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CONVERGENCE OF KOENIGS' SEQUENCES

Abstract

Let f be an interval map defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$ and let $\phi_n(x) = f^n(x)/\lambda^n$. It is shown that if

$$f(x) = \lambda x + \mathcal{O}\left(\frac{|x|}{y\log(y)\cdots\log^{p-1}(y)(\log^p(y))^{1+\varepsilon}}\right)$$

for some $\varepsilon > 0$ and nonnegative integer p where $y = |\log(|x|)|$, then the Koenigs' sequence $\{\phi_n\}$ of f converges uniformly on a neighborhood of 0 to a limit ϕ with $\phi(0) = 0$ and $\phi'(0) = 1$. On the other hand, if $f(0) = 0$ and

$$f(x) = x\left(\lambda - \frac{1}{\log(x)\log(-\log(x))\cdots\log^p(-\log(x))}\right)$$

for sufficiently small $x > 0$ where $0 < \lambda < 1$ and p is a nonnegative integer, then the Koenigs' sequence of f diverges on a small right-neighborhood of 0. It is illustrated by examples that when $\varepsilon = 0$ in the first equation for f given above, the Koenigs' sequence of f can also converge to zero on a neighborhood of 0 or converge to a limit ϕ that is nondifferentiable at 0. It is also shown that when the Koenigs' sequence of a map f converges to a limit ϕ that is differentiable at 0, then $\phi'(0)$ is either 0 or 1.

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1 Introduction

Throughout this paper f^n will denote the n'th iterate of f where f° denotes the identity map and the real numbers $\mathbb{R}$ will be regarded as the underlying topological space. A neighborhood of $0 \in \mathbb{R}$ shall refer to a subinterval of $\mathbb{R}$ having 0 as an interior point. We let $\eta(0)$ denote a neighborhood of 0 with corresponding left-neighborhood defined by $\eta^-(0) = \eta(0) \cap \{x \in \mathbb{R} | x \leq 0\}$ and right-neighborhood $\eta^+(0)$ similarly defined.

Suppose $f : X \to \mathbb{C}$ is an analytic function where X is a neighborhood of the origin in the complex plane, $f(0) = 0$, and $0 < |f'(0)| < 1$. G. Koenigs showed that the Schröder equation $\phi \circ f(z) = \lambda\,\phi(z)$ where λ is a scalar, has a unique local analytic solution ϕ given by

$$\phi(z) = \lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{f^n(z)}{\lambda^n}\,,$$

where $\lambda = f'(0)$, $\phi(0) = 0$, and $\phi'(0) = 1$. The sequence $\{\phi_n\}$ is the Koenigs' sequence of f. If ϕ is an invertible solution of the Schröder equation, then ϕ conjugates f to its linearization $\lambda z\colon\ \phi \circ f \circ \phi^{-1}(z) = \lambda z$. We refer to [1], [2], [3] for further background and references concerning Koenigs' sequences and the Schröder equation.

We consider the Koenigs' sequence $\{\phi_n\}$ associated with an interval map f defined on a neighborhood of a stable fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. If the Koenigs' sequence of f converges on a neighborhood $\eta(0)$ of 0, then the limit ϕ satisfies the Schröder equation with f and λ on $\eta(0)$. It is well-known that if $f \in C^{1+\varepsilon}$ for some $\varepsilon > 0$, then Koenigs' sequence converges in C^1 to a continuously differentiable limit ϕ on a neighborhood of 0 satisfying $\phi(0) = 0$ and $\phi'(0) = 1$. Furthermore, it is also well-known that $f \in C^1$ is not in itself sufficient to guarantee convergence of the Koenigs' sequence of f. An example in [2] included below presents such a situation.

Example 1.1. Let

$$f(x) = \begin{cases} x\left(\lambda - \frac{1}{\log(x)}\right) & \text{if } x \in (0, a] \\ 0 & \text{if } x = 0, \end{cases}$$

where $0 < \lambda < 1$ and $0 < a < e^{1/(\lambda-1)}$. Then f' is strictly increasing and continuous on $[0, a]$ with $f(0) = 0$ and $f'_+(0) = \lambda$. The Koenigs' sequence of f diverges on $(0, a]$. □

A result of Oscar Lanford III presents general conditions for convergence of the Koenigs' sequence of an interval map on a neighborhood of a stable fixed point.

Theorem 1.2 (0. E. Lanford III)**.** *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. If*

$$f(x) = \lambda x + \mathcal{O}(|x|^{1+\varepsilon}) \quad \textit{for some } \varepsilon > 0,$$

then the Koenigs' sequence of f converges uniformly on a neighborhood of 0 to a limit ϕ with $\phi(0) = 0$ and $\phi'(0) = 1$.

By extending Example 1.1 and generalizing Theorem 1.2 we obtain a general convergence result for Koenigs' sequences. The proof hinges on a result of Oscar Lanford III. As an application, examples are obtained which have Koenigs' sequences that diverge, converge to 0 on a neighborhood of the fixed point 0, and converge to a limit that is nondifferentiable at 0. In the final section of this paper, a "0-1" law for Koenigs' sequences is presented which shows that when the Koenigs' sequence of a map converges to a limit ϕ that is differentiable at a fixed point 0, then $\phi'(0)$ is either 0 or 1.

2 Koenigs' Sequences

A well-known result (see for instance [4]) will be useful in the sequel. For a fixed nonnegative integer p the series

$$\sum_{k=k_0}^{\infty} \frac{1}{k \log(k) \cdots \log^{p-1}(k) (\log^p(k))^{1+\varepsilon}}, \quad \varepsilon > 0 \tag{1}$$

converges while the series

$$\sum_{k=k_0}^{\infty} \frac{1}{k \log(k) \cdots \log^{p-1}(k) \log^p(k)} \tag{2}$$

diverges, where k_0 is chosen to ensure that each term of the series is positive and $\log^p(k)$ denotes the p'th composition of $\log(k)$. If k is replaced by $\alpha k + \beta$ where α and β are positive numbers, then with the aid of the integral test it can be seen that (1) remains convergent and (2) remains divergent, where k_0 is replaced by k_0' if necessary. In preparation for the main convergence result for Koenigs' sequences, we will now state and prove a result due to Oscar Lanford III.

Lemma 2.1. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. If $\{\phi_n(x)/x\}$ converges uniformly to a function $\phi(x)/x$ on a deleted neighborhood of 0, then $\{\phi_n(x)\}$ converges uniformly on a neighborhood of 0 to limit ϕ with $\phi(0) = 0$ and $\phi'(0) = 1$.*

Proof. For each nonnegative integer n we have $\phi_n(0) = 0$ and $\phi_n'(0) = 1$, and consequently $\lim_{x\to 0} \phi_n(x)/x = 1$. Since $\{\phi_n(x)/x\}$ converges uniformly on a deleted neighborhood of 0 to $\phi(x)/x$, we conclude that $\phi(x)/x$ is a real-valued function and

$$\lim_{n\to\infty} \lim_{x\to 0} \frac{\phi_n(x)}{x} = \lim_{x\to 0} \lim_{n\to\infty} \frac{\phi_n(x)}{x} = 1,$$

which completes the proof. □

We present a general convergence result for Koenigs' sequences.

Theorem 2.2. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$ and let $y = |\log(|x|)|$. If*

$$f(x) = \lambda x + \mathcal{O}\left(\frac{|x|}{y \log(y) \cdots \log^{p-1}(y)(\log^p(y))^{1+\varepsilon}}\right)$$

for some $\varepsilon > 0$ and nonnegative integer p, then the Koenigs' sequence of f converges uniformly on a neighborhood of 0 to a limit ϕ with $\phi(0) = 0$ and $\phi'(0) = 1$.

Proof. As a consequence of Lemma 2.1, it is enough to show that $\{\phi_n(x)/x\}$ converges uniformly on a deleted neighborhood of 0. Let $y = |\log(|x|)|$, and let $\delta > 0$ be chosen so that $0 < |\lambda| \pm \delta < 1$. Since $f(0) = 0$ and $f'(0) = \lambda$, one can choose a neighborhood $\eta(0)$ so that $\log^m(y) > 0$ on $\eta(0) \setminus \{0\}$ for each integer $0 \leq m \leq p$, and such that the inequality

$$\lambda - \delta < \frac{f(x)}{x} < \lambda + \delta \tag{3}$$

holds on $\eta(0) \setminus \{0\}$. Let $\eta(0)$ and $M > 0$ be additionally chosen so that f satisfies

$$\left|\frac{f(x)}{x} - \lambda\right| \leq \frac{M|\lambda|}{y \log(y) \cdots \log^{p-1}(y)(\log^p(y))^{1+\varepsilon}}, \quad x \in \eta(0) \setminus \{0\}$$

for some $\varepsilon > 0$. Letting $u_k = y(f^k(x)) = |\log(|f^k(x)|)|$ we see that for each $x \in \eta(0) \setminus \{0\}$,

$$\left|\frac{f(f^k(x))}{\lambda f^k(x)}\right| \leq 1 + \frac{M}{u_k \log(u_k) \cdots \log^{p-1}(u_k)(\log^p(u_k))^{1+\varepsilon}}. \tag{4}$$

Using the reorganization $\phi_n(x)/x = \prod_{k=0}^{n-1} \left(f(f^k(x))/\lambda f^k(x)\right)$, it follows from (4) that

$$\left|\frac{\phi_n(x)}{x}\right| \leq \prod_{k=0}^{n-1} \left(1 + \frac{M}{u_k \log(u_k) \cdots \log^{p-1}(u_k)(\log^p(u_k))^{1+\varepsilon}}\right)$$

for each $x \in \eta(0) \setminus \{0\}$. To show that $\{\phi_n(x)/x\}$ converges uniformly on $\eta(0) \setminus \{0\}$, it is enough to prove that the series

$$\sum_{k=0}^{\infty} \frac{1}{u_k \log(u_k) \cdots \log^{p-1}(u_k)(\log^p(u_k))^{1+\varepsilon}} \tag{5}$$

converges uniformly on $\eta(0)\setminus\{0\}$. Since f satisfies (3) on $\eta(0)\setminus\{0\}$, it follows that $|f^k(x)/x| \le (|\lambda| + \delta)^k$ for each $x \in \eta(0) \setminus \{0\}$ and therefore

$$u_k = |\log(|f^k(x)|)| \ge k|\log(|\lambda| + \delta)| + |\log(|x|)|, \quad x \in \eta(0) \setminus \{0\}.$$

Let $\alpha = |\log(|\lambda| + \delta)|$, let $\beta = \inf(|\log(|x|)|)$ for $x \in \eta(0) \setminus \{0\}$, and let $v_k = \alpha k + \beta$. It follows that $\beta > 0$ and

$$\frac{1}{u_k \log(u_k) \cdots \log^{p-1}(u_k)(\log^p(u_k))^{1+\varepsilon}} \le \frac{1}{v_k \log(v_k) \cdots \log^{p-1}(v_k)(\log^p(v_k))^{1+\varepsilon}}$$

for $x \in \eta(0) \setminus \{0\}$. The above inequality and the discussion surrounding (1) together with the Weierstrass M-test show that the series in (5) converges uniformly on $\eta(0) \setminus \{0\}$. $\square$

Replacing the series in (1) with a more general series leads to the following result, which we state without proof.

Theorem 2.3. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. Let $f(x) = \lambda x + x\mathcal{O}(\mu(|\log(|x|)|))$ where μ is a continuous, positive and decreasing function defined on $[k_0, \infty)$ for some positive integer k_0. If $\sum_{k=k_0}^{\infty} \mu(k)$ converges, then the Koenigs' sequence of f converges uniformly on a neighborhood of 0 to a limit ϕ with $\phi(0) = 0$ and $\phi'(0) = 1$.*

Using Theorem 2.2, interval maps that aren't $C^{1+\varepsilon}$ for any $\varepsilon > 0$ can be formulated which have uniformly convergent Koenigs' sequence with limit ϕ satisfying $\phi(0) = 0$ and $\phi'(0) = 1$. Theorem 2.2 and (2) lead to the following construction of an interval map that has divergent Koenigs' sequence.

Example 2.4. Let $y = |\log(x)|$, $0 < \lambda < 1$, and let $a > 0$ be chosen so that $\log^m(y) > 0$ on $(0, a)$ for each integer $0 \le m \le p$. Let

$$f(x) = \begin{cases} x\left(\lambda + \frac{1}{y \log(y) \cdots \log^p(y)}\right) & \text{if } x \in (0, a) \\ 0 & \text{if } x = 0. \end{cases}$$

Then f' is strictly increasing and continuous on $[0, a)$ with $f(0) = 0$ and $f'_+(0) = \lambda$. The Koenigs' sequence of f diverges on a right-neighborhood of 0.

Proof. A simple computation indicates that $f'_+(0) = \lambda$ and that f' is strictly increasing and continuous on $[0, a)$. Let $\delta > 0$ be chosen so that $0 < \lambda \pm \delta < 1$. Choose $0 < b < a$ so that f satisfies (3) on $(0, b]$. The function h defined by

$$h(x) = \begin{cases} \frac{1}{y \log(y) \cdots \log^p(y)} & \text{if } x \in (0, a) \\ 0 & \text{if } x = 0, \end{cases}$$

will be useful for showing that the Koenigs' sequence of f diverges on $(0, b]$. Using the standard reorganization, we obtain

$$\phi_n(x) = x\left(1 + \frac{h(x)}{\lambda}\right) \cdots \left(1 + \frac{h(f^{n-1}(x))}{\lambda}\right).$$

To show that this sequence diverges, it is enough to show that the series $\sum_{k=1}^{\infty} h(f^k(x))$ diverges on $(0, b]$. Since h is positive and strictly increasing on $(0, b]$, it follows from the definition of f that $f^k(x) > x\lambda^k$ for any positive integer k. Therefore

$$\log(f^k(x)) > k\log(\lambda) + \log(x) \quad \text{and} \quad k\log(1/\lambda) - \log(x) > \left|\log(f^k(x))\right|.$$

To simplify the notation, let $\alpha = \log(1/\lambda)$ and $\beta = -\log(x)$. Then α and β are positive numbers with $\alpha k + \beta > |\log(f^k(x))|$ and therefore $\log^m(\alpha k + \beta) > \log^m(|\log(f^k(x))|)$ for each positive integer k and each integer $0 \le m \le p$. We obtain

$$h(f^k(x)) > \frac{1}{(\alpha k + \beta)\log(\alpha k + \beta) \cdots \log^{p-1}(\alpha k + \beta)\log^p(\alpha k + \beta)},$$

and thus, in view of the discussion surrounding (2), the series $\sum_{k=1}^{\infty} h(f^k(x))$ diverges on $(0, b]$. □

3 A "0-1" Law for Koenigs' Sequences

Consider an interval map f defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. In the previous section, a result of Oscar Lanford III shows that if $\{\phi_n(x)/x\}$ converges uniformly to $\phi(x)/x$ on a deleted neighborhood of 0, then $\phi'(0) = 1$. We will now show that if ϕ is differentiable at 0, then $\phi'(0)$ is either 0 or 1. The proof of this result requires some preliminary work, beginning with a result which explains a connection between existence of solutions of the Schröder equation and convergence of Koenigs' sequences.

Lemma 3.1. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. If the Koenigs' sequence of f diverges at each member of a sequence of points $\{x_m\}$ converging to 0, then there does not exist a function ϕ with $\phi(0) = 0$, $\phi'(0) = c \neq 0$, and satisfying $\phi(f(x)) = \lambda\phi(x)$ on a neighborhood of 0.*

PROOF. Suppose the Koenigs' sequence of f diverges at each member of a sequence of points $\{x_m\}$ converging to 0. Assume to obtain a contradiction that there exists a function ϕ such that $\phi(0) = 0$, $\phi'(0) = c \neq 0$, and $\phi(f(x)) = \lambda\phi(x)$ on a neighborhood of 0. Let $\eta(0)$ be a neighborhood of 0 which is contained in the basin of attraction of 0 such that $f(x) \neq 0$ when $x \neq 0$ and $\phi(f(x)) = \lambda\phi(x)$ for each $x \in \eta(0)$. Let m be chosen so large that $x_m \in \eta(0)$. It follows that $\phi(x_m) \neq 0$ and one has the relationship

$$\frac{\phi(f^n(x_m))}{f^n(x_m)} = \frac{\lambda^n}{f^n(x_m)}\phi(x_m), \quad n = 0, 1, 2, \dots . \tag{6}$$

The left side of (6) converges to $c \neq 0$ while the right side either converges to 0 or diverges, which is a contradiction. □

Proposition 3.2. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. If there exists a function ϕ with $\phi'(0) = 1$ satisfying the Schröder equation $\phi(f(x)) = \lambda\phi(x)$ on a neighborhood of 0, then $\phi(0) = 0$ and ϕ is the limit of the Koenigs' sequence of f.*

PROOF. One has to show that if there exists a function ϕ with $\phi'(0) = 1$ satisfying $\phi(f(x)) = \lambda\phi(x)$ for each x in a neighborhood of 0, then ϕ is the limit function of the Koenigs' sequence of f on some neighborhood of 0. Since $\phi(f(0)) = \phi(0) = \lambda\phi(0)$, then $\phi(0) = 0$. Clearly $f^n(0) = 0$ for each $n \geq 0$, and therefore $f^n(0)/\lambda^n \xrightarrow{n} \phi(0) = 0$. Following the argument of the previous lemma, we can see from (6) that $\phi(x)\lambda^n/f^n(x) \xrightarrow{n} 1$ must hold for each nonzero x in a neighborhood of 0. Hence, in this neighborhood ϕ must be the limit of the Koenigs' sequence of f. □

Next, we consider the case when the limit of the Koenigs' sequence of an interval map has vanishing derivative at a fixed point.

Proposition 3.3. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$ and let f have convergent Koenigs' sequence on a neighborhood of 0 with limit ϕ. If $\phi'(0) = 0$, then $\phi = 0$ on a neighborhood of 0.*

PROOF. Let f have convergent Koenigs' sequence on $\eta(0)$ with f nonzero on $\eta(0) \setminus \{0\}$. The limit ϕ of the Koenigs' sequence of f satisfies the Schröder equation with f and λ on $\eta(0)$. Since the left side of (6) converges to 0 for $x_m = x \in \eta(0)$, and $|\lambda^n/f^n(x)|$ converges to either a nonzero finite value or to infinity, it follows that $\phi = 0$ on $\eta(0)$. □

Corollary 3.4. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$. If there is a function ϕ with $\phi'(0) = 0$ satisfying the Schröder equation with f and λ on a neighborhood of 0, then ϕ is not the limit of the Koenigs' sequence of f on any neighborhood of 0 unless the Koenigs' sequence of f converges to 0 on some neighborhood of 0.*

A "0 - 1" law for Koenigs' sequences is obtained as a consequence of Proposition 3.2.

Theorem 3.5. *Let f be defined on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$ and let f have convergent Koenigs' sequence on a neighborhood of 0 with limit ϕ. If ϕ is differentiable at 0, then $\phi'(0) = 1$ unless $\phi = 0$ on some neighborhood of 0.*

PROOF. Assume to obtain a contradiction that ϕ, which is the limit of the Koenigs' sequence of f and which is not identically 0 on any neighborhood of 0, satisfies $\phi(0) = 0$ and $\phi'(0) = c$ where c is different from 0 or 1. Let $\psi(x) = \phi(x)/c$. Proposition 3.2 indicates that the Koenigs' sequence of f converges to ψ, which is a contradiction. □

We now present an example of a continuously differentiable map which has Koenigs' sequence converging to 0 on a neighborhood of a stable fixed point, thus showing that the statement of Theorem 3.5 is the best possible.

Example 3.6. Let

$$f(x) = \begin{cases} x\left(\lambda + \frac{1}{\log(|x|)}\right) & \text{if } 0 < |x| < e^{-1/\lambda} \\ 0 & \text{if } x = 0, \end{cases}$$

where $0 < \lambda < 1$. If $0 < a < e^{-1/\lambda}$ is chosen so that f is strictly increasing on $(0, a)$ and $f(x) > x\lambda/2$ for $x \in (0, a)$, then f' is continuous and positive on $[-a, a]$ with f' strictly increasing on $[-a, 0]$ and strictly decreasing on $[0, a]$. The Koenigs' sequence of f converges to 0 on $[-a, a]$.

PROOF. Differentiating f, it can be seen that f' is continuous and positive on $[-a, a]$ with f' strictly increasing on $[-a, 0]$ and strictly decreasing on $[0, a]$. Since f is an odd function and $\lambda > 0$, it is sufficient to prove that the Koenigs' sequence of f converges to 0 on $[0, a]$. Clearly $\phi(0) = 0$. By reorganization, the Koenigs' sequence of f for $x \in (0, a]$ can be written as

$$\phi_n(x) = x\left(1 + \frac{1}{\lambda \log(x)}\right)\left(1 + \frac{1}{\lambda \log(f(x))}\right) \cdots \left(1 + \frac{1}{\lambda \log(f^{n-1}(x))}\right).$$

In order to show that $\{\phi_n(x)\}$ converges to 0 on $(0,a]$, it is sufficient to show that the sequence $\{1/\phi_n(x)\}$ diverges to infinity on $(0,a]$. We have

$$\frac{1}{\phi_n(x)} = \frac{1}{x}\left(1 - \frac{1}{\lambda\log(x)+1}\right)\cdots\left(1 - \frac{1}{\lambda\log(f^{n-1}(x))+1}\right),$$

which diverges for $x \in (0,a]$ if the series $\sum_{k=0}^{\infty} 1/|\log(f^k(x)) + 1/\lambda\,|$ diverges. It follows from the definition of f and the choice of $a > 0$ that $f(x) > x(\lambda/2)$ and therefore $f^k(x) > x(\lambda/2)^k$ for each $x \in (0,a]$. Then

$$k\log\left(\tfrac{2}{\lambda}\right) - \log(x) - \tfrac{1}{\lambda} > -\log(f^k(x)) - \tfrac{1}{\lambda}\,, \quad x \in (0,a],$$

and therefore

$$\frac{1}{k\log\left(\frac{2}{\lambda}\right) - \log(x) - \frac{1}{\lambda}} < \frac{1}{-\log(f^k(x)) - \frac{1}{\lambda}}\,, \quad x \in (0,a],$$

which proves the statement. □

Lemma 2.1 indicates that in the previous example $\{\phi_n(x)/x\}$ converges pointwise, but not uniformly, on a deleted neighborhood of 0; yet, as we will now show, $\{\phi_n(x)\}$ converges uniformly on a neighborhood of 0.

Proposition 3.7. *Let f be defined and continuous on a neighborhood of a fixed point 0 with $f'(0) = \lambda$ where $0 < |\lambda| < 1$ and let f have convergent Koenigs' sequence $\{\phi_n(x)\}$ on a neighborhood of 0 with limit ϕ. If $\phi'(0) = 0$ and if $\{\phi_n(x)\}$ is monotone on a neighborhood of 0, then $\{\phi_n(x)\}$ converges uniformly on a neighborhood of 0.*

Proof. Proposition 3.3 indicates that the limit function is continuous on a neighborhood of 0. The sequence is monotone and each member of the sequence is continuous on a neighborhood of 0; hence, we conclude by a theorem of Dini (see for example Theorem 7.13 of [5]) that the convergence is uniform on a neighborhood of 0. □

In fact, bimonotonicity of the Koenigs' sequence of an interval map f is sufficient. For the case when $\lambda > 0$, Proposition 5.5 in [2] provides sufficient conditions for monotonicity of the Koenigs' sequence of an interval map f. The preceding results yield the construction of a map which has convergent Koenigs' sequence with limit ϕ that is nondifferentiable at the fixed point 0.

Example 3.8. Let $0 < \lambda < 1$, $0 < a < e^{-1/\lambda}$ and recursively define the sequence $\{a_n\}_{n=0}^{\infty}$ by letting $a_0 = a$, and $a_n = a_{n-1}(\lambda - 1/|\log(a_{n-1})|^2)$ for each $n \geq 1$. Let $b \in (a_1, a_0)$ and recursively define the sequence $\{b_n\}_{n=0}^{\infty}$ by letting $b_0 = b$, and $b_n = b_{n-1}(\lambda - 1/|\log(b_{n-1})|)$ for each $n \geq 1$, where b is chosen so that $b_i \neq a_j$ for any integers $i, j \geq 0$. Let $\varepsilon(x)$ be a differentiable function defined on $(0, a]$ such that $\varepsilon(a_n) = 1$ and $\varepsilon(b_n) = 0$ for each $n \geq 0$, and $0 \leq \varepsilon(x) \leq 1$ for each $x \in (0, a]$. Let

$$f(x) = \begin{cases} x\left(\lambda - \frac{1}{|\log(x)|^{1+\varepsilon(x)}}\right) & \text{if } x \in (0, a] \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is differentiable on $[0, a]$ with $f(0) = 0$ and $f'_+(0) = \lambda$. The Koenigs' sequence of f converges on $[0, a]$ to a limit ϕ with $\phi(0) = 0$ and $0 \leq \phi(x) < x$ for each $x \in (0, a]$, but $\phi'_+(0)$ doesn't exist.

PROOF. It follows from Theorem 2.2 and (6) that

$$\phi_n(f^k(a)) \xrightarrow{n} \phi(f^k(a)) = \lambda^k \phi(a) > 0 \quad \text{and thus} \quad \frac{\phi(f^k(a))}{f^k(a)} \xrightarrow{k} 1\,. \tag{7}$$

The proof of Example 3.6 shows that

$$\phi_n(f^k(b)) \xrightarrow{n} \phi(f^k(b)) = 0, \quad k \geq 0\,. \tag{8}$$

From the definition of f and since $f(x) < \lambda x$ and $0 \leq \varepsilon(x) \leq 1$ on $(0, a]$, then $0 < \phi_{n+1}(x) < \phi_n(x)$ for each $x \in (0, a]$ and $n \geq 0$, and therefore the Koenigs' sequence of f converges to a real-valued function $\phi(x)$ with $0 \leq \phi(x)/x < 1$ for each $x \in (0, a]$. These facts together with (7) and (8) prove the assertion that $\phi'_+(0)$ doesn't exist. □

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