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A NOTE ON A PETTIS-KURZWEIL-HENSTOCK TYPE INTEGRAL IN RIESZ SPACES

Abstract

Recently a connection has been found between the improper Kurzweil-Henstock integral on the real line and the integral over a compact space. In this paper these results are extended to a Pettis-type integral for the case of functions with values in Riesz spaces with “enough” order continuous functionals.

1 Introduction.

In [12] two possibilities are mentioned for defining the improper Kurzweil-Henstock integral on the real line. Their coincidence has been proved in [6]. On the other hand in [1] and [13] the Kurzweil-Henstock integral has been studied for real functions defined on a compact space. In [5] a natural connection was established between these two situations: the improper integral on the real line and the integral on a compact space. Here we introduce and investigate a Pettis-type integral (p-integral) for functions with values in a Dedekind complete Riesz space R such that the space R^* of its order continuous functionals separates the points of R, and we shall show that the above mentioned relation holds even for the p-integral. Furthermore, some convergence-type theorems are proved.

Key Words: Riesz spaces, compact topological spaces, order continuous linear functionals, Henstock-Kurzweil integral, Pettis integral.

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2 Preliminaries.

Let $\mathbb{N}$ be the set of all strictly positive integers, $\mathbb{R}$ be the set of the real numbers and $\mathbb{R}^+$ be the set of all strictly positive real numbers. We begin with some preliminary definitions and results.

Definition 2.1. A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

Definition 2.2. Let R be a Dedekind complete Riesz space. A sequence $(r_n)_n$ of elements of R is said to be *bounded* if there exist s_1, $s_2 \in R$ such that $s_1 \leq r_n \leq s_2$ for all $n \in \mathbb{N}$. Analogously we can define boundedness of a net $(r_\beta)_{\beta\in\Lambda}$ of elements of R, where $(\Lambda, \geq) \neq \emptyset$ is a directed set. Given a bounded sequence $(r_n)_n$ in R, we define

$$\limsup_n r_n = \inf_n [\sup_{m\geq n} r_m] \text{ and } \liminf_n r_n = \sup_n [\inf_{m\geq n} r_m].$$

Given a net $(r_\beta)_\beta$ in R, let

$$\limsup_\beta r_\beta = \inf_\beta [\sup_{\alpha\geq\beta} r_\alpha] \text{ and } \liminf_\beta r_\beta = \sup_\beta [\inf_{\alpha\geq\beta} r_\alpha],$$

provided that these quantities exist in R. We say that $(r_\beta)_\beta$ *order converges* (or simply (o)*-converges*) to $r \in R$ if $r = \limsup_\beta r_\beta = \liminf_\beta r_\beta$, and we write $(o) - \lim_{\beta\in\Lambda} r_\beta = r$. We say that a bounded sequence $(r_n)_n$ in R *order converges* (or simply (o)*-converges*) to $r \in R$ if $r = \limsup_n r_n = \liminf_n r_n$, and we write $(o) - \lim_n r_n = r$.

Let R be as above. A linear functional $g : R \to \mathbb{R}$ is said to be *positive* if $g(r) \geq 0$ for each $r \in R$, $r \geq 0$; *order continuous,* if for every net $(r_\beta)_\beta$ in R such that $(o) - \lim_\beta r_\beta = 0$ we have that $\lim_\beta g(r_\beta) = 0$. We note that a positive functional g is order continuous if and only if $x_\beta \downarrow 0$ in R implies $g(x_\beta) \downarrow 0$ in $\mathbb{R}$, and also if and only if $0 \leq x_\beta \uparrow x$ in R implies $g(x_\beta) \uparrow g(x)$ in $\mathbb{R}$. The vector space of all order continuous linear functionals on R will be denoted by R^*. This space is always a Dedekind complete Riesz space (see [3], p. 55). For example, if $1 \leq p < +\infty$ and $1/p + 1/q = 1$, then $l_p^* = l_q$ and $L_p([0,1]) = L^q([0,1])$. We say that R^* *separates points of* R if for every $r \in R$, $r \neq 0$, there exists $g \in R^*$ such that $g(x) \neq 0$. From now on we always suppose that R is a Dedekind complete Riesz space, such that R^* separates points of R. An example of a Riesz space R satisfying this property, though $R^{**} \neq R$, is the space c_0 of all sequences of real numbers, convergent to zero (see [8]). Recall that $g_1 \geq g_2$ in R^* means $g_1(x) \geq g_2(x)\ \forall\, x \in R$, $x \geq 0$, and that an element $x \in R$ satisfies $x \geq 0$ if and only if $g(x) \geq 0$ holds for each $0 \leq g \in R^*$

(see also [3], Theorem 5.1, p.55). For each $x \in R$, an order continuous linear functional $\widehat{x}$ can be defined on R^* via the formula $\widehat{x}(f) = f(x)$, $f \in R^*$. Thus, a positive operator $x \mapsto \widehat{x}$ can be defined from R into R^{**}. This operator, which we denote by c, is called the *canonical embedding* of R into R^{**}. The map c is one-to-one if and only if R^* separate points of R. In economic models, a way to describe the commodity-price system is the pair (R, R^*), in which the hypothesis that R^* separates points of R is essential (see [2], pp. 100 and 115).

3 The p-integral.

Let X be a Hausdorff compact topological space. If $A \subset X$, then the interior of the set A is denoted by int A.

We shall work with a family $\mathcal{F}$ of compact subsets of X such that $X \in \mathcal{F}$ and closed under intersection and a monotone and additive mapping $\lambda : \mathcal{F} \to [0, +\infty)$. The additivity means that

$$\lambda(A \bigcup B) + \lambda(A \bigcap B) = \lambda(A) + \lambda(B)$$

whenever $A, B, A \bigcup B \in \mathcal{F}$.

By a *partition* (in detail, $(\mathcal{F}, \lambda)$-*partition*) of a nonempty set $A \in \mathcal{F}$ we mean a finite collection $\Pi = \{(A_1, \xi_1), \dots, (A_q, \xi_q)\}$ such that:

(i) $A_1, \dots, A_q \in \mathcal{F}, \bigcup_{i=1}^{q} A_i = A$,

(ii) $\lambda(A_i \bigcap A_j) = 0$ whenever $i \neq j$,

(iii) $\xi_j \in A_j$ $(j = 1, \dots, q)$.

Sometimes, when no confusion can arise, we will indicate by *partition of* A a finite collection $\{A_j : j = 1, \dots, q\}$, satisfying conditions (i) and (ii). If $F : \mathcal{F} \to \mathbb{R}$ is a set function and $\Pi = \{A_j : j = 1, \dots, q\}$ is a partition of $\emptyset \neq A \in \mathcal{F}$, we denote by $\sum_{\Pi} F$ the quantity $\sum_{j=1}^{q} F(A_j)$.

We shall assume that $\mathcal{F}$ *separates points* in the following way: to any $A \in \mathcal{F}$ there exists a sequence $(\mathcal{A}_n)_n$ of partitions of A such that

(i) $\mathcal{A}_{n+1}$ is a refinement of $\mathcal{A}_n$,

(ii) for any $x, y \in A$, $x \neq y$, there exist $n \in \mathbb{N}$ and $B \in \mathcal{A}_n$ such that $x \in B$ and $y \notin B$.

We note that this assumption is fulfilled if $\mathcal{F}$ consists of all compact sets and the topological space X is metrizable or it satisfies the second axiom of countability (see [13]).

A *gauge* on a set $A \in \mathcal{F}$ is a mapping δ assigning to every point $x \in A$ a neighborhood $\delta(x)$ of x. We endow the set of all gauges on A with the order

$$\delta_1 \leq \delta_2 \Longleftrightarrow \delta_1(x) \subset \delta_2(x) \quad \forall\, x \in A.$$

If $\Pi = \{(A_1, \xi_1), \dots, (A_q, \xi_q)\}$ is a partition of A and δ is a gauge on A, then we say that Π is *δ-fine* if $A_j \subset \delta(\xi_j)$ for any $j = 1, 2, \dots, q$.

We obtain a simple example putting $X = [a, b] \subset \mathbb{R}$ with the usual topology, $\mathcal{F}$ =the family of all closed subintervals of X, $\lambda([\alpha, \beta]) = \beta - \alpha$, $a \leq \alpha < \beta \leq b$. Any gauge can be represented by a real function $d : [a, b] \to \mathbb{R}^+$, if we put $\delta(x) = (x - d(x), x + d(x))$.

Another example is the unbounded interval $[a, +\infty] = [a, +\infty) \bigcup \{+\infty\}$ considered as the one-point compactification of the locally compact space $[a, +\infty)$. The base of open sets consists of open subsets of $[a, +\infty)$ and the sets of the type $(b, +\infty) \bigcup \{+\infty\}$, $a \leq b < +\infty$. Any gauge in $[a, +\infty]$ has the form $\delta(x) = (x - d(x), x + d(x))$, if $x \in [a, +\infty] \bigcap \mathbb{R}$, and $\delta(+\infty) = (b, +\infty] = (b, +\infty) \bigcup \{+\infty\}$, where d denotes a positive real-valued function defined on $[a, +\infty)$, and b denotes a real number, with $b \geq a$.

We now define the p-integral on X. If $\Pi = \{(A_1, \xi_1), \dots, (A_q, \xi_q)\}$ is a partition of a set $A \in \mathcal{F}$, and $f : X \to R$, then we define the Riemann sum by

$$S(f, \Pi) = \sum_{j=1}^{q} \lambda(A_j)\, f(\xi_j).$$

We note that the fact that $\mathcal{F}$ separates points guarantees the existence of at least one δ-fine partition for any gauge δ (see [13], [19]).

Definition 3.1. A function $f : X \to R$ is *p-integrable* on a set $A \in \mathcal{F}$, if there exists $I \in R$ such that $\forall\, \varepsilon > 0$ and $\forall\, g \in R^*$ there exists a gauge δ on A such that

$$|g(S(f, \Pi)) - g(I)| \leq \varepsilon \tag{1}$$

whenever Π is a δ-fine partition of A. We denote I by $\int_A f$.

Remark 3.2. We note that, if I_1, $I_2 \in R$ satisfy (1), then $I_1 = I_2$. Indeed, for all $g \in R^*$, $g \geq 0$, and for large enough partition Π, we get

$$|g(I_1) - g(I_2)| \leq |g(I_1) - g(S(f, \Pi))| + |g(S(f, \Pi)) - g(I_2)| \leq 2\,\varepsilon. \tag{2}$$

Since R^* separates the points of R, then we get $I_1 - I_2 = 0$, that is the assertion. □

Remark 3.3. It is easy to check that, in the case $R = \mathbb{R}$, the p-integral coincides with the classical Kurzweil-Henstock integral, as introduced in [5]. In this case, we often will use the term "integrable" instead of "p-integrable".

We now state the main properties of the p-integral.

Proposition 3.4. *If f_1, f_2 are p-integrable on $A \in \mathcal{F}$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is p-integrable on A and*

$$\int_A (c_1 f_1 + c_2 f_2) = c_1 \int_A f_1 + c_2 \int_A f_2.$$

The proof is similar to the one of [12], Theorems 2.5.1 and 2.5.3.

Proposition 3.5. *If $f : X \to R$ is p-integrable on $A \in \mathcal{F}$, then for every $g \in R^*$ the real-valued map $g \circ f$ is integrable on $\mathcal{A}$, and*

$$\int_A g \circ f = g\left(\int_A f\right).$$

Conversely, if $f : X \to R$ is such that $g \circ f$ is integrable on $A \in \mathcal{F}$ for each $g \in R^*$ and there exists $I \in R$ such that

$$\int_A g \circ f = g(I) \quad \forall\, g \in R^*,$$

then f is p-integrable on A, and $\displaystyle\int_A f = I$.

Proof. Fix an arbitrary $g \in R^*$ and a partition Π of A, $\Pi = \{(A_i, \xi_i) : i = 1, \ldots, q\}$. We have

$$\begin{aligned} g(S(f,\Pi)) &= g\left(\sum_{i=1}^{q} \lambda(A_i)\, f(\xi_i)\right) \\ &= \sum_{i=1}^{q} \lambda(A_i)\, g(f(\xi_i)) = S(g \circ f, \Pi). \end{aligned} \tag{3}$$

The assertion follows from (3) and definitions of integrability and p-integrability. □

Proposition 3.6. *If f_1 and f_2 are p-integrable on $A \in \mathcal{F}$ and $f_1 \leq f_2$, then $\int_A f_1 \leq \int_A f_2$.*

Proof. Fix arbitrarily $g \in R^*$, $g \geq 0$. Then $g \circ f_1 \leq g \circ f_2$. By the first part of Proposition 3.5 and Proposition 1.4 of [13] we get that $g \circ f_1$ and $g \circ f_2$ are integrable on A, and

$$\int_A g \circ f_1 \leq \int_A g \circ f_2. \tag{4}$$

Again by Proposition 3.5, we have

$$\int_A g \circ f_l = g\left(\int_A f_l\right), \quad l = 1, 2. \tag{5}$$

From (4) and (5) it follows that

$$g\left(\int_A f_1\right) \leq g\left(\int_A f_2\right). \tag{6}$$

The assertion follows from (6) and arbitrariness of $g \in R^*$. □

A simple consequence of Proposition 3.6 is the following assertion.

Corollary 3.7. *If both f and $|f|$ are p-integrable on $A \in \mathcal{F}$, then*

$$\left|\int_A f\right| \leq \int_A |f|.$$

We now state the following results.

Proposition 3.8. *Let $u \in R$, $u \geq 0$. For every $E \in \mathcal{F}$, the function $f = \chi_E\, u : X \to R$ satisfies the condition*

$$\exists\, I \in R \text{ such that } \forall\, \varepsilon > 0, \exists \text{ gauge}\, \delta \text{ such that } |S(f,\Pi) - I| \leq \varepsilon\, u \tag{7}$$

for all δ-fine partition Π of X.

Proof. It is enough to apply Proposition 1.5., pp. 155–156, of [13], and to use the same technique as in Theorem 3.18 of [6]. □

Proposition 3.9. *Let $f : X \to R$ satisfy condition (7) for suitable I and $u \in R$, $u \geq 0$. Then f is p-integrable on X, and $\int_X f = I$.*

Proof. Let I and u be as in the hypothesis of the proposition. Fix an arbitrary $\varepsilon > 0$ and $g \in R^*$, $g \geq 0$. Then there exists $\eta > 0$ such that

$$\eta\, g(u) \leq \varepsilon. \tag{8}$$

Moreover, by condition (7), in correspondence with η there exists a gauge δ such that

$$|S(f,\Pi) - I| \leq \eta\, u \tag{9}$$

for all δ-fine partitions Π of X. From (8) and (9) it follows that

$$|g(S(f,\Pi)) - g(I)| \leq g(\eta\, u) = \eta\, g(u) \leq \varepsilon \tag{10}$$

for all δ-fine partitions Π of X. The assertion follows from (10). □

Proposition 3.10. *For every $E \in \mathcal{F}$ and $u \in R$ the function $\chi_E\, u$ is p-integrable on X and $\int_X \chi_E\, u = \lambda(E)\, u$.*

Proof. Since R is a Riesz space, we have $u = u^+ - u^-$, where $u^+, u^- \in R$, $u^+ \geq 0$, $u^- \geq 0$. So, we can suppose, without loss of generality, that $u \geq 0$. The assertion follows from Propositions 3.8 and 3.9. □

4 Convergence Theorems

The following theorem generalizes to the context of Riesz spaces and our Pettis-type integral Theorem 3.1 of [5], which was formulated for real-valued functions.

Theorem 4.1. *Let $X = X_0 \bigcup \{x_0\}$ be the one-point compactification of a locally compact space X_0. Let $f : X \to R$ be a function such that $f(x_0) = 0$. Let $(A_n)_n$ be a sequence of sets, such that $A_n \in \mathcal{F}$, $A_n \subset \operatorname{int} A_{n+1}$, $A_{n+1} \setminus \operatorname{int} A_n \in \mathcal{F}$, $\lambda(A_n \setminus \operatorname{int} A_n) = 0$ $(n = 1, 2, \ldots)$, $\bigcup_{n=1}^{\infty} A_n = X_0$. Let f be p-integrable on A_n $(n = 1, 2, \ldots)$ and let there exist in R an element I such that, $\forall\, \varepsilon > 0$, $\forall\, g \in R^*$, there exists an integer n_0 such that*

$$\left| \int_A g \circ f - g(I) \right| \leq \varepsilon \quad \forall\, A \in \mathcal{F},\ X_0 \supset A \supset A_{n_0}.$$

Then f is p-integrable on X and $\displaystyle\int_X f = I$.

Proof. By hypothesis and the first part of Proposition 3.5, we get that $g \circ f$ is integrable on A_n for all $g \in R^*$. Moreover, by Theorem 3.1 of [5], $g \circ f$ is integrable on X and

$$\int_X g \circ f = g(I). \tag{11}$$

The assertion follows by (11) and the second part of Proposition 3.5. □

We now state a monotone convergence Levi-type theorem.

Theorem 4.2. *Let $f_n : X \to R$, $n \in \mathbb{N}$ be p-integrable, $\left(\int_X f_n\right)_n$ be bounded, and suppose that for every $g \in R$, $g \geq 0$, and $\forall\, x \in X$, $g(f_n(x)) \uparrow g(f(x))$. Then f is p-integrable and $\sup_n \int_X f_n = \int_X f$.*

Proof. Fix an arbitrary $g \in R^*$, $g \geq 0$. By hypothesis, we get that the sequence $\left(g\left(\int_X f_n\right)\right)_n$ is bounded. Thus, by [13], Theorem 2.2, pp. 159–162 and the first part of Proposition 3.5, the real-valued function $g \circ f$ is integrable and

$$\begin{aligned} \int_X g \circ f &= \lim_n \int_X g \circ f_n = \sup_n \int_X g \circ f_n \\ &= \sup_n \left[g\left(\int_X f_n\right)\right] = g\left(\sup_n \int_X f_n\right). \end{aligned} \tag{12}$$

The assertion follows from (12) and the second part of Proposition 3.5. □

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