(x_1, x_2), ..., g_n(x_1, x_2, ..., x_n)).$$

The set of triangular maps will be denoted by $C_\Delta(I^n, I^n)$.

In this section we recall well known results on common fixed points for commuting interval maps and commuting triangular maps. We also present several properties relating equicontinuity, pointwise convergence and uniform convergence with the set of periodic points of these maps.

We start with the following property on equicontinuous families of interval maps. Recall that $\{f_\alpha\}_{\alpha \in A} \subset C(X, X)$ is *equicontinuous at* $x \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f_\alpha(x), f_\alpha(y)) < \varepsilon$ for all $\alpha \in A$ and for any $y \in X$ with $\rho(x, y) < \delta$. (Here ρ denote the metric of X.) The family is called equicontinuous if it is equicontinuous at $x \in X$, for all $x \in X$.

We say that $\mathrm{Fix}(f)$ is nondegenerate if it is not a singleton.

Theorem 2.1. *([6],Theorem 2) Let $f \in C(I,I)$. Suppose that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then,*

1. $f \in \mathcal{A} = \{g \in C(I,I) : \mathrm{Fix}(g) = [a_g, b_g],\ a_g \leq b_g\}$.
2. *If* $\mathrm{Fix}(f)$ *is nondegenerate,* $f \in \mathcal{B} = \{g \in C(I,I) : \mathrm{Fix}(g) = \mathrm{P}(g)\}$.

In [14] a version of a result of Cano (see [6],Theorem 1) for two commuting triangular maps was proved.

Theorem 2.2. *([14],Theorem 2.3)Assume that $F, G \in C_{\Delta}(I^2, I^2)$ commute. If either*

1. $\mathrm{Per}(G) = \mathrm{Fix}(G)$, *or*
2. $\pi_1(\mathrm{Fix}(G))$ *is an interval and* $\mathrm{Fix}(g_2(x,\cdot))$ *is an interval for every* $x \in \pi_1(\mathrm{Fix}(G))$, *where* π_1 *denote the canonical projection given by* $\pi_1(x,y) = x$,

then $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) \neq \emptyset$.

Following the notation of [15], for $f, g \in C(X,X)$ we put

$$\mathrm{Coin}(f,g) = \{x \in X : f(x) = g(x)\}.$$

If $\mathrm{Coin}(f,g) \neq \emptyset$ and f, g commute on $\mathrm{Coin}(f,g)$, then we say that f and g are *nontrivially compatible* ([19]). If $X = I$, then the following result holds (Jungck's Theorem).

Theorem 2.3. *([19],Theorem 3.6) A map $g \in C(I,I)$ has a common fixed point with every map $f \in C(I,I)$ which is nontrivially compatible with g if and only if* $\mathrm{P}(g) = \mathrm{Fix}(g)$.

We consider now the results given by Jachymski in [17] on equivalent conditions to guarantee the existence of common fixed points for interval maps.

Theorem 2.4. *([17],Theorem 1) Let g be a continuous self-map of I. The following conditions are equivalent:*

1. $\mathrm{Fix}(g)$ *is a closed interval.*
2. *The family $\{g^n : n \in \mathbb{N}\}$ is equicontinuous on* $\mathrm{Fix}(g)$, *or* $\mathrm{Fix}(g)$ *is a singleton.*
3. *g has a common fixed point with every continuous map $f : I \to I$ that commutes with g on* $\mathrm{Fix}(g)$.

Theorem 2.5. *([17],Theorem 2) Let $g : I \to I$ be continuous. Then the following conditions are equivalent:*

1. $\mathrm{Fix}(g) = \mathrm{P}(g)$.
2. *The sequence $\{g^n\}_{n=1}^{\infty}$ is pointwise convergent on I.*
3. *g has a common fixed point with every continuous map $f : I \to I$ that commutes with g on* $\mathrm{Fix}(f)$.

Theorem 2.6. *([17],Theorem 3) Let $g \in C(I, I)$. Suppose that $\mathrm{Fix}(g)$ is not a singleton. Then, the following conditions are equivalent:*

1. *The family of iterates $\{g^n : n \in \mathbb{N}\}$ is equicontinuous on I.*
2. *The sequence $\{g^n\}_{n=1}^{\infty}$ is uniformly convergent on I.*
3. *g has a common fixed point with every continuous map $f : I \to I$ that commutes with g either on $\mathrm{Fix}(f)$, or on $\mathrm{Fix}(g)$.*

Finally, we introduce Corollary 2.8 of [15]. Notice that it states that in the triangular case Jungck's Theorem is equivalent to condition (3) of Theorem 2.5.

Theorem 2.7. *([15],Corollary 2.8) Let $G \in C_\Delta(I^n, I^n)$. Then the following conditions are equivalent:*

1. $\mathrm{P}(G) = \mathrm{Fix}(G)$.
2. *$C \cap \mathrm{Fix}(G) \neq \emptyset$ for any nonempty closed set $C \subseteq I^n$ such that $G(C) \subseteq C$.*
3. *G has a common fixed point with every $F \in C_\Delta(I^n, I^n)$ that commutes with G on $\mathrm{Fix}(F)$.*
4. *G has a common fixed point with every map $F \in C(I^n, I^n)$ that commutes with G on $\mathrm{Fix}(F)$.*
5. *G has a common fixed point with every triangular map F which is nontrivially compatible with G.*

3 Basic Properties of Cournot Maps

Given two compact metric spaces X, Y, we say that $F : X \times Y \to X \times Y$ is a *Cournot map* if $F(x, y) = (f_2(y), f_1(x))$, where $f_1 : X \to Y$ and $f_2 : Y \to X$ are continuous. It is easy to check that for every $n \geq 0$

$$F^{2n}(x, y) = ((f_2 \circ f_1)^n(x), (f_1 \circ f_2)^n(y)) \qquad (1)$$

and

$$F^{2n+1}(x,y) = ((f_2 \circ f_1)^n(f_2(y)), (f_1 \circ f_2)^n(f_1(x))). \tag{2}$$

Observe that

$$\mathrm{P}(F) = \mathrm{P}(f_2 \circ f_1) \times \mathrm{P}(f_1 \circ f_2) \text{ (see [7])}. \tag{3}$$

We use $C_A(X \times Y)$ to denote the set of Cournot maps from $X \times Y$ into itself. From now on, we denote Cournot maps with capital letters, and their coordinates with the corresponding indexed small letter (for example $G(x,y) = (g_2(y), g_1(x))$).We use π_1, π_2 to denote the canonical projections from $X \times Y$ onto X, and from $X \times Y$ onto Y, respectively.

Suppose that $F, G \in C_A(X \times Y)$ and $F \circ G = G \circ F$. For $i, j \in \{1, 2\}$, $i \neq j$, it follows that

$$f_j \circ g_i = g_j \circ f_i. \tag{4}$$

Moreover,

$$f_i \circ f_j \text{ and } g_i \circ g_j \text{ commute,} \tag{5}$$

$$f_i \circ f_j \circ g_i \circ g_j = (f_i \circ g_j)^2,$$

and

$$g_j \circ f_i \text{ commutes with } g_j \circ g_i \text{ and } f_j \circ f_i, \text{ for } i, j \in \{1, 2\}, i \neq j. \tag{6}$$

Let $F, G \in C_A(X \times Y)$. Then

$$(x_1, x_2) \in \mathrm{Fix}(F) \cap \mathrm{Fix}(G) \text{ iff } x_i = f_j(x_j) = g_j(x_j), i, j \in \{1, 2\}, i \neq j,$$

$$(x_1, x_2) \in \mathrm{Fix}(F) \cap \mathrm{Fix}(G) \text{ gives } x_j \in \mathrm{Fix}(f_j \circ f_i) \cap \mathrm{Fix}(g_j \circ g_i), i \neq j. \tag{7}$$

Concerning the set of fixed points of a Cournot map F, notice that if $x_i \in \mathrm{Fix}(f_j \circ f_i)$, then $f_i(x_i) \in \mathrm{Fix}(f_i \circ f_j)$, for all $i, j \in \{1, 2\}$, $i \neq j$, and $\{(x_1, f_1(x_1)), (f_2(x_2), x_2)\} \subset \mathrm{Fix}(F)$. Moreover, if x_2, y_2 are two different fixed points of $f_1 \circ f_2$, then $(f_2(y_2), x_2)$ is a periodic point of order two for F (similar conclusions hold for two different fixed points of $f_2 \circ f_1$). With the above observations, it is easy to obtain the following.

Proposition 3.1. *Given $F \in C_A(X \times Y)$, the following hold:*

1. $\mathrm{Fix}(F) \subsetneq \mathrm{Fix}(f_2 \circ f_1) \times \mathrm{Fix}(f_1 \circ f_2)$, *if* $\mathrm{Card}(\mathrm{Fix}(f_1 \circ f_2)) \geq 2$.
2. $\mathrm{Fix}(F) = \mathrm{Fix}(f_2 \circ f_1) \times \mathrm{Fix}(f_1 \circ f_2)$, *if* $\mathrm{Card}(\mathrm{Fix}(f_1 \circ f_2)) = 1$.
3. $\mathrm{Fix}(F^2) = \mathrm{Fix}(f_2 \circ f_1) \times \mathrm{Fix}(f_1 \circ f_2)$.

We can add the following results on the set of fixed points. All of them are immediate.

Proposition 3.2. *Let* $F \in C_A(X \times Y)$.

1. *Let* $(x_0, y_0) \in \mathrm{Fix}(F)$. *Then,*

$$\mathrm{Card}(\mathrm{Fix}(F) \cap (\{x_0\} \times Y)) = \mathrm{Card}(\mathrm{Fix}(F) \cap (X \times \{y_0\})) = 1.$$

2. $\mathrm{Fix}(F) = \{(x, f_1(x)) : x \in \mathrm{Fix}(f_2 \circ f_1)\} = \{(f_2(y), y) : y \in \mathrm{Fix}(f_1 \circ f_2)\}$.

3. $\mathrm{Card}(\mathrm{Fix}(F)) = \mathrm{Card}(\mathrm{Fix}(f_2 \circ f_1)) = \mathrm{Card}(\mathrm{Fix}(f_1 \circ f_2))$.

4. *For* $i, j \in \{1, 2\}$, $i \neq j$, $\pi_i(\mathrm{Fix}(F)) = \mathrm{Fix}(f_j \circ f_i)$, *and* $f_i(\pi_i(\mathrm{Fix}(F))) = \pi_j(\mathrm{Fix}(F))$.

Proposition 3.3. *Let* $F, G \in C_A(X \times Y)$ *be such that* $F \circ G = G \circ F$. *For* $i, j \in \{1, 2\}$, $i \neq j$, *we put*

$$A_i = \mathrm{Fix}(f_j \circ f_i) \cap \mathrm{Fix}(g_j \circ g_i).$$

1. *The applications* $g_i : A_i \to A_j, f_i : A_i \to A_j$ *are bijective.*

2. *The applications* $h_{ji} : A_i \to A_i$ *are bijective, where* h_{ji} *denotes one of the following maps:* $f_j \circ g_i$, $f_j \circ f_i$, $g_j \circ g_i$.

3. *The equality* $(f_j \circ g_i)^2|_{A_i} = (g_j \circ f_i)^2|_{A_i} = \mathrm{Identity}|_{A_i}$ *holds.*

Notice that there exists commuting Cournot maps on I^2 without sharing fixed points.

Proposition 3.4. *There exist* $G_1, G_2 \in C_A(I^2)$ *such that* $G_1 \circ G_2 = G_2 \circ G_1$ *and* $\mathrm{Fix}(G_1) \cap \mathrm{Fix}(G_2) = \emptyset$.

PROOF. We consider interval maps f_1, f_2 with $f_1 \circ f_2 = f_2 \circ f_1$ and $\mathrm{Fix}(f_1) \cap \mathrm{Fix}(f_2) = \emptyset$. (According to [4] or [16], these exist.) Notice that $f_2 \circ f_1 \circ f_1 = f_1 \circ f_2 \circ f_1$ and $f_2 \circ f_2 \circ f_1 \circ f_1 = f_2 \circ f_1 \circ f_2 \circ f_1$. We define $G_1, G_2 \in C_A(I^2)$ as

$$G_1(x, y) = (f_1(y), x), G_2(x, y) = ((f_2 \circ f_1 \circ f_1)(y), (f_2 \circ f_1)(x)).$$

It is straightforward to see that

$$(G_2 \circ G_1)(x, y) = ((f_2 \circ f_1 \circ f_1)(x), (f_2 \circ f_1 \circ f_1)(y)),$$

and

$$(G_1 \circ G_2)(x, y) = ((f_1 \circ f_2 \circ f_1)(x), (f_2 \circ f_1 \circ f_1)(y)),$$

so G_1 and G_2 commute. Let $(x,y) \in \mathrm{Fix}(G_1) \cap \mathrm{Fix}(G_2)$. Since $(x,y) \in \mathrm{Fix}(G_1)$, we have $x = f_1(y)$, $y = x$. Hence $x \in \mathrm{Fix}(f_1)$. On the other hand, $G_2(x,y) = (x,y)$ implies $x = (f_2 \circ f_1 \circ f_1)(y)$, $x = y = (f_2 \circ f_1)(x)$. From this, we obtain

$$\begin{aligned} f_2(x) =& f_2((f_2 \circ f_1 \circ f_1)(y)) = (f_2 \circ f_1 \circ f_2 \circ f_1)(x) \\ =& (f_2 \circ f_1)((f_2 \circ f_1)(x)) = (f_2 \circ f_1)(x) = x. \end{aligned}$$

So, $x \in \mathrm{Fix}(f_2)$, and $x \in \mathrm{Fix}(f_1) \cap \mathrm{Fix}(f_2)$, a contradiction. Therefore, $\mathrm{Fix}(G_1) \cap \mathrm{Fix}(G_2) = \emptyset$. □

We need the following results on periodic structure of Cournot maps, whose proof can be found in [3]. Remember that Sharkovskii's ordering is given by

$$\begin{aligned} 3 >_s 5 >_s 7 >_s \ldots >_s 2 \cdot 3 >_s 2 \cdot 5 >_s \ldots \cdots >_s 2^2 \cdot 3 >_s 2^2 \cdot 5 >_s \ldots \\ \ldots >_s 2^k \cdot 3 >_s 2^k \cdot 5 >_s \cdots >_s 2^3 >_s 2^2 >_s 2 >_s 1, \end{aligned}$$

and Sharkovskii's Theorem (see [24]) establishes for any $f \in C(I,I)$ that either $\mathrm{Per}(f) = S(m) = \{k : m >_s k\} \cup \{m\}$, with $m \in \mathbb{N}$, or $\mathrm{Per}(f) = S(2^\infty) = \left\{2^i : i = 0,1,2,\ldots\right\}$.

Theorem 3.5. *Let $F \in C_A(I^2)$.*

1. *F has at least two different fixed points if and only if $f_2 \circ f_1$ possesses at least two different fixed points.*
2. *$2 \in \mathrm{Per}(F)$ if and only if F has at least two different fixed points.*
3. *Either $\mathrm{Per(F)} = \mathrm{S_2(m)}$ or $\mathrm{Per}(F) = S_2(m) \cup \{2\}$, where $m \in \mathbb{N} \cup \{2^\infty\}$ and*

$$S_2(m) = \{pt : p \in \{1,2\}, t \in (S(m) \setminus \{1\}), \gcd(t, \frac{2}{p}) = 1\} \cup \{1\},$$

where $S(m)$ is an initial segment of Sharkovskii's ordering and $\gcd(s,t)$ denote the greatest common divisor of two positive integers s,t.

In the following sections we will try to extend the results of Section 2 on common fixed points from the interval case or the triangular case to the Cournot case, with $X \times Y = I^2 = [0,1]^2$. More precisely, we prove the extension of results of Cano ([6]). We show that Jungck's Theorem ([19]), which is also true in $C_\Delta(I^n, I^n)$ ([14], [15]), also works in the Cournot case if we modify the hypothesis in a suitable way. We obtain that the results on equivalent conditions involving common fixed points, obtained by Jachymski in [17], only can be partially translated to our case. And finally we see that Jungck's Theorem and Jachymski's result of Theorem 2.5, which are equivalent in the triangular case ([15]), are independent for commuting Cournot maps.

4 Extension of Cano's Results

In order to translate Theorem 2.1 to $C_A(I^2)$, we define

$$\mathfrak{A} = \{F \in C_A(I^2) : \mathrm{Fix}(F) \text{ is connected}\},$$
$$\mathfrak{B} = \{F \in C_A(I^2) : \mathrm{Fix}(F^2) = \mathrm{P}(F)\}.$$

Theorem 4.1. *Let $F \in C_A(I^2)$. Suppose that $\{F^n\}_{n=1}^{\infty}$ is equicontinuous. Then*

1. $F \in \mathfrak{A}$.

2. If $\mathrm{Fix}(F)$ is nondegenerate, $F \in \mathfrak{B}$.

Proof. If $\{F^n\}_{n=1}^{\infty}$ is equicontinuous, so is $\{F^{2m}\}_{m=1}^{\infty}$. According to (1) we deduce that $\{(f_2 \circ f_1)^m\}_{m=1}^{\infty}$ and $\{(f_1 \circ f_2)^m\}_{m=1}^{\infty}$ are equicontinuous. By Theorem 2.1, $\mathrm{Fix}(f_2 \circ f_1) = [a_1, a_2] = J$, $a_1 \leq a_2$, and $\mathrm{Fix}(f_1 \circ f_2) = [b_1, b_2] = K$, $b_1 \leq b_2$. By Proposition 3.2 we have that J is nondegenerate iff K is nondegenerate. If $J = \{a\}$, $K = \{b\}$, from Proposition 3.1 we have $\mathrm{Fix}(F) = \{(a, b)\}$; so $F \in A$. If both J and K are nondegenerate, by Proposition 3.2 we obtain

$$\mathrm{Fix}(F) = \{(x, f_1(x)) : x \in J\} = \{(f_2(y), y) : y \in K\}.$$

Therefore, $\mathrm{Fix}(F)$ is connected, and $F \in \mathfrak{A}$.

Now, suppose that $\mathrm{Fix}(F)$ is nondegenerate. Then J and K are also nondegenerate. By Theorem 2.1, $\mathrm{Fix}(f_2 \circ f_1) = \mathrm{P}(f_2 \circ f_1)$, $\mathrm{Fix}(f_1 \circ f_2) = \mathrm{P}(f_1 \circ f_2)$. Since (3) and Proposition 3.1 hold, we deduce $\mathrm{P}(F) = \mathrm{Fix}(F^2)$. □

We remark that in the Cournot case we cannot state that $\mathrm{P}(F) = \mathrm{Fix}(F)$ whenever $\mathrm{Fix}(F)$ is nondegenerate. In this case $\mathrm{Fix}(F) \subsetneq \mathrm{Fix}(F^2)$ since $\mathrm{Card}(\mathrm{Fix}(F)) \geq 2$, and $\mathrm{P}(F)$ contains periodic points of order two (see Theorem 3.5). For example, consider $F(x, y) = (y, x)$. Then $\mathrm{Fix}(F) = \{(x, x) : x \in I\}$ is nondegenerate but $\mathrm{P}(F) = I^2$.

Next, we prove that Theorem 2.2 can be extended in some sense to the Cournot case.

Theorem 4.2. *Let $F, G \in C_A(I^2)$, $F \circ G = G \circ F$. If either $\mathrm{P}(G) = \mathrm{Fix}(G)$ or $\pi_i(\mathrm{Fix}(G))$ is an interval for $i = 1, 2$, then $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) \neq \emptyset$.*

Proof. **1**. Assume that $\mathrm{P}(G) = \mathrm{Fix}(G)$. Then $2 \notin \mathrm{Per}(G)$, and according to Proposition 3.5, $\mathrm{Card}(\mathrm{Fix}(g_i \circ g_j)) = 1$ for $i, j \in \{1, 2\}$, $i \neq j$. Let z_0 be the unique fixed point of $g_2 \circ g_1$. By Proposition 3.2 we find $\mathrm{Fix}(G) = \{(z, g_1(z))\}$. Notice that $g_1(z)$ is the unique fixed point of $g_1 \circ g_2$. We wish to show that $F(z, g_1(z)) = (z, g_1(z))$, and then $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) \neq \emptyset$.

On the one hand, since F and G commute,

$$\begin{aligned} F(z, g_1(z)) =& F(G(z, g_1(z))) = G(F(z, g_1(z))) \\ =& ((g_2 \circ f_1)(z), (g_1 \circ f_2)(g_1(z))). \end{aligned} \tag{8}$$

On the other hand, a direct calculation gives

$$F(z, g_1(z)) = ((f_2 \circ g_1)(z), f_1(z)). \tag{9}$$

From (8) and (9) we deduce $f_1(z) = (g_1 \circ f_2)(g_1(z))$, and using (4) we obtain

$$f_1(z) = (g_1 \circ f_2)(g_1(z)) = (g_1 \circ g_2)(f_1(z)).$$

Hence $f_1(z)$ is a fixed point of $g_1 \circ g_2$. Therefore, $f_1(z) = g_1(z)$, $z = g_2(f_1(z))$. Again by (8) and (9) we get $F(z, g_1(z)) = (z, g_1(z))$.

2. Notice that by Proposition 3.2, $\pi_1(\mathrm{Fix}(G)) = \mathrm{Fix}(g_2 \circ g_1)$ is an interval iff $\pi_2(\mathrm{Fix}(G)) = \mathrm{Fix}(g_1 \circ g_2)$ is an interval. Assume $\pi_1(\mathrm{Fix}(G)) = [a, b]$ is an interval. Then $\mathrm{Fix}(g_1 \circ g_2) = g_1([a, b])$. By (5), $g_2 \circ g_1$ and $f_2 \circ f_1$ commute. Then Theorem 1 of [6] implies $A := \mathrm{Fix}(g_2 \circ g_1) \cap \mathrm{Fix}(f_2 \circ f_1) \neq \emptyset$. If $A = \{x_0\}$, according to Proposition 3.3 we deduce

$$\mathrm{Fix}(g_1 \circ g_2) \cap \mathrm{Fix}(f_1 \circ f_2) = \{g_1(x_0)\} = \{f_1(x_0)\}.$$

Since $g_1(x_0) = f_1(x_0)$, we obtain $(x_0, g_1(x_0)) \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F)$. Now suppose $\mathrm{Card}(A) \geq 2$. By Proposition 3.3 we know $(g_2 \circ f_1)^2(x_0) = x_0$. If $(g_2 \circ f_1)(x) = x$ for some $x \in A$, then $g_1(x) = f_1(x)$, and we go on as above. Assume then that there exist $x_1, x_2 \in A$, $x_1 < x_2$, such that $(g_2 \circ f_1)(x_i) = x_j$, $i, j \in \{1, 2\}$, $i \neq j$. By continuity, there exists $p \in (x_1, x_2) \cap \mathrm{Fix}(g_2 \circ f_1)$. Since $(x_1, x_2) \subset [a, b] = \mathrm{Fix}(g_2 \circ g_1)$, $p \in \mathrm{Fix}(g_2 \circ g_1)$ holds. From this and using (4), (6) and Proposition 3.3, we have

$$\begin{aligned} (f_2 \circ f_1)((g_2 \circ g_1)(p)) =& (f_2 \circ f_1)(p) = (g_2 \circ g_1)((f_2 \circ f_1)(p)) \\ =& (g_2 \circ f_1)((g_2 \circ f_1)(p)) = (g_2 \circ f_1)^2(p) = p; \end{aligned}$$

so $p \in \mathrm{Fix}(f_2 \circ f_1)$. We conclude that $f_1(p) = g_1(p)$, so $(p, g_1(p)) \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F)$. □

5 Extension of Jungck's Theorem

Jungck's Theorem can be extended to the Cournot case if we replace $\mathrm{Fix}(g)$ by $\mathrm{Fix}(G^2)$.

Theorem 5.1. *Let $G \in C_A(I^2)$. Then* $\mathrm{P}(G) = \mathrm{Fix}(G^2)$ *if and only if* $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$ *holds for all $F \in C_A(I^2)$ nontrivially compatible with G.*

PROOF. Suppose that $F \circ G = G \circ F$ on $\mathrm{Coin}(F,G) \neq \emptyset$. Let $(x,y) \in \mathrm{Coin}(F,G)$. It is not difficult to see that $\{G^n(x,y)\}_{n=0}^{\infty} \subseteq \mathrm{Coin}(F,G)$. In particular, $G^{2n}(x,y) = ((g_2 \circ g_1)^n(x), (g_1 \circ g_2)^n(y)) \in \mathrm{Coin}(F,G)$. Since $\mathrm{P}(G) = \mathrm{Fix}(G^2)$, from (3) and Proposition 3.1 we have $\mathrm{P}(g_j \circ g_i) = \mathrm{Fix}(g_j \circ g_i)$ for $i,j \in \{1,2\}$, $i \neq j$. According to [25],Chapter 4, Th.4.2, it follows that $(g_2 \circ g_1)^n(x) \to x_g$ and $(g_1 \circ g_2)^n(y) \to y_g$, when $n \to \infty$, for some $x_g \in \mathrm{Fix}(g_2 \circ g_1)$ and $y_g \in \mathrm{Fix}(g_1 \circ g_2)$. Since $\mathrm{Coin}(F,G)$ is obviously a closed set and $G^{2n}(x,y) \to (x_g, y_g)$, we deduce that $F(x_g,y_g) = G(x_g,y_g)$. (In particular, $f_1(x_g) = g_1(x_g)$.)

Since $\mathrm{Per}(G) = \{1,2\}$ and $(x_g,y_g) \in \mathrm{P}(G) \cap \mathrm{Coin}(F,G)$, we have

$$\begin{aligned}(x_g,y_g) =& G^2(x_g,y_g) = G(G(x_g,y_g)) = G(F(x_g,y_g)) = F(G(x_g,y_g))\\ =& F(F(x_g,y_g)) = F^2(x_g,y_g);\end{aligned}$$

so $(x_g,y_g) \in \mathrm{Fix}(G^2) \cap \mathrm{Fix}(F^2)$. In particular, $x_g \in \mathrm{Fix}(f_2 \circ f_1)$. Finally, since $f_1(x_g) = g_1(x_g)$ and $(f_2 \circ f_1)(x_g) = x_g$, it is easily seen that $(x_g, g_1(x_g)) \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F)$.

Now, suppose that $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$ holds for all $F \in C_A(I^2)$ nontrivially compatible with G. We wish to prove that $\mathrm{P}(G) = \mathrm{Fix}(G^2)$. Suppose that $\mathrm{P}(G) \supset \mathrm{Fix}(G^2)$. Since (3) and Proposition 3.1 hold, there exists a periodic point $u \in I$ of order 2 for $g_2 \circ g_1$. Moreover, let $v \in I$ be a fixed point of $g_1 \circ g_2$. Notice that (u,v) is a periodic point of period 4 of G. Let $\mathrm{Orb}_G(u,v)$ denote its finite orbit.

We define $F \in C_A(I^2)$ in the following way. We put

$$\begin{aligned}&f_1(u) = g_1(u), f_1(g_2(v)) = v, f_1((g_2 \circ g_1)(u)) = g_1((g_2 \circ g_1)(u)),\\ &f_2(v) = g_2(v), f_2(g_1(u)) = g_2(g_1(u)), f_2((g_1 \circ g_2 \circ g_1)(u)) = u,\end{aligned}$$

and we continuously extend f_1, f_2 such that $f_1(x) \neq g_1(x)$, $f_2(y) \neq g_2(y)$ for all $x \in I \setminus \{u, g_2(v), g_2(g_1(u))\}$ and $y \in I \setminus \{v, g_1(u), (g_1 \circ g_2 \circ g_1)(u)\}$. Notice that $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset$. It is clear that $\mathrm{Coin}(F,G) = \mathrm{Orb}_G(u,v)$, and $F \circ G = G \circ F$ on $\mathrm{Coin}(F,G)$. By hypothesis, $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) \neq \emptyset$, a contradiction. □

In the statement of Jungck's Theorem we cannot replace $\mathrm{Fix}(g)$ by $\mathrm{Fix}(G)$, as the following example shows.

Example 5.2. Consider $G(x,y) = (1-y, 1-x)$. It is clear that

$$\begin{aligned} \text{Fix}(G) =& \{(x, 1-x) : x \in I\} = \{(1-y, y) : y \in I\}, \\ \text{Fix}(G^2) =& I^2 = \text{P}(G). \end{aligned}$$

Let $F \in C_A(I^2)$ be nontrivially compatible with G. We are going to show that F and G share a common fixed point. However, $\text{Fix}(G) \subset \text{P}(G)$. Observe that $(x,y) \in \text{Coin}(F,G)$ if and only if

$$f_1(x) = 1-x, f_2(y) = 1-y. \tag{10}$$

Let $(x,y) \in \text{Coin}(F,G)$. Then $(F \circ G)(x,y) = (G \circ F)(x,y)$ implies

$$f_1(1-y) = 1 - f_2(y), f_2(1-x) = 1 - f_1(x). \tag{11}$$

From (10) and (11), we obtain $f_1(1-y) = y, f_2(y) = 1-y$. This yields $Z = (1-y, y) \in \text{Coin}(F,G)$. Moreover, $Z \in \text{Fix}(G)$. Then $F(Z) = G(Z) = Z$, so $Z \in \text{Fix}(G) \cap \text{Fix}(F)$.

6 Extension of Jachymski's Results

Now, we try to extend to the Cournot case the results given in [17] on equivalent conditions to guarantee the existence of common fixed points for interval maps.

Theorem 6.1. *Let $G \in C_A(I^2)$. The following conditions are equivalent:*

1. $\text{Fix}(G)$ *is a connected set.*
2. $\{G^n : n \in \mathbb{N}\}$ *is equicontinuous on* $\text{Fix}(G)$, *or* $\text{Fix}(G)$ *is a singleton.*
3. *G has a common fixed point with every $F \in C_A(I^2)$ which commutes with G on* $\text{Fix}(G)$.

Proof. (1) $\Rightarrow$ (2) Assume that $\text{Fix}(G)$ is a connected set. If $\text{Fix}(G)$ is a singleton, there is nothing to prove. Suppose then that $\text{Card}(\text{Fix}(G)) \geq 2$. According to Proposition 3.2 we have that $\text{Fix}(g_i \circ g_j)$ is a closed interval for $i,j \in \{1,2\}$, $i \neq j$. From Theorem 2.4 we deduce that $\{(g_i \circ g_j)^n : n \in \mathbb{N}\}$ is equicontinuous on $\text{Fix}(g_i \circ g_j)$, $i,j \in \{1,2\}$, $i \neq j$. By (1), $\{G^{2n} : n \in \mathbb{N}\}$ is equicontinuous on $\text{Fix}(g_2 \circ g_1) \times \text{Fix}(g_1 \circ g_2) = \text{Fix}(G^2)$, in particular on $\text{Fix}(G)$. The continuity of G implies that $\{G^{2n+1} : n \in \mathbb{N}\}$ is also equicontinuous on $\text{Fix}(G)$. Since $\lim_{n\to\infty} G^{2n}(Z) = \lim_{n\to\infty} G^{2n+1}(Z) = Z$ for all $Z \in \text{Fix}(G)$, we finally obtain that $\{G^m : m \in \mathbb{N}\}$ is equicontinuous on $\text{Fix}(G)$.

(2) $\Rightarrow$ (1) Suppose that $\{G^n : n \in \mathbb{N}\}$ is equicontinuous on $\mathrm{Fix}(G)$. Then $\{G^{2n} : n \in \mathbb{N}\}$ is also. From (1), $\{(g_i \circ g_j)^n : n \in \mathbb{N}\}$ is equicontinuous on $\mathrm{Fix}(g_i \circ g_j)$, for $i, j \in \{1, 2\}$, $i \neq j$. By Theorem 2.4, for $i, j \in \{1, 2\}$, $i \neq j$, we obtain that $\mathrm{Fix}(g_i \circ g_j)$ is a closed interval. From Proposition 3.2 we conclude that $\mathrm{Fix}(G)$ is connected.

(1) $\Rightarrow$ (3) Suppose that $\mathrm{Fix}(G)$ is connected. Let $F \in C_A(I^2)$ commute with G on $\mathrm{Fix}(G)$. We must prove that $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$. Since $F \circ G = G \circ F$ on $\mathrm{Fix}(G)$, it follows that $F^2 \circ G^2 = G^2 \circ F^2$ on $\mathrm{Fix}(G)$. (Notice that $F(Z) \in \mathrm{Fix}(G)$ for all $Z \in \mathrm{Fix}(G)$.) In this case, given $x \in \mathrm{Fix}(g_2 \circ g_1)$, it follows that $(g_2 \circ g_1 \circ f_2 \circ f_1)(x) = (f_2 \circ f_1 \circ g_2 \circ g_1)(x)$ since $(x, g_1(x)) \in \mathrm{Fix}(G)$. As $\mathrm{Fix}(G)$ is connected, $\mathrm{Fix}(g_i \circ g_j)$ is a closed interval, for $i, j \in \{1, 2\}$, $i \neq j$, and according to Theorem 2.4, we obtain that $A := \mathrm{Fix}(g_2 \circ g_1) \cap \mathrm{Fix}(f_2 \circ f_1) \neq \emptyset$. Let $z \in A$. By Proposition 3.2 and since $g_1(z) \in \mathrm{Fix}(f_1 \circ f_2)$, we obtain that $Z = (z, g_1(z)) \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F^2) \neq \emptyset$. Recall that $F(Z) \in \mathrm{Fix}(G)$. Again by Proposition 3.2, if $\mathrm{Fix}(G)$ is connected, then it is homeomorphic to a closed interval of I, $\mathrm{Fix}(G) \overset{h}{\approx} \Gamma$. Suppose $F(Z) \neq Z = F^2(Z)$. Then $h(F(Z)) \neq h(Z) = h(F^2(Z))$. Let $\varphi = h \circ F \circ h^{-1} : \Gamma \to \Gamma$. Then φ is continuous and well defined since $F : \mathrm{Fix}\, G \to \mathrm{Fix}\, G$. As

$$\varphi(h(Z)) = h(F(Z)) \neq h(F^2(Z)) = (h \circ F)(F(Z)) = \varphi(h(F(Z))),$$

and $\varphi(h(Z)) = h(F(Z))$, $\varphi(h(F(Z)) = \varphi(h(Z))$, we deduce that there exists $w \in \Gamma$ such that $\varphi(w) = w$. Then $F(h^{-1}(w)) = h^{-1}(w) := W$, so $W \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F)$, which completes the proof.

(3) $\Rightarrow$ (1) Now suppose that $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$ for every $F \in C_A(I^2)$ which commutes with G on $\mathrm{Fix}(G)$. We are going to prove that $\mathrm{Fix}(G)$ is connected. On the contrary, suppose that $\mathrm{Fix}(G)$ is not connected. Let $\mathrm{Fix}(G) = \bigcup_{\alpha \in \chi} C_\alpha$, where C_α is a closed connected component of $\mathrm{Fix}(G)$. Let $C := \bigcup_{\alpha \in \chi} C_\alpha$. According to Proposition 3.2 we have that $\pi_i(C_\alpha) \cap \pi_i(C_\beta) = \emptyset$ for all $\alpha, \beta \in \chi$, $\alpha \neq \beta$, $i = 1, 2$, and $\pi_i(C) = \bigcup_{\alpha \in \chi} \pi_i(C_\alpha)$ for $i = 1, 2$. Set $D_i := \pi_i(C)$, $i = 1, 2$. Now we construct two interval maps f_i, $i = 1, 2$, in the following way. We choose two different connected components C_{α_0}, C_{α_1} and two points $(p_1, p_2) \in C_{\alpha_0}$, $(q_1, q_2) \in C_{\alpha_1}$. Define $f_i(\pi_i(C_\alpha)) = p_{i+1(\mathrm{mod}2)} \in \pi_{i+1(\mathrm{mod}2)}(C_{\alpha_0})$ for any index $\alpha \neq \alpha_0$, $f_i(\pi_i(C_{\alpha_0})) = q_{i+1(\mathrm{mod}2)} \in \pi_{i+1(\mathrm{mod}2)}(C_{\alpha_1})$, and we complete continuously the definition of f_i over $I \setminus D_i$, $i = 1, 2$. Then we define the Cournot map $F(x, y) = (f_2(y), f_1(x))$. In this case, it is easy to check that $F \circ G = G \circ F$ on $\mathrm{Fix}(G)$ since $F\,|_C \colon C \to C$. Moreover, it is clear that $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset$, which contradicts the statement of (3). Therefore, $\mathrm{Fix}(G)$ is connected. $\square$

As with Theorem 2.4, in statement (2) of Theorem 6.1 we cannot remove the condition "$\mathrm{Fix}(G)$ is a singleton". For instance, consider $G(x, y) =$

$(y, g(x))$, where g is the interval map defined in Example 1 of [17], namely $g(x) = 1$ if $x \in [0, \frac{1}{4}]$, $g(x) = -2x + \frac{3}{2}$ if $x \in [\frac{1}{4}, \frac{3}{4}]$, and $g(x) = 0$ if $x \in [\frac{3}{4}, 1]$. In this case, $\mathrm{Fix}(G) = \{(\frac{1}{2}, \frac{1}{2})\}$, but the family $\{G^n\}_{n=1}^{\infty}$ is not equicontinuous at the point $(\frac{1}{2}, \frac{1}{2})$.

We continue with the possible extension of Theorem 2.5 to Cournot maps. Grinč proved in [14] that for triangular maps, Jachymski's result remains true for the equivalence (1) $\Leftrightarrow$ (3). However, this is not true for Cournot maps, since in this situation [(1) $\Leftrightarrow$ (2)] $\Rightarrow$ (3), but in general (3) $\Rightarrow$ (1) is false.

Theorem 6.2. *Let $G \in C_A(I^2)$. Then the following properties are equivalent:*

1. $\mathrm{Fix}(G) = \mathrm{P}(G)$.

2. $\{G^n\}_{n=1}^{\infty}$ *is pointwise convergent on* I^2.

Proof. (1) $\Rightarrow$ (2) Suppose $\mathrm{P}(G) = \mathrm{Fix}(G)$. From Proposition 3.5, $2 \notin \mathrm{Per}(G)$ and G has a unique fixed point. Now, by Proposition 3.1

$$\mathrm{Fix}(G) = \mathrm{Fix}(g_2 \circ g_1) \times \mathrm{Fix}(g_1 \circ g_2).$$

Since $\mathrm{P}(G) = \mathrm{P}(g_2 \circ g_1) \times \mathrm{P}(g_1 \circ g_2)$, Theorem 2.5 states that the sequences $\{(g_2 \circ g_1)^n\}_{n=1}^{\infty}$ and $\{(g_1 \circ g_2)^n\}_{n=1}^{\infty}$ are pointwise convergent on I. By (1) and (2) this implies that $\{G^{2n}\}_{n=1}^{\infty}$ and $\{G^{2n+1}\}_{n=1}^{\infty}$ are also pointwise convergent on I^2. We must prove that $\{G^n\}_{n=1}^{\infty}$ is pointwise convergent; that is, $\lim_{n\to\infty} G^{2n}(x, y) = \lim_{n\to\infty} G^{2n+1}(x, y)$ for all $(x, y) \in I^2$. Consider

$$\lim_{n\to\infty} G^{2n}(x, y) = (\lim_{n\to\infty} (g_2 \circ g_1)^n(x), \lim_{n\to\infty} (g_1 \circ g_2)^n(y)) = (u, v).$$

According to [25], Chapter 4, Th.4.2, u and v are fixed points of $g_2 \circ g_1$ and $g_1 \circ g_2$, respectively. Then, (u, v) is the unique fixed point of G, and by continuity of G,

$$\lim_{n\to\infty} G^{2n+1}(x, y) = G(\lim_{n\to\infty} G^{2n}(x, y)) = G(u, v) = (u, v).$$

(2) $\Rightarrow$ (1) Assume that $\{G^n\}_{n=1}^{\infty}$ is pointwise convergent on I^2. Then, from (1) we have that $\{(g_2 \circ g_1)^n\}_{n=1}^{\infty}$ and $\{(g_1 \circ g_2)^n\}_{n=1}^{\infty}$ are pointwise convergent on I; so $\mathrm{P}(g_2 \circ g_1) = \mathrm{Fix}(g_2 \circ g_1)$ and $\mathrm{P}(g_1 \circ g_2) = \mathrm{Fix}(g_1 \circ g_2)$ since Theorem 2.5 holds. From here we obtain

$$\mathrm{P}(G) = \mathrm{P}(g_2 \circ g_1) \times \mathrm{P}(g_1 \circ g_2) = \mathrm{Fix}(g_2 \circ g_1) \times \mathrm{Fix}(g_1 \circ g_2) = \mathrm{Fix}(G^2).$$

To finish we have to show that $\mathrm{Fix}(G^2) = \mathrm{Fix}(G)$; that is, G has no periodic points of order two. Let $(x, y) \in \mathrm{Fix}(G^2)$. Then $\lim_{n\to\infty} G^{2n+1}(x, y) = \lim_{n\to\infty} G^{2n}(x, y) = (x, y)$. On the other hand, from the continuity of G

$$\lim_{n\to\infty} G^{2n+1}(x, y) = G(\lim_{n\to\infty} G^{2n}(x, y)) = G(x, y).$$

Hence $(x,y) \in \mathrm{Fix}(G)$. □

Theorem 6.3. *Let $G \in C_A(I^2)$. Suppose $\mathrm{Fix}(G) = \mathrm{P}(G)$. Then G has a common fixed point with every $F \in C_A(I^2)$ which commutes with G on $\mathrm{Fix}(F)$.*

PROOF. If $\mathrm{Fix}(G) = \mathrm{P}(G)$, then G has no periodic points of order two, and according to Theorem 3.5 this implies

$$\mathrm{Card}(\mathrm{Fix}(g_2 \circ g_1)) = \mathrm{Card}(\mathrm{Fix}(g_1 \circ g_2)) = 1. \tag{12}$$

Moreover, by (3) and Proposition 3.1,

$$\begin{aligned} \mathrm{P}(g_2 \circ g_1) &= \mathrm{Fix}(g_2 \circ g_1), \\ \mathrm{P}(g_1 \circ g_2) &= \mathrm{Fix}(g_1 \circ g_2). \end{aligned} \tag{13}$$

So, $\mathrm{Fix}(G) = \mathrm{Fix}(g_2 \circ g_1) \times \mathrm{Fix}(g_1 \circ g_2) = \{(x_0, y_0)\}$. Let $F \in C_A(I^2)$, and assume that $G \circ F = F \circ G$ on $\mathrm{Fix}(F)$. Now, we are going to prove that $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$.

First, we claim that $g_i \circ g_j$ and $f_i \circ f_j$ commute on $\mathrm{Fix}(f_i \circ f_j)$, for $i, j \in \{1, 2\}$, $i \neq j$. Let $z \in \mathrm{Fix}(f_2 \circ f_1)$. (The case $\mathrm{Fix}(f_1 \circ f_2)$ is analogous.) Then, $(z, f_1(z)) \in \mathrm{Fix}(F)$. By hypothesis, $F \circ G = G \circ F$ on $\mathrm{Fix}(F)$; so

$$(f_2 \circ g_1)(z) = (g_2 \circ f_1)(z) = (f_1 \circ g_2)(f_1(z)) = (g_1 \circ f_2)(f_1(z)) = g_1(z).$$

At the same time, since $(F \circ G)(z, f_1(z)) = G(z, f_1(z))$, $G(z, f_1(z)) \in \mathrm{Fix}(F)$ holds, and it is immediate to obtain

$$G^n(z, f_1(z)) \in \mathrm{Fix}(F) \text{ for all } n \geq 0.$$

In particular, $G^2(z, f_1(z)) = ((g_2 \circ g_1)(z), (g_1 \circ g_2)(f_1(z))) \in \mathrm{Fix}(F)$. Then, $(g_2 \circ g_1)(z) \in \mathrm{Fix}(f_2 \circ f_1)$, and since $z \in \mathrm{Fix}(f_2 \circ f_1)$, it follows that

$$(f_2 \circ f_1 \circ g_2 \circ g_1)(z) = (g_2 \circ g_1)(z) = (g_2 \circ g_1 \circ f_2 \circ f_1)(z).$$

Therefore, $f_2 \circ f_1$ and $g_2 \circ g_1$ commute on $\mathrm{Fix}(f_2 \circ f_1)$. This proves the claim.

According to (13), and the above claim, Theorem 2.5 states that $\mathrm{Fix}(f_i \circ f_j) \cap \mathrm{Fix}(g_i \circ g_j) \neq \emptyset$, for $i, j \in \{1, 2\}$, $i \neq j$. Moreover, from (12) and (13)

$$\begin{aligned} \mathrm{Fix}(g_2 \circ g_1) \cap \mathrm{Fix}(f_2 \circ f_1) &= \{x_0\}, \\ \mathrm{Fix}(g_1 \circ g_2) \cap \mathrm{Fix}(f_1 \circ f_2) &= \{y_0\}. \end{aligned}$$

Since $\{f_1(x_0), g_1(x_0)\} \subseteq \mathrm{Fix}(g_1 \circ g_2) \cap \mathrm{Fix}(f_1 \circ f_2)$, we deduce $y_0 = f_1(x_0) = g_1(x_0)$, and similarly $x_0 = f_2(y_0) = g_2(y_0)$. Thus, it is easy to show that $(x_0, f_1(x_0)) \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F)$. □

In order to establish that in general (3) $\Rightarrow$ (1) of Theorem 2.5 is not true for the Cournot case, we need the following result. After this, we will show a counterexample to (3) $\Rightarrow$ (1).

Lemma 6.4. *Let $F, G \in C_A(I^2)$. Suppose $F \circ G = G \circ F$ on $\mathrm{Fix}(F)$.*

1. $G^k(x,y) \in \mathrm{Fix}(F)$ for all $(x,y) \in \mathrm{Fix}(F)$ and for all $k \geq 0$.

2. $F^n \circ G^n = G^n \circ F^n$ on $\mathrm{Fix}(F)$.

Proof. (1) Let $Z = (x,y) \in \mathrm{Fix}(F)$. Then $(G \circ F)(Z) = G(Z)$. Since F and G commute on $\mathrm{Fix}(F)$, $(G \circ F)(Z) = G(Z) = (F \circ G)(Z) = F(G(Z))$; so $G(Z) \in \mathrm{Fix}(F)$. Reasoning in an inductive way, $G^k(x,y) \in \mathrm{Fix}(F)$ for all $k \geq 0$.

(2) It is an immediate consequence of (1). □

Example 6.5. Let $G(x,y) = (y^2, 1-x^2)$. Then $(g_1 \circ g_2)(x) = 1 - x^4$, $(g_2 \circ g_1)(x) = (1-x^2)^2$, and it is easy to check that $\mathrm{Per}(g_2 \circ g_1) = \mathrm{Per}(g_1 \circ g_2) = \{1,2\}$ and $\mathrm{Card}(\mathrm{Fix}(g_2 \circ g_1)) = 1$. According to Theorem 3.5, we obtain $\mathrm{Per}(G) = \{1,4\}$, and by Proposition 3.1 $\mathrm{Fix}(G^2) = \mathrm{Fix}(G)$. Moreover,

$$\begin{aligned}\mathrm{Fix}(g_2 \circ g_1) =& \{x_G\} = \{0.52488859\ldots\},\\ \mathrm{Fix}(g_1 \circ g_2) =& \{y_G\} = \{0.72449195\ldots\},\end{aligned}$$

and x_G and y_G are repelling for $g_2 \circ g_1$, $g_1 \circ g_2$, respectively. (Consult Chapter 1 of [25] for the notions of repelling and attracting cycles.) Both of $g_2 \circ g_1$ and $g_1 \circ g_2$ have a unique periodic orbit of order two, $\{0,1\}$, which is an attracting cycle. By [25], Chapter 4, Th. 4.2, given $x \neq x_G$ it follows that $\{(g_2 \circ g_1)^n(x)\}_{n=0}^{\infty} \to \{0,1\}$, and given $y \neq y_G$, $\{(g_1 \circ g_2)^n(x)\}_{n=0}^{\infty} \to \{0,1\}$. Following with the description of the dynamics of G, we can establish that $\mathrm{Fix}(G) = \{(x_G, y_G)\}$, and G only possesses two periodic orbits of order four; namely,

$$\begin{aligned}O_1 =& \{(0,0),(0,1),(1,1),(1,0)\} &= \mathrm{Orb}_G(0,0),\\ O_2 =& \{(0,y_G),(x_G,1),(1,y_G),(x_G,0)\} &= \mathrm{Orb}_G(0,y_G).\end{aligned}$$

Given $(x,y) \in I_{x_G} \cup \widetilde{I}_{y_G} := (\{x_G\} \times I) \cup (I \times \{y_G\})$, $(x,y) \neq (x_G, y_G)$, it is simple to prove that

$$\{G^n(x,y)\}_{n=0}^{\infty} \to \mathrm{Orb}_G(0, y_G). \tag{14}$$

Given $(x,y) \in I^2 \setminus (I_{x_G} \cup \widetilde{I}_{y_G})$, now

$$\{G^n(x,y)\}_{n=0}^{\infty} \to \mathrm{Orb}_G(0,0). \tag{15}$$

Next, assume that $F \in C_A(I^2)$ which implies $F \circ G = G \circ F$ on $\mathrm{Fix}(F)$. We wish to prove that F and G share a fixed point, in fact $(x_G, y_G) \in \mathrm{Fix}(F)$. However, $\mathrm{P}(G) \supset \mathrm{Fix}(G)$.

If $(x_G, y_G) \in \mathrm{Fix}(F)$, the proof is complete. If $(x_G, y_G) \notin \mathrm{Fix}(F)$, we will obtain a contradiction. Let $(a, b) \in \mathrm{Fix}(F)$. (We know that $\mathrm{Fix}(F) \neq \emptyset$.) Suppose $(a, b) \neq (x_G, y_G)$. Since $F \circ G = G \circ F$ on $\mathrm{Fix}(F)$, Lemma 6.4 implies $G^n(a, b) \in \mathrm{Fix}(F)$, for all $n \geq 0$. Since $\mathrm{Fix}(F)$ is obviously a closed set, according to (14) and (15), either $\mathrm{Orb}_G(0, y_G) \subset \mathrm{Fix}(F)$, or $\mathrm{Orb}_G(0, 0) \subset \mathrm{Fix}(F)$. In both cases, we obtain a contradiction to the result of Proposition 3.2. Therefore, the unique fixed point of F is (x_G, y_G) and $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) \neq \emptyset$.

Now we continue with the extension of Theorem 2.6. The first observation is concerned with uniform convergence.

Proposition 6.6. *Let $G \in C_A(I^2)$. Suppose that* $\mathrm{Fix}(G)$ *is not a singleton. Then the sequence $\{G^n : n \in \mathbb{N}\}$ is not uniformly convergent on I^2.*

PROOF. Suppose that $\{G^n : n \in \mathbb{N}\}$ would be uniformly convergent on I^2. In particular, $\{G^n : n \in \mathbb{N}\}$ would be pointwise convergent on I^2. From Theorem 6.2, it follows that $\mathrm{Fix}(G) = \mathrm{P}(G) = \mathrm{Fix}(G^2)$. Finally, according to Proposition 3.1 we obtain $\mathrm{Card}(\mathrm{Fix}(G)) = 1$, a contradiction. □

Hence, in the extension of Theorem 2.6 to the Cournot case, we must omit condition (2).

Theorem 6.7. *Let $G \in C_A(I^2)$. Suppose that* $\mathrm{Fix}(G)$ *is not a singleton. If $\{G^n : n \in \mathbb{N}\}$ is equicontinuous on I^2, then* $\mathrm{Fix}(G^2) = \mathrm{P}(G) \supsetneq \mathrm{Fix}(G)$.

PROOF. Assume $\{G^n\}_{n=1}^{\infty}$ is equicontinuous on I^2. According to Theorem 4.1, $\mathrm{Fix}(G)$ is connected, and $\mathrm{Fix}(G^2) = \mathrm{P}(G)$ since $\mathrm{Fix}(G)$ is nondegenerate. □

The converse result is false. To prove this consider the following example.

Example 6.8. Let $G(x, y) = (y, g(x))$, where $g(x) = 2x^2$ if $x \in [0, \frac{1}{2}]$, and $g(x) = x$ if $x \in [\frac{1}{2}, 1]$. It is easy to see that $\mathrm{Fix}(G)$ is not a singleton, and $\mathrm{Fix}(G^2) = \mathrm{P}(G)$. Moreover, $\mathrm{Fix}(G) \subset \mathrm{P}(G)$. However, $\{G^n\}_{n=1}^{\infty}$ is not equicontinuous at $(\frac{1}{2}, \frac{1}{2})$ since $\{g^n\}_{n=1}^{\infty}$ is not equicontinuous at $\frac{1}{2}$. (If $\{g^n\}_{n=1}^{\infty}$ were equicontinuous at $\frac{1}{2}$, the pointwise convergence to the map $\widetilde{g} : I \to I$, given by $\widetilde{g}(x) = 0$ if $x \in [0, \frac{1}{2})$, $\widetilde{g}(x) = x$ if $x \in [\frac{1}{2}, 1]$, would imply that $\widetilde{g}$ is continuous at $\frac{1}{2}$, impossible.) Observe that $\mathrm{Fix}(G)$ is not connected.

In order to obtain a converse result for the above result, we must suppose that $\mathrm{Fix}(G)$ is a connected set.

Proposition 6.9. *Let $G \in C_A(I^2)$. Suppose that* $\mathrm{Fix}(G)$ *is not a singleton. The following properties are equivalent.*

1. $\{G^n\}_{n=1}^{\infty}$ *is equicontinuous on* I^2.

2. $\mathrm{Fix}(G^2) = \mathrm{P}(G)$ *and* $\mathrm{Fix}(G)$ *is connected.*

PROOF. Theorems 4.1 and 6.7 show (1) $\Rightarrow$ (2). It remains to prove (2) $\Rightarrow$ (1). If $\mathrm{Fix}(G)$ is connected, from Proposition 3.2 we obtain that $\mathrm{Fix}(g_i \circ g_j)$ is an interval, for $i, j \in \{1,2\}$, $i \neq j$. According to Theorem 2.4, the families $\{(g_i \circ g_j)^n\}_{n=1}^{\infty}$ are equicontinuous on $\mathrm{Fix}(g_i \circ g_j)$, for $i, j \in \{1,2\}$, $i \neq j$. Since $\mathrm{Fix}(G^2) = \mathrm{P}(G)$, it follows that $\mathrm{Fix}(g_i \circ g_j) = \mathrm{P}(g_i \circ g_j)$ (see (3) and Proposition 3.1). By Theorem 2.5, this implies that the sequences $\{(g_i \circ g_j)^n\}_{n=1}^{\infty}$ are pointwise convergent. Moreover, given $x \in I$ there is $p(x) \in \mathrm{Fix}(g_i \circ g_j)$ such that $\lim_{n\to\infty}(g_i \circ g_j)^n(x) = p(x)$ ([25],Chapter 4, Th.4.2).

First, we claim that $\{(g_2 \circ g_1)^n\}_{n=1}^{\infty}$ is equicontinuous at $x \in I$. (The proof is completely analogous for $\{(g_1 \circ g_2)^n\}_{n=1}^{\infty}$.) Assume $\mathrm{Fix}(g_2 \circ g_1) = [a,b]$, $a < b$. Let $x \in I$, and $p = p(x)$ as above. Suppose $p \in (a,b)$. Then there exists $m_0 = m_0(x) \in \mathbb{N}$ such that $(g_2 \circ g_1)^{m_0}(x) = p$. (Notice that $(a,b) \subset \mathrm{Fix}(g_2 \circ g_1)$.) On other hand, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|(g_2 \circ g_1)^i(x) - (g_2 \circ g_1)^i(z)\right| < \min\{\varepsilon, |p-a|, |p-b|\},$$

for $i = 1, \ldots, m_0$, whenever $|x - z| < \delta$. In this case,

$$\left|(g_2 \circ g_1)^j(x) - (g_2 \circ g_1)^j(z)\right| = |p(x) - p(z)| < \varepsilon,$$

for all $j \geq m_0$, and this completes the proof of the equicontinuity for $x \in I$ if $p(x) \in (a,b)$.

Now, suppose that $p = p(x) \in \{a,b\}$. Without loss of generality we can assume that $p = a$. If $a = 0$, we proceed as in the case $p \in (a,b)$. Hence, assume $a > 0$. Since $\{(g_2 \circ g_1)^n\}_{n=1}^{\infty}$ is equicontinuous at $a \in \mathrm{Fix}(g_2 \circ g_1)$ (see Theorem 2.4), for $\varepsilon > 0$ there is $\delta > 0$, $\delta < 2\varepsilon$, such that $|w - a| < \delta$ implies $|(g_2 \circ g_1)^n(w) - a| < \frac{\varepsilon}{2}$ for all $n \geq 1$. For this $\delta > 0$, since $a = p(x)$, there exists $m_1 \in \mathbb{N}$ with $|(g_2 \circ g_1)^n(x) - a| < \frac{\delta}{2} < \delta$ for $n \geq m_1$. Now, for $\frac{\delta}{2}$ there is δ_1 such that $|x - z| < \delta_1$ implies $\left|(g_2 \circ g_1)^i(x) - (g_2 \circ g_1)^i(z)\right| < \frac{\delta}{2} < \varepsilon$ for $i = 1, \ldots, m_1$. Notice that for $i = m_1$,

$$\begin{aligned}|(g_2 \circ g_1)^{m_1}(z) - a| &< |(g_2 \circ g_1)^{m_1}(z) - (g_2 \circ g_1)^{m_1}(x)| \\ &\quad + |(g_2 \circ g_1)^{m_1}(x) - a| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

This leads to

$$|(g_2 \circ g_1)^n((g_2 \circ g_1)^{m_1}(z)) - a| < \frac{\varepsilon}{2}, \text{ for all } n \geq 1.$$

On the other hand, $|(g_2 \circ g_1)^{m_1}(x) - a| < \frac{\delta}{2} < \delta$ implies

$$|(g_2 \circ g_1)^n((g_2 \circ g_1)^{m_1}(x)) - a| < \frac{\varepsilon}{2}, \text{ for all } n \geq 1.$$

Finally, if $|x - z| < \delta_1$, for $k \geq m_1 + 1$, we find

$$\begin{aligned}\left|(g_2 \circ g_1)^k(x) - (g_2 \circ g_1)^k(z)\right| &\leq \left|(g_2 \circ g_1)^{k-m_1}((g_2 \circ g_1)^{m_1}(x)) - a\right| \\ &+ \left|(g_2 \circ g_1)^{k-m_1}((g_2 \circ g_1)^{m_1}(z)) - a\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore, we have proved for any $x \in I$ that $\{(g_2 \circ g_1)^n\}_{n=1}^{\infty}$ is equicontinuous at x; so is equicontinuous on I. This completes the claim.

Observe that the compactness of I and the continuity of $(g_2 \circ g_1)$ imply that the family $\{(g_2 \circ g_1)^n\}_{n=1}^{\infty}$ is uniformly equicontinuous on I. Since $\{(g_i \circ g_j)^n\}_{n=1}^{\infty}$ are equicontinuous for $i, j \in \{1, 2\}$, $i \neq j$, from (1) we obtain that $\{G^{2n}\}_{n=1}^{\infty}$ is also equicontinuous on I^2. Moreover, as an immediate consequence of the continuity of G, the family $\{G^{2n+1}\}_{n=1}^{\infty}$ is also equicontinuous. Since the finite union of equicontinuous families is equicontinuous, we conclude that $\{G^m\}_{m=1}^{\infty}$ is equicontinuous on I^2. □

Suppose that $\mathrm{Fix}(G)$ is connected, with $\mathrm{Card}(\mathrm{Fix}(G)) > 1$ and $\mathrm{P}(G) = \mathrm{Fix}(G^2)$, for $G \in C_A(I^2)$. According to the above result, the sequence $\{G^n\}_{n=1}^{\infty}$ is equicontinuous, and the iterates $\{(g_i \circ g_j)^n\}_{n=1}^{\infty}$ are pointwise convergent ([25], Chapter 4, Theorem 2). However, $\{G^n\}_{n=1}^{\infty}$ is not pointwise convergent, since in the contrary case, for any $(p, g_1(q)) \in I^2$, where $p, q \in \mathrm{Fix}(g_2 \circ g_1)$, $p \neq q$, we would obtain (observe that $G^2(p, g_1(q)) = (p, g_1(q))$)

$$(p, g_1(q)) = \lim_{n\to\infty} G^{2n}(p, g_1(q)) = \lim_{n\to\infty} G^{2n+1}(p, g_1(q)) = G(p, g_1(q));$$

so $G(p, g_1(p)) = (p, g_1(q))$, but this is not possible since $p \neq q$. If $\mathrm{Card}(\mathrm{Fix}(G))$ $= 1$ and $\mathrm{P}(G) = \mathrm{Fix}(G^2) = \mathrm{Fix}(G)$, then $\{G^n\}_{n=1}^{\infty}$ is also uniformly convergent.

Proposition 6.10. *Let $G \in C_A(I^2)$. Suppose* $\mathrm{Card}(\mathrm{Fix}(G)) = 1$. *The following conditions are equivalent.*

1. $\mathrm{Fix}(G) = \mathrm{P}(G)$.

2. $\{G^n\}_{n=1}^{\infty}$ *is pointwise convergent on I^2.*

3. $\{G^n\}_{n=1}^{\infty}$ *is uniformly convergent on* I^2*.*

PROOF. (1) $\Leftrightarrow$ (2) See Theorem 6.2.
(3) $\Rightarrow$ (2) It is immediate.
(1) $\Rightarrow$ (3) Suppose that $\mathrm{Fix}(G) = \mathrm{P}(G)$; so $\{G^n\}_{n=1}^{\infty}$ is pointwise convergent on I^2 to the constant map $\widetilde{G}(x, y) = (a, b)$, where a and b are the unique fixed points of $g_2 \circ g_1$ and $g_1 \circ g_2$, respectively. (We use Proposition 3.1, (3), and [25], Chapter 4, Th.4.2.) Given $\varepsilon > 0$, since a is an attracting fixed point of $g_2 \circ g_1$, there exists an open neighborhood $U \subseteq (a - \varepsilon, a + \varepsilon)$ such that $(g_2 \circ g_1)^n(x) \in U$ for all $x \in U$, and for all $n \in \mathbb{N}$. Now for $m \in \mathbb{N}$, we put

$$U_m = \{x \in I : (g_2 \circ g_1)^m(x) \in U\}.$$

Obviously, by continuity of $(g_2 \circ g_1)^m$, each U_m is an open set of I, and $\bigcup_m U_m \supseteq I$. Moreover, $U_m \subseteq U_{m+1}$ for all $m \in \mathbb{N}$. Since I is compact, there is a finite recovering of I. This means that there exists $m_k \in \mathbb{N}$ such that $I \subseteq U_{m_k}$. Hence $|(g_2 \circ g_1)^{m_k}(x) - a| < \varepsilon$ for all $x \in I$. This yields

$$|(g_2 \circ g_1)^n(x) - a| < \varepsilon \text{ for all } n \geq m_k \text{ and for all } x \in I.$$

Therefore, $\{(g_2 \circ g_1)^m\}_{m=1}^{\infty}$ is uniformly convergent to the constant map $g(x) = a$. Similarly, it can be proved that $\{(g_1 \circ g_2)^m\}_{m=1}^{\infty}$ is uniformly convergent to the constant map $f(x) = b$. By (1) we deduce that $\{G^{2n}\}_{n=1}^{\infty}$ is uniformly convergent to $\widetilde{G}(x, y) = (a, b)$, and by the continuity of G, also $\{G^{2n+1}\}_{n=1}^{\infty}$ is uniformly convergent to the same map since $\{(a, b)\} = \mathrm{Fix}(G)$. □

Notice that in the last result, all of three equivalent conditions imply that $\{G^n\}_{n=1}^{\infty}$ is an equicontinuous family on I^2. In general the converse result is not true. For instance, consider the Cournot map $G(x, y) = (y, 1 - x)$. It is clear that $\mathrm{Card}(\mathrm{Fix}(G)) = 1$, and that $\{G^n\}_{n=1}^{\infty}$ is an equicontinuous family on I^2 since $\{G^n\}_{n=1}^{\infty} = \{G_1, G_2, G_3, G_4\}$, where $G_1 = G$ and

$$G_2(x, y) = (1 - x, 1 - y),\ G_3(x, y) = (1 - y, x),\ G_4(x, y) = (x, y).$$

However, $\mathrm{Fix}(G) \neq \mathrm{P}(G)$ since the point $(0, 0)$ has period four, and the family $\{G^n\}_{n=1}^{\infty}$ is neither uniformly convergent nor pointwise convergent.

Proposition 6.11. *Let* $G \in C_A(I^2)$*. Suppose* $\mathrm{Card}(\mathrm{Fix}(G)) = 1$*. If* $\mathrm{Fix}(G) = \mathrm{P}(G)$*, then* G *has a common fixed point with every* $F \in C_A(I^2)$ *that commutes with* G *either on* $\mathrm{Fix}(F)$*, or on* $\mathrm{Fix}(G)$*.*

PROOF. It is a consequence of Theorems 6.3 and 6.1. □

The converse result is false. To see this, consider the Cournot map of Example 6.5, $G(x,y) = (y^2, 1-x^2)$. Then $\mathrm{P}(G) \supsetneqq \mathrm{Fix}(G^2) = \mathrm{Fix}(G) = \{(x_G, y_G)\} = \{X_G\}$, with $\mathrm{Per}(G) = \{1,4\}$. It was proved that G has a common fixed point with every $F \in C_A(I^2)$ that commutes with G on $\mathrm{Fix}(F)$. The same situation holds if F commutes with G on $\mathrm{Fix}(G) = \{X_G\}$. If $(F \circ G)(X_G) = (G \circ F)(X_G)$, then $F(X_G) = G(F(X_G))$, and since X_G is the unique fixed point of G, this implies that $F(X_G) = X_G$; so $X_G \in \mathrm{Fix}(G) \cap \mathrm{Fix}(F)$. However, $\mathrm{Fix}(G) \neq \mathrm{P}(G)$.

We now return to the extension of Theorem 2.6, concerning the case in which $\mathrm{Fix}(G)$ is not a singleton.

Theorem 6.12. *Let $G \in C_A(I^2)$. Suppose that $\mathrm{Fix}(G)$ is not a singleton, and $\mathrm{Fix}(G^2) = \mathrm{P}(G)$. Then $\{G^n : n \in \mathbb{N}\}$ is equicontinuous on I^2 if and only if $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$ for every $F \in C_A(I^2)$ that commutes with G on $\mathrm{Fix}(G)$.*

Proof. Suppose that $\{G^n : n \in \mathbb{N}\}$ is equicontinuous on I^2. In particular, the family is equicontinuous on $\mathrm{Fix}(G)$, and by Theorem 6.1 we obtain the second part of the statement. Suppose that $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$ for every $F \in C_A(I^2)$ that commutes with G on $\mathrm{Fix}(G)$. From Theorem 6.1 and Proposition 6.9 we obtain that $\{G^n : n \in \mathbb{N}\}$ is equicontinuous on I^2. □

7 Extension of Theorem 2.7. Connection between Jungck's Theorem and a Jachymski's Result

We can translate only partially the equivalent conditions of Theorem 2.7 to the Cournot case.

Theorem 7.1. *Let $G \in C_A(I^2)$.*

(a) The following conditions are equivalent:

1. $\mathrm{P}(G) = \mathrm{Fix}(G)$.
2. *$C \cap \mathrm{Fix}(G) \neq \emptyset$ for any non-empty closed set $C \subseteq I^2$ such that $G(C) \subseteq C$.*
3. *G has a common fixed point with every continuous map $F : I^2 \to I^2$ that commutes with G on $\mathrm{Fix}(F)$.*

(b) The equivalent conditions of (a) imply that G has a common fixed point with every $F \in C_A(I^2)$ that commutes with G on $\mathrm{Fix}(F)$.

(c) The equivalent conditions of (a) imply that G has a common fixed point with every $F \in C_A(I^2)$ which is nontrivially compatible with G.

PROOF. (a) (2) $\Leftrightarrow$ (3) The equivalence holds according to Proposition 1 of [17].

(1) $\Rightarrow$ (2) Suppose $\mathrm{P}(G) = \mathrm{Fix}(G)$. Let $C \neq \emptyset$ a closed set of I^2 such that $G(C) \subseteq C$. We wish to prove that $C \cap \mathrm{Fix}(G) \neq \emptyset$. Since $G^2(C) \subseteq G(C) \subseteq C$, and G^2 is a triangular map, from Corollary 3.1 of [14] we obtain $C \cap \mathrm{Fix}(G^2) \neq \emptyset$. Finally, from $\mathrm{Fix}(G) = \mathrm{P}(G)$ it follows that $\mathrm{Fix}(G^2) = \mathrm{Fix}(G)$, so $C \cap \mathrm{Fix}(G) \neq \emptyset$.

(2) $\Rightarrow$ (1) Suppose that (2) holds. First, we will prove that G has only fixed points. Let P a periodic orbit of order $k > 1$ of G. Then $G(P) \subseteq P$, P is closed and non-empty. By hypothesis, $P \cap \mathrm{Fix}(G) \neq \emptyset$, but this contradicts that P is a periodic orbit of order $k > 1$. Hence, G has only fixed points. Moreover, $\mathrm{Card}(\mathrm{Fix}(F)) = 1$, since if G has at least two different fixed points according to Theorem 3.5 we find periodic points of order two. Therefore, $\mathrm{Fix}(G) = \mathrm{P}(G)$.

To prove (b) and (c) see Theorems 6.3 and 5.1, respectively. □

Notice that the converse results of (b) and (c) are not true (consult Example 6.5 and Example 5.2).

To finish the extension of Theorem 2.7, we must determine if there is some relation in the Cournot case between Jachymski's Theorem 2.5 (the relation (1) $\Leftrightarrow$ (3)) and Jungck's Theorem. The answer is negative, as the following examples show.

Example 7.2. Let $G(x,y) = (1-y, 1-x)$. According to Example 5.2, this map satisfies Jungck's Theorem; that is, $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) \neq \emptyset$ for any $F \in C_A(I^2)$ nontrivially compatible with G. However, we are going to prove that G does not satisfy Jachymski's result; namely, there exists $F \in C_A(I^2)$ such that $F \circ G = G \circ F$ on $\mathrm{Fix}(F)$ but $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset$.
For this purpose, consider $F(x,y) = (f_2(y), f_1(x))$, where

$$f_1(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [0, \frac{1}{2}] \\ -4x^2 + 6x - \frac{7}{4} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{1}{2} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{4}] \\ x + \frac{1}{4} & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ \frac{3}{4} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

It is straightforward to see that

$$(f_2 \circ f_1)(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\ -4x^2 + 6x - \frac{3}{2} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{3}{4} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

and $\mathrm{Fix}(f_2 \circ f_1) = \{\frac{1}{2}, \frac{3}{4}\}$. By Proposition 3.2, $\mathrm{Fix}(F) = \{(\frac{1}{2}, \frac{1}{4}), (\frac{3}{4}, \frac{1}{2})\}$. Since $\mathrm{Fix}(G) = \{(x, 1-x) : x \in I\}$, it is clear that $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset$. However, F and G commute on $\mathrm{Fix}(F)$:

$$F(G(\frac{1}{2}, \frac{1}{4})) = F(\frac{3}{4}, \frac{1}{2}) = (\frac{3}{4}, \frac{1}{2}) = G(\frac{1}{2}, \frac{1}{4}) = G(F(\frac{1}{2}, \frac{1}{4})),$$
$$F(G(\frac{3}{4}, \frac{1}{2})) = F(\frac{1}{2}, \frac{1}{4}) = (\frac{1}{2}, \frac{1}{4}) = G(\frac{3}{4}, \frac{1}{2}) = G(F(\frac{3}{4}, \frac{1}{2})).$$

Example 7.3. Consider now $G(x, y) = (y^2, 1 - x^2)$. By Example 6.5, we know that $\mathrm{Fix}(G) \cap \mathrm{Fix}(F) \neq \emptyset$ for any $F \in C_A(I^2)$ which commutes with G on Fix(F). (G satisfies Jachymski's Theorem 2.5.) However, we will show that it does not verify Jungck's Theorem; that is, we will prove that there is $F \in C_A(I^2)$ nontrivially compatible with G such that $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset$. Define $F(x, y) = (f_2(y), f_1(x)) = (y, \frac{1}{x+1} - \frac{x}{2})$. A direct computation gives $\mathrm{Coin}(F, G) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The maps F and G are nontrivially compatible,

$$G(F(0, 0)) = G(0, 1) = (1, 1) = F(0, 1) = F(G(0, 0)),$$
$$G(F(0, 1)) = G(1, 1) = (1, 0) = F(1, 1) = F(G(0, 1)),$$
$$G(F(1, 0)) = G(0, 0) = (0, 1) = F(0, 0) = F(G(1, 0)),$$
$$G(F(1, 1)) = G(1, 0) = (0, 0) = F(1, 0) = F(G(1, 1)).$$

However, $\mathrm{Fix}(F) \cap \mathrm{Fix}(G) = \emptyset$.

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