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THE SHARP RIESZ-TYPE DEFINITION FOR THE HENSTOCK-KURZWEIL INTEGRAL

Abstract

In this paper, we prove that if f is Henstock-Kurzweil integrable on a compact subinterval $[a, b]$ of the real line, then the following conditions are satisfied: (i) there exists an increasing sequence $\{X_n\}$ of closed sets whose union is $[a, b]$; (ii) $\{f\chi_{X_n}\}$ is a sequence of Lebesgue integrable functions on $[a, b]$; (iii) the sequence $\{f\chi_{X_n}\}$ is Henstock-Kurzweil equi-integrable on $[a, b]$. Subsequently, we deduce that the gauge function in the definition of the Henstock-Kurzweil integral can be chosen to be measurable, and an indefinite Henstock-Kurzweil integral generates a sequence of uniformly absolutely continuous finite variational measures.

1 Introduction

E. J. McShane in [9] developed the Lebesgue integration on an interval $I \subset \mathbb{R}^n$ using the monotone convergence of step functions. In dimension one, it is well-known (see [5] or [6]) that if f is Denjoy-Perron integrable on a compact subinterval $[a, b]$ of the real line $\mathbb{R}$, then it can be defined as a controlled convergent sequence of step functions. Since the Controlled Convergence Theorem is equivalent to the equi-integrability theorem (see [4, Theorem 5.4]), it is natural to ask the following question : given that f is Henstock-Kurzweil integrable on $[a, b]$, can $[a, b]$ be decomposed into a countable union of closed sets $\{X_n\}$ so that for each n, $f\chi_{X_n}$ is Lebesgue integrable on $[a, b]$, and $\{f\chi_{X_n}\}$ is Henstock-Kurzweil equi-integrable on $[a, b]$? In this paper, we shall give an affirmative answer to the above problem (see Theorem 3.7). The importance of this equi-integrability theorem lies in the construction of a topology $\mathcal{J}$ on

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the space $\mathcal{HK}([a,b])$, namely the space of all Henstock-Kurzweil integrable functions on $[a,b]$, so that the resulting space $(\mathcal{HK}([a,b]),\mathcal{J})$ is complete. See [3] for more details. Moreover, we deduce that the gauge function in the definition of the Henstock-Kurzweil integral can be chosen to be measurable, and an indefinite Henstock-Kurzweil integral generates a sequence of uniformly absolutely continuous finite variational measures (Corollary 3.9).

2 Preliminaries

Unless stated otherwise, the following conventions and notations will be used. The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}$ with its usual norm. For $x \in \mathbb{R}$ and $r > 0$, the open ball $B(x,r)$ is the open interval centered at x with sides equal to $2r$. For a set $Z \subset \mathbb{R}$, we denote by χ_{Z}, $\text{int}(Z)$, $\overline{Z}$ and $\text{diam}(Z)$ the characteristic function, interior, closure and diameter of Z, respectively. The expressions "absolutely continuous", "measure", "measurable" refer to the one-dimensional Lebesgue measure μ_1. A set $Z \subset \mathbb{R}$ is called *negligible* whenever $\mu_1(Z) = 0$. Given two subsets X, Y of $\mathbb{R}$, we say that X and Y are nonoverlapping if their intersection is negligible. A function is always real-valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$. If Z is a measurable subset of $\mathbb{R}$, $\mathcal{L}(Z)$ will denote the space of all Lebesgue integrable functions on Z. If $f \in \mathcal{L}(Z)$, the Lebesgue integral of f over Z will be denoted by $(L)\int_Z f$.

An *interval* is a compact nondegenerate interval of $\mathbb{R}$, and $[a,b]$ is a fixed interval. $\mathcal{I}$ is the family of all nondegenerate subintervals of $[a,b]$. If $I \in \mathcal{I}$, we shall write $\mu_1(I)$ as $|I|$. A function F defined on $\mathcal{I}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each nonoverlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$.

A *partition* is a collection $P = \{(I_i,\xi_i)\}_{i=1}^{p}$, where $I_1, I_2, \ldots, I_p$ are nonoverlapping intervals, and $\xi_i \in I_i$ for $i = 1,2,\ldots,p$. Given $Z \subseteq [a,b]$, a positive function δ on Z is called a *gauge* on Z. We say that a partition is

(i) a partition *in* Z if $\bigcup_{i=1}^{p} I_i \subset Z$;

(ii) a partition *of* Z if $\bigcup_{i=1}^{p} I_i = Z$;

(iii) *anchored* in Z if $\{\xi_1,\xi_2,\ldots,\xi_p\} \subset Z$;

(iv) δ-*fine* if $I_i \subset B(\xi_i,\delta(\xi_i))$ for each $i = 1,2,\ldots,p$.

In view of the Cousin's lemma [7, Theorem 2.3.1], the following definition is meaningful.

Definition 2.1. A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* if there exists $A \in \mathbb{R}$ such that for any given $\epsilon > 0$, there exists a gauge δ on $[a,b]$ such that

$$\left| \sum_{i=1}^{p} f(\xi_i)\,|I_i| - A \right| < \epsilon \tag{2.1}$$

for each δ-fine partition $\{(I_i,\xi_i)\}_{i=1}^{p}$ of $[a,b]$. Here A is called the Henstock-Kurzweil integral of f over $[a,b]$, and we write $A = (HK)\int_a^b f$. If $[a,b] = E$, we also write A as $(HK)\int_E f$.

Remark 2.2. (a) The linear space of all Henstock-Kurzweil integrable functions on $[a,b]$ is denoted by $\mathcal{HK}([a,b])$.

(b) It follows from [7, Theorem 2.5.14] that if $f \in \mathcal{HK}([a,b])$, then $f \in \mathcal{HK}(J)$ for each subinterval J of $[a,b]$. The interval function $F : J \mapsto (HK)\int_J f$ is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite $\mathcal{HK}$-integral, of f. By [7, Theorem 2.5.12], F is an additive interval function on $\mathcal{I}$.

(c) By [7, Theorem 3.13.3], we see that $\mathcal{L}([a,b]) \subset \mathcal{HK}([a,b])$. Furthermore, $(L)\int_a^b f = (HK)\int_a^b f$ for each $f \in \mathcal{L}([a,b])$.

(d) If f is a nonnegative, Henstock-Kurzweil integrable on $[a,b]$, then it follows from [7, Theorem 3.13.3] that $f \in \mathcal{L}([a,b])$.

We have the following important Saks-Henstock Lemma [7, Theorem 3.2.1].

Theorem 2.3. *(Saks-Henstock). If F is the indefinite $\mathcal{HK}$-integral of a function f on $[a,b]$, then for $\epsilon > 0$, there exists a gauge δ on $[a,b]$ such that for any δ-fine partition $\{(I_i,\xi_i)\}_{i=1}^{p}$ in $[a,b]$, we have $\sum_{i=1}^{p} |f(\xi_i)\,|I_i| - F(I_i)| < \epsilon$.*

If F is the indefinite $\mathcal{HK}$-integral of a function f on $[a,b]$, then it follows from Saks-Henstock Lemma and [7, p.81−82] that F is continuous in the sense that $F(I) \to 0$ as the measure of the interval I tends to zero. Thus, the space $\mathcal{HK}([a,b])$ of all Henstock-Kurzweil integrable functions on $[a,b]$ may be equipped with the Alexiewicz norm $\|\cdot\|$, where $\|f\| = \sup\left\{\left|(HK)\int_I f\right|\right\}$ where the supremum is taken over all subintervals I of $[a,b]$.

Denoting the ordinary derivative of F at $x \in [a,b]$ by $F'(x)$, an application of the Vitali covering theorem [13, p.109] that $F'(x)$ exists for almost all $x \in [a,b]$ with $F' = f$ almost everywhere. In particular, the measurability of f follows easily.

Let F be an interval function on $\mathcal{I}$, and X be an arbitrary subset of $[a,b]$. If δ is a gauge on X, we set $V(F,X,\delta) := \sup_P \sum_{i=1}^{p} |F(I_i)|$ where the supremum is taken over all δ-fine partitions $P = \{(I_i,\xi_i)\}_{i=1}^{p}$ anchored in X.

We put $V_{\mathcal{HK}}F(X) := \inf_\delta V(F,X,\delta)$ where the infimum is taken over all gauges δ on X. Then, the extended real-valued set function $V_{\mathcal{HK}}F(\cdot)$ has the property that $V_{\mathcal{HK}}F$ is a metric outer measure. See, for example, [14].

The following Radon-Nikodym Theorem holds for the Henstock-Kurzweil integral.

Theorem 2.4. *If $f \in \mathcal{HK}([a,b])$ with F being its indefinite $\mathcal{HK}$ integral, then*

$$V_{\mathcal{HK}}F(X) = (L)\int_X |f|$$

for each measurable subset X of $[a,b]$.

PROOF. This follows from [12, Theorem 8] and [12, Proposition 10]. □

3 Main Results

The first theorem is essentially a reformulation of [6, Lemma 15.5] and [6, Lemma 6.18], whose proofs depend on the theory of Denjoy-Perron integration. We shall prove it without reference to the theory of Denjoy-Perron integration [13].

Theorem 3.1. *If $f \in \mathcal{HK}([a,b])$, then there exists an increasing sequence $\{Y_n\}$ of closed sets that satisfies the following conditions:*

(a) $\bigcup_{n=1}^{\infty} Y_n = [a,b]$;

(b) $f \in \mathcal{L}(Y_n)$ *for each* $n \in \mathbb{Z}^+$;

(c) *the series* $\sum_{k=1}^{\infty} \|f\chi_{[c_k^{(n)},d_k^{(n)}]}\|$ *converges, where* $\{[c_k^{(n)},d_k^{(n)}]\}$ *is the collection of subintervals of* $[a,b]$ *contiguous to* Y_n.

PROOF. Let F denotes the indefinite Henstock-Kurzweil integral of f on $[a,b]$. By Theorem 2.3, for $\epsilon = 1$, there exists a gauge δ on $[a,b]$ such that for any δ-fine partition $\{(I_i,\xi_i)\}_{i=1}^{p}$ in $[a,b]$, we have

$$\sum_{i=1}^{p} |f(\xi_i)\,|I_i| - F(I_i)| < 1. \tag{3.1}$$

For each positive integer n, we put

$$Y_n = \overline{\{x \in [a,b] : |f(x)| < n \text{ and } \delta(x) > \frac{1}{n}\}}.$$

Since f is real-valued and δ is strictly positive on $[a,b]$, it is clear that (a) holds.

In order to prove (b), given any positive integer n, we choose a $\frac{1}{n}$-fine partition $\{([u_i,v_i],\xi_i)\}_{i=1}^{p}$ anchored in Y_n. We distinguish the following cases:

(i) If $\xi_i \in (u_i,v_i)$ or $\xi_i = u_i = a$ or $\xi_i = v_i = b$ for some $i \in \{1,2,\ldots,p\}$, we choose $x_i \in \{x \in [a,b] : |f(x)| < n \text{ and } \delta(x) > \frac{1}{n}\} \cap (u_i,v_i)$;

(ii) If $a < u_i = \xi_i$ for some $i \in \{1,2,\ldots,p\}$, we choose $y_i \in (u_i,v_i)$ and $x_i \in \{x \in [a,b] : |f(x)| < n \text{ and } \delta(x) > \frac{1}{n}\}$ so that $u_i \in (x_i,y_i)$ and the sequence of intervals $\{[x_i,y_i]\}_{i=1}^{p}$ are pairwise disjoint with

$$|F([u_i,v_i]) - F([x_i,y_i])| < \frac{1}{p}.$$

(iii) If $v_i = \xi_i < b$ for some $i = 1,2,\ldots,p$, we choose $x_i \in (u_i,v_i)$ and $y_i \in \{x \in [a,b] : |f(x)| < n \text{ and } \delta(x) > \frac{1}{n}\}$ so that $v_i \in (x_i,y_i)$ and the sequence of intervals $\{[x_i,y_i]\}_{i=1}^{p}$ are pairwise disjoint with

$$|F([u_i,v_i]) - F([x_i,y_i])| < \frac{1}{p}.$$

Put

$T_1 = \{i \in \{1,2,\ldots,p\} : \xi_i \in (u_i,v_i) \text{ or } \xi_i = u_i = a \text{ or } \xi_i = v_i = b\}$.
$T_2 = \{i \in \{1,2,\ldots,p\} : a < u_i = \xi_i\}$.
$T_3 = \{i \in \{1,2,\ldots,p\} : v_i = \xi_i < b\}$.

Then it follows from (i), (ii), (iii) and (3.1) that

$$\begin{aligned}\sum_{i=1}^{p}|F([u_i,v_i])| &= \sum_{j=1}^{3}\sum_{i\in T_j}|F([u_i,v_i])|\\ &< \sum_{i\in T_1}|F([u_i,v_i])| + \sum_{i\in T_2}|F([x_i,y_i])| + 1 + \sum_{i\in T_3}|F([x_i,y_i])| + 1\\ &< n(b-a)+1+n(b-a)+1+1+n(b-a)+1+1\end{aligned}$$

proving that

$$V(F,Y_n,\frac{1}{n}) \leq 3n(b-a)+5. \tag{3.2}$$

Since $V_{\mathcal{HK}}F(Y_n) \leq V(F,Y_n,\frac{1}{n})$, (b) follows from (3.2) and Theorem 2.4.

In order to prove (c), it suffices to observe that there exists a positive integer N such that

$$\sum_{k=N}^{\infty}(d_k^{(n)} - c_k^{(n)}) < 1 \text{ and } \sum_{k=N}^{\infty}\|f\chi_{[c_k^{(n)},d_k^{(n)}]}\| \leq 2V(F,Y_n,\frac{1}{n}) < \infty. \qquad \square$$

The next theorem is the Harnack extension for the Henstock-Kurzweil integral.

Theorem 3.2. [2, Theorem 9.22] *Let X be a closed subset of $[a,b]$ with $\{[c_k,d_k]\}$ being the collection of subintervals of $[a,b]$ contiguous to X. Suppose the following conditions are satisfied :*

(a) $f\chi_X \in \mathcal{HK}([a,b])$;

(b) $f \in \mathcal{HK}([c_k,d_k])$ *for each positive integer k;*

(c) *the series* $\sum_{k=1}^{\infty}\|f\chi_{[c_k,d_k]}\|$ *converges;*

then $f \in \mathcal{HK}([a,b])$ and the equality

$$(HK)\int_c^d f = (HK)\int_c^d f\chi_X + \sum_{k=1}^{\infty}(HK)\int_{c_k}^{d_k} f\chi_{[c,d]}$$

holds for each subinterval $[c,d]$ of $[a,b]$.

Lemma 3.3. *Let X be a closed subset of $[a,b]$ with $\{[c_k,d_k]\}$ being the collection of subintervals of $[a,b]$ contiguous to X. Suppose the following conditions are satisfied :*

(i) $f \in \mathcal{HK}([a,b])$*;*

(ii) $f\chi_X \in \mathcal{HK}([a,b])$*;*

(iii) *the series* $\sum_{k=1}^{\infty} \|f\chi_{[c_k,d_k]}\|$ *converges;*

(iv) $\{[u_i,v_i]\}_{i=1}^{q} \subset [a,b]$ *is a finite sequence of nonoverlapping intervals satisfying the condition that at least one of the endpoints of each* $[u_i,v_i]$ *belongs to* X*.*

Then

$$\sum_{i=1}^{q}\left|(HK)\int_{u_i}^{v_i}(f-f\chi_X)\right| \le \sum_{k=1}^{N}\sum_{i=1}^{q}\left|(HK)\int_{u_i}^{v_i} f\chi_{[c_k,d_k]}\right| + 2\sum_{k=N+1}^{\infty}\|f\chi_{[c_k,d_k]}\|$$

for each $N \in \mathbb{Z}^+$*.*

PROOF. By (i), $f \in \mathcal{HK}([c_k,d_k])$ for each positive integer k. In view of (ii), (iii) and Theorem 3.2, we have $(HK)\int_{u_i}^{v_i}(f-f\chi_X) = \sum_{k=1}^{\infty}(HK)\int_{c_k}^{d_k} f\chi_{[u_i,v_i]}$ for each $i=1,2,\ldots,q$. Thus, we have

$$\begin{aligned}
&\left|(HK)\int_{u_i}^{v_i}(f-f\chi_X)\right| \\
\le &\sum_{k=1}^{\infty}\left|(HK)\int_{c_k}^{d_k} f\chi_{[u_i,v_i]}\right| \\
\le &\sum_{k=1}^{N}\left|(HK)\int_{c_k}^{d_k} f\chi_{[u_i,v_i]}\right| + \sum_{k=N+1}^{\infty}\left|(HK)\int_{c_k}^{d_k} f\chi_{[u_i,v_i]}\right|
\end{aligned}$$

giving

$$\begin{aligned}&\sum_{i=1}^{q}\left|(HK)\int_{u_i}^{v_i}(f-f\chi_X)\right|\\ &\leq\sum_{k=1}^{N}\sum_{i=1}^{q}\left|(HK)\int_{c_k}^{d_k}f\chi_{[u_i,v_i]}\right|+\sum_{k=N+1}^{\infty}\sum_{i=1}^{q}\left|(HK)\int_{c_k}^{d_k}f\chi_{[u_i,v_i]}\right|\\ &\leq\sum_{k=1}^{N}\sum_{i=1}^{q}\left|(HK)\int_{u_i}^{v_i}f\chi_{[c_k,d_k]}\right|+2\sum_{k=N+1}^{\infty}\|f\chi_{[c_k,d_k]}\|\end{aligned}$$

by (iv), since each interval $[c_k, d_k]$ can intersect with at most two intervals belonging to the set $\{[u_i, v_i]\}_{i=1}^{q}$. □

Theorem 3.4. *Let X be a closed subset of $[a, b]$ with $\{[c_k, d_k]\}$ being the collection of subintervals of $[a, b]$ contiguous to X. Suppose the following conditions are satisfied:*

(i) $f \in \mathcal{HK}([a, b])$*;*

(ii) $f\chi_X \in \mathcal{HK}([a, b])$*;*

(iii) *the series* $\sum_{k=1}^{\infty} \|f\chi_{[c_k,d_k]}\|$ *converges;*

then given $\epsilon > 0$, there exists a constant gauge δ on X such that for any δ-fine partition $\{([u_i, v_i], \xi_i)\}_{i=1}^{p}$ anchored in X, we have

$$\sum_{i=1}^{p}\left|(HK)\int_{u_i}^{v_i}f\chi_X-(HK)\int_{u_i}^{v_i}f\right|<\epsilon.$$

PROOF. By (iii), for $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$\sum_{k=N+1}^{\infty}\|f\chi_{[c_k,d_k]}\|<\frac{\epsilon}{4}. \tag{3.3}$$

By (i), $f \in \mathcal{HK}([c_k, d_k])$ for each k. Since $f \in \mathcal{HK}([c_i, d_i])$ for each $i = 1, 2, \ldots, N$, it follows from the continuity of indefinite $\mathcal{HK}$-integral that there exists $\eta_i > 0$ such that whenever $[u, v] \subseteq [c_i, d_i]$ satisfying $v - u < \eta_i$, we have

$$\left|(HK)\int_{u}^{v}f\right|<\frac{\epsilon}{4N}. \tag{3.4}$$

Define a constant gauge δ on $[a,b]$ by $\delta = \min_{i=1,2,\ldots,N} \eta_i$. An application of Lemma 3.3, (3.3) and (3.4) shows that for any δ-fine partition $P = \{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X, we have

$$\begin{aligned}&\sum_{i=1}^{p} \left|(HK)\int_{u_i}^{v_i} (f - f\chi_X)\right| \\ \le&\sum_{k=1}^{N}\sum_{i=1}^{p} \left|(HK)\int_{u_i}^{v_i} f\chi_{[c_k,d_k]}\right| + 2\sum_{k=N+1}^{\infty} \|f\chi_{[c_k,d_k]}\| \\ <&2N\frac{\epsilon}{4N} + 2\frac{\epsilon}{4} = \epsilon.\end{aligned}$$ □

In what follows, we shall write a decreasing null sequence of positive numbers $\{\epsilon_n\}$ as $\epsilon_n \downarrow 0$.

Theorem 3.5. *If $f \in \mathcal{HK}([a,b])$, then for $\epsilon_n \downarrow 0$, there exists an increasing sequence $\{X_n\}$ of closed sets such that*

(i) $\bigcup_{n=1}^{\infty} X_n = [a,b]$;

(ii) $f \in \mathcal{L}(X_n)$ *for each* $n \in \mathbb{Z}^+$;

(iii) *for each positive integer n, there exists a partition $P_n = \{(I_i, \xi_i)\}_{i=1}^{p}$ of $[a,b]$ such that the inequality*

$$\sum_{i=1}^{p}\sum_{J\subseteq I_i} \left|(L)\int_{J\cap X_n} f - (HK)\int_J f\right| < \epsilon_n$$

holds whenever $\{J\}$ is a finite sequence of non-overlapping subintervals of $[a,b]$ satisfying $J \cap X_n \neq \emptyset$ for all n.

PROOF. Since $\epsilon_n \downarrow 0$, we may assume that $\epsilon_n = \frac{1}{n}$. Since f is Henstock-Kurzweil integrable on $[a,b]$, there exists an increasing sequence $\{Y_k\}$ of closed sets satisfying all the conditions of Theorem 3.1. By Theorem 3.4, for each $k \in \mathbb{Z}^+$, there exists a constant gauge δ_k' on Y_k such that for any δ_k'-fine partition $\{(I_i, \xi_i)\}_{i=1}^{q}$ anchored in Y_k, we have

$$\sum_{i=1}^{q} \left|(L)\int_{I_i\cap Y_k} f - (HK)\int_{I_i} f\right| < \frac{1}{k}.$$

Next, we want to choose $\{X_n\}$ from $\{Y_k\}$ so that the required properties hold.

Let $p(n,k) = 2^k n$. Define a gauge δ_n on $[a,b]$ by

$$\delta_n(\xi) = \begin{cases} \delta_{p(n,1)'}(\xi) & \text{if } \xi \in Y_{p(n,1)}, \\ \min\{\delta'_{p(n,k)}, \operatorname{dist}(\xi, Y_{p(n,k-1)})\} & \text{if } \xi \in Y_{p(n,k)} \setminus Y_{p(n,k-1)} \\ & \text{for some } k \geq 2. \end{cases}$$

Since δ_n-fine partitions of $[a,b]$ exist, we may fix a δ_n-fine partition $P_n = \{(I_i,\xi_i)\}_{i=1}^p$ of $[a,b]$. For simplicity, we put

$$Q_1 = Y_{p(n,1)} \text{ and } Q_k = Y_{p(n,k)} \setminus Y_{p(n,k-1)} \text{ for } k \geq 2.$$

Next, we put

$$X_n = \bigcup_{k=1}^{\infty} \{I \cap Y_{p(n,k)} : (I,\xi) \in P_n \text{ with } \xi \in Q_k\}.$$

The above union is a finite one because P_n only has finitely many terms. Thus X_n is closed as each Y_k is closed.

Define $k(n) = \max\{k : (I,\xi) \in P_n \ \text{ and } \ \xi \in Q_k\}$. Since $\{Y_k\}$ is an increasing sequence of closed sets whose union is $[a,b]$, we have $Y_{p(n,k(n))} \supseteq X_n$. By the definition of δ_n and the compactness of $Y_{p(n,1)}$, the δ_n-fine partition $P_n = \{(I_i,\xi_i)\}_{i=1}^p$ must cover $Y_{p(n,1)}$. Hence $Y_{p(n,1)} \subseteq X_n$. Thus, we have $Y_{p(n,1)} \subseteq X_n \subseteq Y_{p(n,k(n))}$ and $f \in \mathcal{L}(X_n)$ because X_n is measurable. Observe also that if $(I,\xi) \in P_n$ with $\xi \in Q_k$ for some positive integer k, then $I \cap X_n = I \cap Y_{p(n,k)}$. Note that each $(I,\xi) \in P_n$ may have its associated points ξ belonging to Q_1 only. Without loss of generality, we may suppose that each $(I,\xi) \in P_n$ has its associated point ξ belongs to $Q_{s_1}, Q_{s_2}, \ldots, Q_{s_l}$ for some positive integers $s_1 < s_2 < \cdots < s_l$ with $s_1 = 1$. Let $\{J\}$ be a finite sequence of non-overlapping subintervals of $[a,b]$ with $J \subseteq I_i$ for some $i = 1, 2, \ldots, p$, and $J \cap X_n \neq \emptyset$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{p} \sum_{J \subseteq I_i} \left| (L) \int_{J \cap X_n} f - (HK) \int_J f \right| \\
&= \sum_{i=1}^{p} \sum_{k=1}^{l} \sum_{J \subseteq I_i : \xi_i \in I_i \cap Q_{s_k}} \left| (L) \int_{J \cap X_n} f - (HK) \int_J f \right| \\
&= \sum_{i=1}^{p} \sum_{k=1}^{l} \sum_{J \subseteq I_i : \xi_i \in I_i \cap Q_{s_k}} \left| (L) \int_{J \cap Y_{p(n,s_k)}} f - (HK) \int_J f \right| \\
&= \sum_{k=1}^{l} \sum_{i=1}^{p} \sum_{J \subseteq I_i : \xi_i \in I_i \cap Q_{s_k}} \left| (L) \int_{J \cap Y_{p(n,s_k)}} f - (HK) \int_J f \right| \\
&< \sum_{k=1}^{l} \frac{1}{n 2^{s_k}} < \frac{1}{n}.
\end{aligned}$$

It is easy to see that there exists an increasing subsequence of $\{X_n\}$, denoted again by $\{X_n\}$, such that $\bigcup_{n=1}^{\infty} X_n = [a, b]$. □

Corollary 3.6. [10, Theorem 2] *If $f \in \mathcal{HK}([a, b])$, then the following condition is satisfied: Given $\epsilon_n \downarrow 0$, there exists a sequence $\{X_n\}$ of closed sets in $[a, b]$ such that:*

(i) *$a, b \in X_1$, $X_n \subseteq X_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} X_n = [a, b]$;*

(ii) *$f \in \mathcal{L}(X_n)$ for each n;*

(iii) *for each positive integer n, if a finite sequence $\{I_i\}_{i=1}^{q}$ of non-overlapping intervals contained in $[a, b]$ satisfies the condition that at least one of the endpoints of each I_i belong to X_n, then we have*

$$\sum_{i=1}^{q} \left| (L) \int_{I_i \cap X_n} f - (HK) \int_{I_i} f \right| < \epsilon_n.$$

Theorem 3.7. *If $f \in \mathcal{HK}([a, b])$, then there exists an increasing sequence $\{X_n\}$ of closed sets whose union is $[a, b]$, $\{f \chi_{X_n}\} \subset \mathcal{L}([a, b])$ and $\{f \chi_{X_n}\}$ satisfies the following conditions:*

(i) *$f \chi_{X_n} \to f$ everywhere on $[a, b]$;*

(ii) *for $\epsilon > 0$, there exists a measurable gauge δ, independent of n, on $[a,b]$ such that for every δ-fine partition $P = \{(I_i, \xi_i)\}_{i=1}^{p}$ of $[a,b]$, we have*

$$\left|\sum_{i=1}^{p} f(\xi_i)\chi_{X_n}(\xi_i)\,|I_i| - (L)\int_a^b f\chi_{X_n}\right| < \epsilon$$

for all $n \in \mathbb{Z}^+$. In particular, $\{f\chi_{X_n}\}$ is Henstock-Kurzweil equi-integrable on E.

PROOF. By Corollary 3.6, we choose $\{X_k\}$ corresponding to $\epsilon_k = \frac{1}{k^2}$ and put $f_k = f\chi_{X_k}$ for $k \in \mathbb{Z}^+$. An application of [11, Proposition 4] and [11, Lemma 7(iii)] shows that for $\epsilon > 0$, there exists a measurable gauge δ_k on $[a,b]$ such that for any δ_k-fine partition $P_1 = \{(I_i, \xi_i)\}_{i=1}^{p_1}$ in $[a,b]$, we have

$$\sum_{i=1}^{p_1}\left|f_k(\xi_i)\,|I_i| - \int_{I_i} f_k\right| < \frac{\epsilon}{2^{k+2}}. \tag{3.5}$$

We may also assume that for each $x \in [a,b]$, the sequence $\{\delta_k(x)\}$ is non-increasing. Choose a positive integer $N \geq 2$ such that

$$\sum_{k=N}^{\infty}\frac{1}{k^2} < \frac{\epsilon}{4}. \tag{3.6}$$

Let $\{[c_i^{(N)}, d_i^{(N)}]\}$ be the sequence of subintervals of $[a,b]$ contiguous to X_N and put $\eta = \frac{1}{4}\min_{1\leq i\leq N}\left|d_i^{(N)} - c_i^{(N)}\right|$.

Define a gauge δ on $[a,b]$ by

$$\delta(\xi) = \begin{cases} \min\{\delta_N(\xi), \eta\} & \text{if } \xi \in X_1, \\ \min\{\delta_N(\xi), \eta\} & \text{if } \xi \in X_1, \\ \min\{\delta_N(\xi), \operatorname{dist}(\xi, X_{k-1},), \eta\} & \text{if } \xi \in X_k \setminus X_{k-1} \text{ for some } 2 \leq k \leq N, \\ \min\{\delta_k(\xi), \operatorname{dist}(\xi, X_{k-1}), \eta\} & \text{if } \xi \in X_k \setminus X_{k-1} \text{ for some } k > N. \end{cases}$$

Then δ is a measurable gauge on $[a,b]$ with

$$\delta(\xi) \leq \delta_N(\xi) \text{ for each } \xi \in X_N \tag{3.7}$$

and

$$\delta(\xi) \leq \delta_k(\xi) \text{if } \xi \in X_k \setminus X_{k-1} \text{ for some } k > N. \tag{3.8}$$

Claim. The sequence $\{f_n\}$ is Henstock-Kurzweil equi-integrable with this function δ. Given a δ-fine partition $P = \{(I_i, \xi_i)\}_{i=1}^{p}$ of $[a, b]$, we put

$$N_0 = \max\{i : (I, \xi) \in P \text{ with } \xi \in X_i.\}.$$

Then by our definition of η and δ, any δ-fine cover of X_N cannot be a cover of $[c_i^{(N)}, d_i^{(N)}]$ for $i = 1, 2, 3, \ldots, N$, so we have $N_0 > N$.

Subclaim 1. $\sum_{i=1}^{p} \left| f(\xi_i)\,|I_i| - (HK)\int_{I_i} f \right| < \frac{\epsilon}{2}$.

Let $S_N = \{i : \xi_i \in X_N\}$ and $S_k = \{i : \xi_i \in X_k \backslash X_{k-1}\}$ for each $k > N$. Then it follows from (3.5) to (3.8) that we have

$$\begin{aligned}
&\sum_{i=1}^{p} \left| f(\xi_i)\,|I_i| - (HK)\int_{I_i} f \right| \\
&\le \sum_{i\in S_N} \left| f(\xi_i)\,|I_i| - (HK)\int_{I_i} f \right| + \sum_{k=N+1}^{N_0} \sum_{i\in S_k} \left| f(\xi_i)\,|I_i| - (HK)\int_{I_i} f \right| \\
&\le \sum_{i\in S_N} \left| f_N(\xi_i)\,|I_i| - (L)\int_{I_i} f_N \right| + \sum_{i\in S_N} \left| (L)\int_{I_i} f_N - (HK)\int_{I_i} f \right| \\
&+ \sum_{k=N+1}^{N_0} \sum_{i\in S_k} \left| f_k(\xi_i)\,|I_i| - (L)\int_{I_i} f_k \right| + \sum_{k=N+1}^{N_0} \sum_{i\in S_k} \left| (L)\int_{I_i} f_k - (HK)\int_{I_i} f \right| \\
&< \frac{\epsilon}{2^{N+2}} + \frac{1}{N^2} \\
&+ \sum_{k=N+1}^{N_0} \sum_{i\in S_k} \left| f_k(\xi)\,|I_i| - (L)\int_{I_i} f_k \right| + \sum_{k=N+1}^{N_0} \sum_{i\in S_k} \left| (L)\int_{I_i} f_k - (HK)\int_{I_i} f \right| \\
&< \frac{\epsilon}{2^{N+2}} + \frac{1}{N^2} + \sum_{k=N+1}^{N_0} \frac{\epsilon}{2^{k+2}} + \sum_{k=N+1}^{N_0} \frac{1}{k^2} < \frac{\epsilon}{2}.
\end{aligned}$$

The next two subclaims will enable us to prove that $\{f_n\}$ is Henstock-Kurzweil equi-integrable on $[a, b]$.

Subclaim 2. For each $n = 1, 2, \ldots, N$, we have

$$\sum_{i=1}^{p} \left| f_n(\xi_i)\,|I_i| - (L)\int_{I_i} f_n \right| < \frac{\epsilon}{2^{n+2}}.$$

By our definition of δ, $(\xi-\delta(\xi),\xi+\delta(\xi))\subset(a,b)\backslash X_n$ whenever $\xi\notin X_n$, so

$$\begin{aligned}
&\sum_{i=1}^{p}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|\\
&=\sum_{i:\xi_i\in X_n}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|+\sum_{i:\xi_i\notin X_n}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|\\
&=\sum_{i:\xi_i\in X_n}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|<\frac{\epsilon}{2^{n+2}}.
\end{aligned}$$

Subclaim 3. For each integer n with $n>N$, $\sum_{i=1}^{p}\left|f_n(\xi_i)\,|I_i|-\int_{I_i}f_n\right|<\epsilon$.
Since $f=f_n$ on X_n, we have

$$\begin{aligned}
&\sum_{i=1}^{p}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|\\
&=\sum_{i:\xi_i\in X_n}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|+\sum_{i:\xi_i\notin X_n}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|\\
&=\sum_{i:\xi_i\in X_n}\left|f_n(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|\\
&=\sum_{i:\xi_i\in X_n}\left|f(\xi_i)\,|I_i|-(L)\int_{I_i}f_n\right|\\
&\leq\sum_{i:\xi_i\in X_n}\left|f(\xi_i)\,|I_i|-(HK)\int_{I_i}f\right|+\sum_{i:\xi_i\in X_n}\left|(L)\int_{I_i}f_n-(HK)\int_{I_i}f\right|\\
&<\frac{\epsilon}{2}+\frac{1}{n^2}<\frac{\epsilon}{2}+\sum_{k=N}^{\infty}\frac{1}{k^2}<\epsilon.
\end{aligned}$$

From subclaims 2 and 3, we have, for all positive integer n,

$$\left|\sum_{i=1}^{p}f_n(\xi_i)\,|I_i|-(L)\int_a^b f_n\right|<\epsilon. \qquad\square$$

From the subclaim 1 of the proof of Theorem 3.7 and the measurability of the δ function, we obtain the following corollary, which was proved differently in [6, Theorem 10.3] or [2, Theorem 9.24].

Corollary 3.8 *If $f \in \mathcal{HK}([a,b])$, then for $\epsilon > 0$, the function δ from the definition of the Henstock-Kurzweil integral can be chosen to be measurable.*

Corollary 3.9 *If $f \in \mathcal{HK}([a,b])$ with F being its indefinite $\mathcal{HK}$ integral, then there exists a sequence $\{F_n\}$ of additive interval functions on $\mathcal{I}$ satisfying the following conditions:*

(i) $V_{\mathcal{HK}} F_n([a,b]) < \infty$ *for each n;*

(ii) *given that $Z \subset E$ is negligible and $\epsilon > 0$, there exists a gauge δ, independent of n, on Z such that*

$$V(F_n, Z, \delta) < \epsilon \text{ for all } n.$$

We remark that the proofs of Corollary 3.6, Theorem 3.7 and Corollary 3.9 do not generalize to the higher dimensional interval $E := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m]$ of $\mathbb{R}^m$. Indeed, the proof is based on Theorem 3.4, for which no satisfactory analogue in higher dimensions is known. As a result, we have the following conjecture.

Conjecture 1. If the sequence of additive interval functions $\{F_n\}$ satisfies the following condition:

> Given that $Z \subset E$ is negligible and $\epsilon > 0$, there exists a gauge δ, independent of n, on Z such that
>
> $$V(F_n, Z, \delta) < \epsilon \text{ for all } n$$

then there exists a sequence of functions $\{f_n\}$ on E satisfying the following conditions:

(i) for each $n = 1, 2, \ldots$, F_n is the indefinite $\mathcal{HK}$-integral of f_n on E;

(ii) $\{f_n\}$ is Henstock-Kurzweil equi-integrable on E.

If we assume that $F_n \equiv F$ for all n, then conjecture 1 turns out to be true. This result was obtained for $m = 1$ in [1], and for $m \geq 1$ in [8].

The proof of Corollary 3.6 is real-line dependent. As a result, it is natural to ask whether the next conjecture is true for $m \geq 2$.

Conjecture 2. If $f \in \mathcal{HK}(E)$, then given $\epsilon_n \downarrow 0$, there exists an increasing sequence $\{X_n\}$ of closed sets satisfying the following conditions:

(i) $\bigcup_{n=1}^{\infty} X_n = E$;

(ii) $f \in \mathcal{L}(X_n)$ for each n;

(iii) for each n, there exists a positive constant η_n such that whenever $\{(I_i, \xi_i)\}_{i=1}^{p}$ is a η_n-fine partition anchored in X_n, we have

$$\sum_{i=1}^{p} \left| (L) \int_{I_i \cap X_n} f - (HK) \int_{I_i} f \right| < \epsilon_n.$$

Since the proof of Theorem 3.7 depends on Corollary 3.6, it is also natural to ask whether the following analogue of Theorem 3.7 holds if $m \geq 2$.

Conjecture 3. If $f \in \mathcal{HK}(E)$, then there exists an increasing sequence $\{X_n\}$ of closed sets $\{X_n\}$ satisfying the following conditions:

(i) $X_n \subseteq E$ for all n;

(ii) $Z := E \setminus \bigcup_{k=1}^{\infty} X_k$ has m-dimensional Lebesgue measure zero;

(iii) $f \in \mathcal{L}(X_n)$ for each n;

(iv) $\{f\chi_{X_n \cup Z}\}$ is Henstock-Kurzweil equi-integrable on E.

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