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QUASICONTINUITY AND MEASURABILITY OF FUNCTIONS OF TWO VARIABLES

Abstract

In this article we establish some conditions concerning the sections f^y of a function $f : \mathbb{R}^2 \to \mathbb{R}$ having Lebesgue measurable sections f_x which imply the measurability of f. The first condition is more general than condition $\mathcal{A}$ introduced in [5].

Let $\mathbb{R}$ be the set of all reals and let $D \subset \mathbb{R}$ be a nonempty set.

A function $h : D \to \mathbb{R}$ is quasicontinuous [resp. upper semi-quasicontinuous] {lower semi-quasicontinuous}([6, 7]) at a point $x \in D$ if for every positive real η and for every open interval I containing x there is an open interval $J \subset I$ such that $J \cap D \neq \emptyset$ and $h(J \cap D) \subset (h(x) - \eta, h(x) + \eta)$ [resp. $h(J \cap D) \subset (-\infty, h(x) + \eta)$] $\{h(J \cap D) \subset (h(x) - \eta, \infty)\}$.

Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ ($D_l(A, x)$) of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h} \text{ respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

The family, T_d, of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

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holds, is a topology called the density topology ([1, 11]). The sets $A \in T_d$ are measurable.

In [5] the family, $\mathcal{A}$, of all functions $g : \mathbb{R} \to \mathbb{R}$ for which the sets $D(g)$ of all discontinuity points of g are nowhere dense and for each nonempty set $E \subset D(g)$ belonging to the density topology, T_d, the restricted function $g \restriction E$ is quasicontinuous, was introduced and the following theorem is proved:

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that all sections f_x, $x \in \mathbb{R}$, are (Lebesgue) measurable. If all sections f^y, $y \in \mathbb{R}$, are quasicontinuous and belong to the family $\mathcal{A}$ then f is (Lebesgue) measurable as the function of two variables.*

In this article, I generalize this theorem.

Let $\mathcal{E}$ be the family of all functions $g : \mathbb{R} \to \mathbb{R}$ which are upper and lower semi-quasicontinuous at each point $x \in \mathbb{R}$ such that for each nonempty set $E \subset D(g)$ belonging to the density topology the restricted function $g \restriction E$ is upper and lower semi-quasicontinuous at each point $u \in E$.

It is obvious to observe that the family of all quasicontiuous functions $g \in \mathcal{A}$ is a nowhere dense subset in the space of all functions $g \in \mathcal{E}$ with the metric

$$\rho_C(g, h) = \min(1, \sup_{x \in \mathbb{R}} |g(x) - h(x)|)$$

of uniform convergence. So, the following theorem is a generalization of Theorem 1.

Theorem 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the sections f_x, $x \in \mathbb{R}$, are measurable. If the sections $f^y \in \mathcal{E}$ for $y \in \mathbb{R}$ then f is measurable as the function of two variables.*

In the proof of this theorem we apply the following Lemma which is a particular case of Davies Lemma from [2] and a remark on functions from the family $\mathcal{E}$.

Lemma 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If for every positive real η and for each measurable set $A \subset \mathbb{R}^2$ of positive measure there is a measurable set $B \subset A$ of positive measure such that $\operatorname{osc}_B f \leq \eta$ then the function f is measurable.*

Remark 1. *Let $g \in \mathcal{E}$ be a bounded function and let $A \subset \mathbb{R}$ be a nonempty set belonging to T_d. For each positive real η there is an open interval I such that $I \cap A \neq \emptyset$ and $\operatorname{osc}_{(I \cap A)} g < \eta$.*

PROOF. If $A \setminus D(g) \neq \emptyset$ then g is continuous at a point belonging to A and in this case the proof is obvious. So suppose that $A \subset D(g)$. The function

g is bounded on A then we find a point $x \in A$ such that $g(x) > \sup_A g - \frac{\eta}{4}$. Since the restricted function g/A is lower semi-quasicontinuous at x, there is an open interval I with $I \cap A \neq \emptyset$ and $g(I \cap A) \subset (g(x) - \frac{\eta}{4}, \infty)$. But

$$g(I \cap A) \subset (-\infty, g(x) + \frac{\eta}{4}),$$

so $\operatorname{osc}_{(I \cap A)} g \leq \frac{\eta}{2} < \eta$.

PROOF OF THEOREM 2. Without loss of the generality we may assume that f is bounded, since in the contrary case we can consider the function $arctgf$.

We will show that the function f satisfies the assumptions of the above Lemma. Let $A \subset \mathbb{R}^2$ be a set of positive measure and let η be a positive real. For $x, y \in \mathbb{R}$ denote by $A_x = \{u \in \mathbb{R}; (x, u) \in A\}$ the vertical section of the set A corresponding to x and respectively by $A^y = \{t \in \mathbb{R}; (t, y) \in A\}$ the horizontal section of the set A corresponding to y. Moreover let

$$K = \{(x, y) \in A; x \text{ is a density point of } A^y\},$$

$$E = \{(x, y) \in K; x \in D(f^y)\}$$

and let $H = K \setminus E$. Denote by μ_2 the Lebesgue measure in $\mathbb{R}^2$ and observe that by well known theorem from Saks monograph [9] (p. 130-131) $\mu_2(A \setminus K) = 0$.

Now we will consider two cases.

Case I. The set H is not of measure 0.

Then for every point $(x, y) \in H$ there are open intervals $I(x, y) \ni x$ and $J(x, y)$ with rational endpoints such that $\mu(I(x, y) \cap K^y) > 0$ and $d(J(x, y) < \frac{\eta}{4}$ ($d(J(x, y))$ denotes the length of the interval $J(x, y)$) and $f^y(I(x, y)) \subset J(x, y)$. Let $I_1, I_2, \ldots, I_n, \ldots$ be a sequence of all open intervals with rational endpoints, let $J_1, \ldots, J_n, \ldots$ be an enumeration of all open intervals with $d(J_n) < \frac{\eta}{4}$ and for $n, m = 1, 2, \ldots$ let

$$A_{n,m} = \{(x, y) \in H; I(x, y) = I_n \text{ and } J(x, y) = J_m\}.$$

Then $H = \bigcup_{n,m=1}^{\infty} A_{n,m}$, and consequently there is a pair of positive integers j, k for which the set $A_{j,k}$ is not of measure zero. Let

$$V = \{y; \exists_x (x, y) \in A_{j,k}\}$$
$$U = \{y; y \text{ is an outer density point of } V\}$$

and $X = K \cap (I_j \times U)$. The set X is measurable and by Fubini's Theorem it is of positive measure. Put $B = X \cap f^{-1}(L_k)$, where $L_k = [a_k, b_k]$ is the closed interval having the same center as J_k and the length equal η. Evidently, $K \cap (I_j \times V) \subset B$.

We will prove that the set $X \setminus B$ is of measure zero. Really, if the set $X \setminus B$ is of positive outer measure, then by the equality

$$X \setminus B = (X \cap f^{-1}((-\infty, a_k))) \cup (X \cap f^{-1}((b_k, \infty))),$$

at least one of the sets on the right side of the above equality is of positive outer measure. Assume that the set $Y = X \cap f^{-1}((-\infty, a_k))$ is not of measure zero. By the upper semi-quasicontinuity of the sections f^y for each point $(x, y) \in Y$ there is an open interval $K(x, y) \subset I_j$ with rational endpoints such that for $t \in K(x, y)$ we have $f(t, y) < a_k$. Let K_1 be an open interval such that the set

$$Z = \{(x, y) \in Y; K(x, y) = K_1\}$$

is of positive outer measure. Put $W = \{y \in \mathbb{R}; \exists_x (x, y) \in Z\}$ and let $v \in K_1$ be a point. Then for $y \in W$ we have $f(v, y) \in \mathbb{R} \setminus L_k$ and for $y \in V$ the relation $f(v, y) \in J_k \subset L_k$ holds.

Observe that $V, W \subset U$, the set W is of positive outer measure, every point of the set U is an outer density point of the set V and $f_v(W) \subset \mathbb{R} \setminus L_k$ and $f_v(V) \subset L_k$. This contradicts to the measurability of the section f_v.

Let $B \subset X$ and $\mu_2(X \setminus B) = 0$. Then the set $B \subset A$ is measurable and $\mu_2(B) > 0$ and $\operatorname{osc}_B f \le \eta$.

Case II. The set H is of measure 0.

In this case we put $F_1 = \{(x, y) \in E; x \text{ is a density point of } E^y\}$. Since all sections $f^y \in \mathcal{E}$, $y \in \mathbb{R}$, analogously as in case I by Remark 1 we find open intervals I, J and a set $P \subset \mathbb{R}$ such that $d(J) < \frac{\eta}{4}$, P is not of measure zero, $I \cap (F_1)^y \neq \emptyset$ for $y \in P$ and $f(x, y) \in J$ for $(x, y) \in F_1 \cap (I \times P)$. Let

$$Z = \{y;\ \text{the outer density of the set } P \text{ at } y \text{ is equal } 1\}$$

and $S = F_1 \cap (I \times Z)$. Then the set S is measurable and by Fubini's Theorem $\mu_2(S) > 0$. Put $U = \{(x, y) \in S; f(x, y) \in \mathbb{R} \setminus L\}$, where L is the open interval of length equal η having the same center as J.

We will prove that $\mu_2(U) = 0$. Really, in the opposite case since the sections $f^y \in \mathcal{E}$, there are an open interval $I_1 \subset I$ and a set $B_1 \subset Z$ which is not of measure zero such that $\mu(I \cap S_x) > 0$ for $y \in B_1$, and $f(x, y) \in \mathbb{R} \setminus J$ for $(x, y) \in S \cap (I_1 \times B_1)$. If $x \in I_1$ is a point such that $\mu(S_x) > 0$ then we obtain a contradiction with the measurability of the section f_x.

So, $\mu_2(U) = 0$, the set $B = S \setminus U \subset K \subset A$ is measurable, $\mu_2(B) > 0$ and $\operatorname{osc}_B f \le \eta$. $\square$

Remark 2. *It is obvious to observe that the family of all functions* $g : \mathbb{R}^2 \to \mathbb{R}$ *satisfying the hypothesis of Theorem 1 is a nowhere dense subset in the space of all functions satisfying the hypothesis of Theorem 2 with the metric* ρ_C *of uniform convergence.*

Before the formulation of the next theorem we recall two examples of nonmeasurable functions of two variables.

Example 1. (W. Sierpiński [10]). There is a nonmeasurable set $A \subset \mathbb{R}^2$ such that for every straight line p the inequality $card(A \cap p) \leq 2$ is true. Let ϕ be the characteristic function of the set A.

Example 2. (R. O. Davies [2]) Martin's Axiom (Continuum Hypothesis) implies that there is a nonmeasurable function $\psi : \mathbb{R}^2 \to [0, 1]$ such that the sections ψ^y, $y \in \mathbb{R}$, are approximately continuous and for each $x \in \mathbb{R}$ the set $\{t \in \mathbb{R}; f(x, t) \neq 0\}$ is of measure zero (is countable).

Remark 3. *Observe that for every countable set* $G \subset \mathbb{R}$ *there is a set* $H \subset \mathbb{R}$ *of measure zero such that the restricted sections* $\{\phi^y/(\mathbb{R}\setminus H)\}_{y\in G}$ *and* $\{\psi^y/(\mathbb{R}\setminus H)\}_{y\in G}$ *are equal zero.*

We will say that the sections f^y, $y \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy the condition

$(*)$ if for each positive real η, for each nonempty open set $U \subset \mathbb{R}$ and for all $u, v \in \mathbb{R}$ the condition

$$\mathrm{cl}(\{x \in U; |f(x, u) - f(x, v)| \leq \eta\}) \supset U$$

(cl denotes the closure operation) implies the inequality

$$|f(t, u) - f(t, v)| \leq \eta \text{ for all } t \in U.$$

In [8] O'Malley introduced the topology T_r generated by the basis of all sets A belonging to the density topology which are simultaneously F_σ-sets and G_δ-sets.

Observe that if the sections f^y are T_r-continuous then they satisfy the condition $(*)$.

We will say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the condition $(**)$ if the sections f^y, $y \in \mathbb{R}$, are of the first Baire class, the sections f_x, $x \in \mathbb{R}$ are measurable and for each nonempty set $A \in T_d$ there is a point $y \in A$ such that for each nonempty set $B \in T_d$ and for each positive real η there are a nonempty set $F \subset A$ belonging to T_d and a sequence $(u_n)_n$ such that $y \in F$, $B \cap \mathrm{int}(\mathrm{cl}(\{u_n; n \geq 1\})) \neq \emptyset$ and $|f(u_n, z) - f(u_n, t)| \leq \eta$ for all $z, t \in F$ and $n = 1, 2, \ldots$ (int denotes the interior operation).

Theorem 3. *If a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the conditions $(*)$ and $(**)$ then f is measurable.*

Proof. We will prove that f satisfies the hypothesis of Lemma 1. For this fix a real $\eta > 0$ and a measurable set $H \subset \mathbb{R}^2$ of positive measure. There is a set $G \subset H$ such that $\mu_2(H \setminus G) = 0$ and for each point $(x, y) \in G$ the sections $G_x, G^y \in T_d$ ([3]). Since the sections f^y, $y \in \mathbb{R}$, are Baire 1 class, for each point $y \in Pr_Y(G) = \{y; \exists_x (x, y) \in G\}$ there is an open interval $I(y)$ with rational endpoints such that $I(y) \cap G^y \neq \emptyset$ and $\operatorname{osc}_{(I \cap G^y)} f^y < \frac{\eta}{4}$. The family of intervals with rational endpoints is countable, so there is an open interval I for which the set $M = \{y \in Pr_Y(G); I(y) = I\}$ is of positive outer measure. Let A be the set of all outer density points of the set M. The set A is nonempty and belongs to T_d. Put $P = G \cap (I \times A)$. The set P is measurable, and by Fubini's theorem, $\mu_2(P) > 0$. So, there is a nonempty set $K \in T_d$ which is contained in the set $\{y \in Pr_Y(P); \mu(P^y) > 0\}$. By the hypothesis $(**)$ there is a point $v \in K$ such that for each nonempty set $B \in T_d$ there are a nonempty set $F \in T_d$ and a sequence $(u_n)_n$ such that $v \in F \subset K$ and $|f(u_n, t) - f(u_n, y)| < \frac{\eta}{4}$ for all $t, y \in F$ and $n \geq 1$ and $B \cap \operatorname{int}(\operatorname{cl}(\{u_n; n \geq 1\})) \neq \emptyset$. Find a nonempty set $F \in T_d$ and a sequence $(u_n)_n$ satisfying the above conditions for the set $B = P^v$. Let $E = P \cap (I \times F)$. Then $E \subset H$ is a measurable set of positive measure and for arbitrary points $(x, y), (s, t) \in E$, we have

$$|f(x, y) - f(s, t)| \leq |f(x, y) - f(s, y)| + |f(s, y) - f(s, t)| < \frac{\eta}{4} + \frac{\eta}{4} = \frac{\eta}{2}.$$

So, $\operatorname{osc}_E f \leq \frac{\eta}{2} < \eta$. □

Observe that the functions ϕ and ψ from Examples 1 and 2 satisfy the condition $(**)$ but they are not measurable. So they do not satisfy the condition $(*)$.

An open problem is the following.

Problem. ([4], Probl. 1232.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the sections f^y, $y \in \mathbb{R}$, are T_r-continuous and the sections f_x, $x \in \mathbb{R}$, are measurable. Must the function f be measurable as the function of two variables?

The next example shows that there are T_r-continuous functions $g : \mathbb{R} \to \mathbb{R}$ which do not belong to the family $\mathcal{E}$. So, the answer to the above problem does not result from Theorem 2. Theorem 3 gives only a partial answer to this problem.

Example 3. Let $C \subset (0, 1)$ be a Cantor set of positive measure and let $(I_n)_n$ be an enumeration of all components of the set $(0, 1) \setminus C$. In every component

$I_n = (a_n, b_n)$, $n \geq 1$, of the open set $(0,1) \setminus C$ we find a closed interval $J_n = [c_n, d_n] \subset I_n$ and a continuous function $f_n : (a_n, b_n) \to [0,1]$ such that: $f_n(J_n) = [0,1]$ and $f_n((a_n, b_n) \setminus J_n) = \{1\}$, and

$$\frac{d_n - c_n}{\min(c_n - a_n, b_n - d_n)} < \frac{1}{n}.$$

Putting

$$g(x) = \begin{cases} f_n(x) & \text{for } x \in I_n,\ n \geq 1 \\ 1 & \text{otherwise} \end{cases}$$

we obtain an approximately continuous function which is continuous at each point $x \in \mathbb{R} \setminus C$ and such that the restricted function $g \restriction C$ is also continuous. So g is in the class B_1^* (see [8]) and consequently T_r-continuous.

Let $A \subset C$ is a nonempty set belonging to T_d and for every index $n \geq 1$ let $K_n \subset J_n$ be an open interval for which $f(K_n) \subset [0, \frac{1}{3}]$. Then the set

$$M = A \cup \bigcup_n K_n \in T_d,$$

but the restricted function g/M is not upper semi-quasicontinuous at points $x \in A$. So the function g is not in the class $\mathcal{E}$.

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